SOFTWARE FOR CONFORMAL BOUNDARY OPERATORS, T-CURVATURES, AND CONFORMAL FRACTIONAL LAPLACIANS OF ODD ORDER

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Abstract. This document contains some guidelines for understanding and using the computer software files that accompany this document. The author used the these files, together with *Mathematica* and John M. Lee's *Ricci* software package, to compute many of the symbolic formulae appearing in Gover and Peterson's article "Conformal boundary operators, T-curvatures, and conformal fractional Laplacians of odd order."

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1. Overview

Several computer software files should accompany this document. These software files relate to the article "Conformal boundary operators, T-curvatures, and conformal fractional Laplacians of odd order," by A. Rod Gover and Lawrence J. Peterson, [3]. This document contains some guidelines for reading, understanding, and using these software files. The files, along with this document, are available on the University of North Dakota Scholarly Commons. These Scholarly Commons are located at

`https://commons.und.edu`

The second author of [3] developed most of the software files that accompany this document, and he then used these files to derive and verify several of the symbolic formulae that appear in that article. In doing this, he used *Mathematica* together with John M. Lee's *Ricci* software package. See [6, 5]. *Mathematica* is a computer algebra system that one may purchase and install on most personal computers. *Ricci* is a *Mathematica* package that one may use together with *Mathematica* to perform tensor calculus operations in differential geometry. *Ricci* runs with *Mathematica*. One may only use *Ricci* if one is also using *Mathematica*.

Twenty-two of the software files are *Mathematica* "notebooks." These notebooks contain *Ricci* and *Mathematica* software commands which the author of this document used in his work with [3]. The names of all twenty-two notebooks end with the suffix "`.nb`." All twenty-two notebooks contain commands which ask *Mathematica* to read in various other files. The names of all of these files end in the suffix "`.m`." These files should all accompany this document. One of these files is the file `Ricci1.61.m`. This file contains the source code for the *Ricci* software package, [5]. Lee has granted everyone permission to copy, modify, and redistribute the *Ricci* package *subject to certain restrictions*. He states these restrictions at the top of the `Ricci1.61.m` file. You may view this file with most file reader programs. Each of the twenty-two notebooks reads in the the `Ricci1.61.m` file.

Further information on *Ricci*, including a user's manual, may be available at Lee's Web site. Lee's Web site is currently located at

`https://sites.math.washington.edu/~lee`

One may also obtain information about *Ricci* from *Ricci* itself. To do this, one begins by opening a notebook and issuing the command to read in the `Ricci1.61.m` file. One then types in and runs a command beginning with the "?" symbol and followed by the name of a *Ricci* command. For example, the command "`?TensorSimplify`" returns a brief description of *Ricci*'s `TensorSimplify` command.

This document includes a list of the file names of the twenty-two notebooks that we discussed above and a brief description of the contents of each of these notebooks. It also contains a list of many of the other files that accompany this document, along with brief descriptions of the contents of these files. In addition, this document contains some technical notes which may help the reader to download the notebooks and accompanying files and actually run the notebooks with *Mathematica*. (To "run" a notebook means to have *Mathematica* execute the software commands in the notebook.) One section of this document provides a warning which may help readers avoid problems in future work sessions with *Mathematica*. This document also contains a section that discusses the sign conventions and notation of [2]; this will help the reader understand some of the notebooks. The document contains a disclaimer statement and some notes on possible future versions of *Mathematica*. As of the time of this writing, an updated version of [3] is under review for publication in a journal. The final section of the present document discusses this updated version and the verification of the symbolic formulae in that updated version.

The authors of [3] collaborated on the writing of their article, and they also discussed the software work related to the article. But except for the file `Ricci1.61.m`, the second author of [3] was the sole developer of the software code that accompanies this document. So for the remainder of the present document and throughout the twenty-two *Mathematica* notebooks that accompany it, the words "I," "my," and "me" will refer to the author of the present document. The words "we," "our," and "us" will refer to Gover and me or to the reader and me.

2. Disclaimer

Rod Gover and I are not responsible for the consequences of any errors in this document or the accompanying files. I am nevertheless confident that the statements and formulae in the updated versions of [3] are correct.

3. Downloading, reading, and using the files

Depending on the software available on one's computer, it may be possible for a person to read my twenty-two notebooks and the associated files without saving them to a computer hard drive or other storage device. If the name of a file ends in the "`.m`" suffix, one should be able to read the file with most text-reading or text-editing programs. To read the notebooks easily, however, one will need a working copy of *Mathematica*. If *Mathematica* is not available, it may be possible to read the notebooks by using the "Wolfram Player." The Wolfram Player may allow one to view *Mathematica* notebooks, but it may not allow a person to actually run the commands in the notebooks. For further information on the Wolfram Player, visit

`http://www.wolfram.com/player`

To run my notebooks, one will need a computer and a working copy of *Mathematica* on that computer. I ran all twenty-two notebooks with *Mathematica* 11.0.1. The notebooks may not run correctly with older or newer versions of *Mathematica*. See the section on "Future versions of *Mathematica*" later on in this document.

To run the notebooks, one should download the notebooks and all of the other files that accompany this document. The main reason for downloading the notebooks and the accompanying files is the fact that all twenty-two notebooks contain commands which ask *Mathematica* to read in external files. These commands contain the locations, or directory paths, of the external files. If the commands are to work effectively, one will likely need to first download the external files.

My *Ricci* notebooks and files, as well as the file `Ricci1.61.m`, are all located in one combined file located on the University of North Dakota Scholarly Commons. The University of North Dakota Scholarly Commons is a repository of scholarly work administered by the Chester Fritz Library at the University of North Dakota. The Scholarly Commons are located on the Web at the location I provided earlier in this document. After arriving at Scholarly Commons, one should search for "Lawrence J. Peterson." One may also search for the title of this document.

If one wishes to download the combined file, *one needs to do this carefully*. One would download the combined file and "extract" the twenty-two notebooks and the other files from it. This extraction process would create a new directory structure on one's computer or storage device. This new directory structure would contain all of the notebooks and other files. There is a slight danger, however, that the downloading and extraction process will overwrite previously existing files on the computer. The file extraction process will create a new directory with the name `bop`. This could cause a problem if one's computer has a previously existing file, directory, or folder with this name. To reduce the chances of problems, one should create a new directory on his or her computer. One should do this before downloading the combined file. One should then download the combined file by clicking on the appropriate button on the Scholarly Commons page. One should store the combined file in the new directory.

The name of the combined file is `bop.zip`. After downloading `bop.zip`, a Linux user should be able to use the `unzip` command to extract the notebooks and other files from the `bop.zip` file. The `unzip` command takes the files in `bop.zip` and uses them to create a directory structure on the user's computer. The new directory structure contains the extracted files. I executed the `unzip` command on my computer as follows. First, I opened a window containing the command prompt. I then navigated to the directory on which I had saved the `bop.zip` file. Finally, I entered the following command at the command prompt:

```
unzip bop.zip
```

This command created a new directory structure within the directory containing the `bop.zip` file. This new directory structure contained my twenty-two notebooks and the other related files.

It may also be possible for a Linux user to extract the files from `bop.zip` by working through a directory navigator and file viewer program. In doing this, one still needs to be aware of my above warning concerning the overwriting of existing files on the computer. In any case, the user would navigate to the directory containing the `bop.zip` file and then click on the file name ("`bop.zip`").

Users of Microsoft Windows may also be able to extract the files by working with a file navigator program. If the Windows user clicks on the file `bop.zip`, a menu for "Extract" may appear. By choosing the correct options, it may be possible to have Windows extract the files and create a directory or folder structure containing the files. The same warning applies.

Similar procedures and warnings may apply to other operating systems.

As I noted above, all twenty-two of my notebooks contain commands to read in other files. Such commands begin with the characters "`<<`∼." When one encounters such commands, it will be obvious that the commands are designed to read in other files. One may need to edit these commands to ensure that they are consistent with the directory (or folder) structure on the computer. When one does this editing, one may find it helpful to locate occurrences of the characters "`<<`∼" by using the *Mathematica* "Find" command. This command should be available through the "Edit" menu near the top of the computer screen. After one edits the text-reading commands, these commands may still not work properly. If one encounters problems running these edited commands, it may help to delete the first few characters at the beginning of the commands and then type them in again. After one completes this process, the commands may work properly.

Warning: In the present document (this guidelines document), the symbol "∼" denotes the tilde character that appears on most computer keyboards. Some readers may read an on-line version of this guidelines document. In this case, copying and pasting operations involving the ∼ symbol may not work properly. After the pasting operation, the reader may need to delete the ∼ symbol and type it in again.

When a user of Microsoft Windows runs a command asking *Mathematica* to read in a file, *Mathematica* will likely reject the "∼" character. If one replaces the "∼" character with a capital letter followed by a colon (":"), this might solve the problem. The capital letter should refer to the computer drive containing the twenty-two notebooks and the related files.

4. Warning concerning default the output format type

If one chooses to run any of my twenty-two notebooks, one should set *Mathematica*'s default output format type to "`OutputForm`." This will allow *Mathematica* to format Ricci output properly. Each of my twenty-two notebooks contains a command which sets the default output format type to "`OutputForm`." When one uses this command, however, one needs to be careful. *Mathematica* may record the value of the default output format type in its configuration files. In the future, when a person starts a new work session with *Mathematica*, *Mathematica* may use the default output format type that it recorded during the previous *Mathematica* work session. **If this default output format type is "OutputForm," then, in the future work session, *Mathematica* may not display graphical output.** This could be a problem for people who use *Mathematica* for purposes other than work with Ricci.

To allow *Mathematica* to display graphical output in future *Mathematica* work sessions, it is very important to set the default output format type to either "`TraditionalForm`" or "`StandardForm`." (With *Mathematica* 11.0.1, one may also set the default output format type to "`InputForm`," but this may produce undesirable results.) One may change the default output format type at the conclusion of a work session with any of my twenty-two notebooks. Each of my twenty-two notebooks contains commands to set the default output format type to either "`TraditionalForm`" or "`StandardForm`." These commands appear at the end of each notebook. Alternatively, one may update the default output format type at the beginning of a future work session with *Mathematica*.

Further information appears near the beginning of the notebook `Alias.nb`. The reader should read the special warning statement at the beginning the `Alias.nb` notebook.

5. Future versions of *Mathematica*

I ran all twenty-two of my *Ricci* notebooks on an Ubuntu Linux system with *Mathematica* 11.0.1. Future versions of *Mathematica* may not be compatible with *Ricci*. If a compatibility problem should occur, *Mathematica* will likely display an error message at the time that one runs the command to read in the file `Ricci1.61.m`. If one encounters compatibility problems involving newer versions of *Mathematica*, one should consult John M. Lee's Web site to see if he has developed a newer updated version of *Ricci* that is compatible with the latest version of *Mathematica*. See [5]. If a suitable updated version of *Ricci* is not available, it may be possible for people to solve compatibility problems by modifying the *Ricci* source code themselves. But before doing this, people should read the comments at the top of the file `Ricci1.61.m`. It may also be possible to solve the compatibility problems by running appropriate *Mathematica* commands before one runs the command to read in the file `Ricci1.61.m`.

Compatibility problems may occur if new versions of *Mathematica* contain "reserved" or "protected" words that match identifiers in the *Ricci* source code. To solve such compatibility problems, it may be helpful to begin by studying the comments and code near the top of the `Ricci1.61.m` file. If one modifies the *Ricci* source code, one should be sure to document the changes in the manner described at the top of the `Ricci1.61.m` file.

The software that accompanies this document (the notebooks and the files whose names end in "`.m`") includes many files other than `Ricci1.61.m`. These other files also contain definitions that may be incompatible with future versions of *Mathematica*. If these files cause compatibility problems in the future, it may be possible to solve these problems by using techniques similar to the ones I have described above.

6. Contents of files

This section gives a brief description of the contents of each of my twenty-two *Mathematica* notebook files. It also describes the contents of a few of the other files that should accompany this document. All of the notebooks are contained in the directory (or folder) ~`/bop/nbk`. The other files are in other directories. I will begin by listing the name of each notebook along with a brief description of the contents of the notebook.

Alias.nb: At certain points in my work, I need to encode intrinsic derivatives into "alias" tensor identifiers. This encoding process helps us to distinguish between intrinsic derivatives and extrinsic derivatives. It also helps us with the process of converting intrinsic derivatives into expressions involving extrinsic derivatives. In the notebook `Alias.nb`, I test some of the rules and definitions that I use to perform this encoding process.

BigFour.nb: In this notebook, I confirm the correctness of the formula for $\delta_{1,3}$ (in dimensions $n \geq 5$) given in Figure 3 of [3]. This figure is in Section 8 of [3].

CQ2017.nb: Here I construct Chang and Qing's third-order operator P_3^{CQ}. A displayed equation involving this operator appears near the end of Section 8.2 of [3]. I verify this equation in this notebook.

CQ2019T.nb: See Section 8, below, for information on this notebook.

CQ2020B.nb: See Section 8, below, for information on this notebook.

EllTwo.nb: In Section 8.2 of [3], Gover and I use the notation $\ell_2 f$ to denote $L^{AB} D_A D_b f$, where f is a density of weight w. A display near the beginning of this section of [3] contains a formula for $\ell_2 f$. In this notebook, I verify the correctness of the formula in this display.

IDLXIDL.nb: In this notebook, I compute the scalar invariant

$$\frac{1}{n-5}(\bar{D}_A L_{BC})\bar{D}^A L^{BC} \ .$$

I use the resulting formula in the notebook `BigFour.nb`.

IntDCheck.nb: In this notebook, I test and illustrate some commands dealing with intrinsic derivatives. Specifically, I test and illustrate each of the following commands: `NbSigHatDen`, `NbSigHatEta`, `NbNbSigHatDen`, `NbSigHatEtaEta`, `NbNbSigHatEtaEta`, and `LapSigHatDen`.

IntTen.nb: This notebook contains tests of some rules for intrinsic tensors. Specifically, it contains tests involving the following tensors on the submanifold Σ: the Riemannian curvature tensor, the Ricci tensor, the scalar curvature tensor, the Schouten tensor, and the tensor J.

LIBoxLfW.nb: In this notebook, I construct $L^{AB}\bar{\Box}L_{AB}f$, where f is a density of weight w. The resulting formula will be useful in the notebook `BigFour.nb`.

NewQ2.nb: In this notebook, I compute $Q_g(\delta_2)$. I do so twice. First, I use Definition 7.5 of [3] to compute $Q_g(\delta_2)$. I then use Theorem 7.12.

NewQ3.nb: Here I compute $Q_g(\delta_{1,2})$. I do so twice. I begin by using Definition 7.5 of [3]. I then use Theorem 7.11 to compute the curvature a second time.

NewQ4.nb: Here I compute $Q_g(\delta_{1,3})$ twice. I use Definition 7.5 and Theorem 7.11 of [3].

NewQ5.nb: I compute $Q_g(\delta_{2,3})$ by using Definition 7.5 of [3] and again by using Theorem 7.11.

O5Form.nb: In this notebook, I show that if f is density of weight $2 - n/2$, then $\delta_{2,3}f = -60\delta P_4 f$. Here P_4 is the Paneitz operator. I also show that if $f \in \mathcal{E}[1 - n/2]$, then $\delta_{2,3}f = -5\delta_3\Box f$.

OFived.nb: I use this notebook to construct a symbolic formula for $\delta_{2,3}$.

OFourd.nb: In this notebook, I construct the fourth-order operator $\delta_{1,3}$.

OThreed.nb: Here I compute a formula for $\delta_{1,2} : \mathcal{E}[w] \to \bar{\mathcal{E}}[w - 3]$. In Section 1 of [3], we discuss the operators δ_3^G of [4]. These operators exist in all dimensions $n \geq 4$. In `Othreed.nb`, I show that $\delta_3^G : \mathcal{E}[(4 - n)/2] \to \bar{\mathcal{E}}[(-2-n)/2]$ and $\delta_{1,2} : \mathcal{E}[(4-n)/2] \to \bar{\mathcal{E}}[(-2-n)/2]$ have the same leading term, up to a nonzero scale.

OThreeF.nb: In this notebook, I verify the correctness of the formula for $\delta_{1,2}f$ given in Figure 2 of [3].

OTwo.nb: This notebook contains computations relating to Section 8.1 of [3]. In this notebook, I compute a formula for δ_2. I also verify the correctness of the last two displayed equations in Section 8.1 of [3].

TestConv.nb: In this notebook, I show that [3] and my *Ricci* work use the same sign convention for the Riemannian curvature tensor and that they place the indices on this tensor in an equivalent way.

TracSFF.nb: In this notebook, I verify the correctness of my *Ricci* rule for expanding the tractor second fundamental form L_{AB}. To do this, I use the formula for L_{AB} given in Proposition 3.7 of [3].

I now list a few of my other files, along with a brief description of the contents of each of these other files. I also give the directory path to each of these files. If the reader downloads the files, the directory paths may change. See my earlier warning concerning the copying and pasting of the "∼" symbol.

∼/bop/bnb/Fourth.m: This file contains a symbolic formula for $\delta_{1,3}$.

∼/bop/bnb/OFive.m: This file contains a symbolic formula for $\delta_{2,3}$.

∼/bop/bnb/OThree.m: This file contains a symbolic formula for $\delta_{1,2}$.

~/bop/bnb/OTwo.m: This file contains a symbolic formula for δ_2.

~/bop/bnb/QQThree.m: This file contains a symbolic formula for the curvature quantity $Q_g(\delta_{1,2})$.

~/bop/bnb/QQTwo.m: This file contains a symbolic formula for $Q_g(\delta_2)$.

~/bop/cnb/IDLXIDL.m: This file contains an expanded symbolic formula for the following scalar invariant:

$$\frac{1}{n-5}(\bar{D}_A L_{BC})\bar{D}^A L^{BC} \ .$$

~/bop/cnb/IPanW.m: This file contains an expanded symbolic formula for

$$-Y^A \overline{\Box} \bar{D}_A f \ ,$$

where f is a density of weight w.

~/bop/cnb/LIBoxLfW.m: This file contains an expanded symbolic formula formula for

$$L^{AB} \overline{\Box} L_{AB} f \ ,$$

where f is a density of weight w.

~/bop/cnb/NewQ5.m: This file contains a symbolic formula for $Q_g(\delta_{2,3})$.

~/bop/cnb/QQFour.m: This file contains a symbolic formula for $Q_g(\delta_{1,3})$.

~/bop/cnb/RawPull.m: This file contains an expression used in the notebook `BigFour.nb`.

~/bop/fnb/CQP3Op.m: This file contains an expanded symbolic formula for Chang and Qing's third-order operator P_3^{CQ} as acting on a smooth function f along the boundary of Σ.

~/bop/fnb/CQT.m: This file contains an expanded symbolic formula for Chang and Qing's T-curvature.

The directories ~`/bop/gen/dtm`, ~`/bop/gen/dtm2`, and ~`/bop/gen/dtm3` contain still more files. These files contain definitions relating to the work in my twenty-two notebooks. Most of these definitions involve commands from the *Ricci* package, [5]. In many of these definitions, I define my own commands for use in the twenty-two notebooks. I also use *Ricci*'s `DefineTensor` command to define various tensors. For information on these commands and tensors, one may examine the definitions of these commands and tensors in the files of the `dtm`, `dtm2`, and `dtm3` directories. The `dtm3` directory contains the file `Ricci1.61.m`.

My notebooks also use several commands defined by the *Ricci* package itself. For example, the notebooks use the following commands: `CommuteCovD`, `DefineRule`, `FactorConstants`, `NewDummy`, `SuperSimplify`, `TensorExpand`, and `TensorSimplify`. These commands are all defined in the *Ricci* software package. Further information on these commands is available in the *Ricci* user's manual and from the *Ricci* package itself. I discuss these sources of information in Section 1, above.

7. Chang and Qing's notation and sign conventions

Figure 1 describes Chang and Qing's notation for various tensors and operators

Chang and Qing's Notation		Our Notation
N	(the inward unit geodesic normal)	n^a (the inward unit normal)
L_{ab}	(the second fundamental form)	$-L_{ab}$
Δ	(the Laplacian)	$-\Delta = -\nabla_a \nabla^a$
τ	(the scalar curvature)	Sc
ρ_{ab}	(the Ricci tensor)	Rc_{ab}
R^i_{jkl}	(the Riemannian curvature tensor)	$R^i{}_{jkl}$
H	(a multiple of the mean curvature)	$-3H$
J	($J = \tau/6$)	J (in dimension $n = 4$)
$G^a_b = R^a_{NbN}$		$G^a{}_b = R^a{}_{ibj} n^i n^j$
$F = G^a_a = R^a_{NaN}$		$F = G^a{}_a = R^a{}_{iaj} n^i n^j = \mathrm{Rc}_{ij} n^i n^j$
tr L^3		$-L_a{}^b L_b{}^c L_c{}^a$

Figure 1. Chang and Qing's notation and the corresponding notation of [3] (our notation)

as described in [2], along with our corresponding notation for the same tensors and operators. (Throughout this section of this document, "our notation" will refer to the notation of [3]. The April 2020 updated version of [3] uses the same notation as [3].) The information in Figure 1 may help the reader to understand the contents of the notebooks `CQ2017.nb` and `CQ2019T.nb`. In what follows, I will justify some of my conclusions concerning Chang and Qing's notation.

I begin by considering Chang and Qing's sign convention for the second fundamental form. We work in a local coordinate system $(x^1, \ldots, x^n)$ on a collar of Σ. On page 332 of [2], Chang and Qing say that they will "first recall the framework set up" by Branson and Gilkey in [1] and "at the same time introduce all notations." On page 335 of [2], Chang and Qing then say that $L_{ab} = -\frac{1}{2} N g_{ab}$. Branson and Gilkey define L_{ab} in the same way. Here N is the inward unit geodesic normal in a collar of Σ. Along Σ, N is the same as our n^a. By [1], we may assume that $\partial/\partial x^n = N$ along Σ. (We use n to denote the m of [1].) Thus in our notation, $\partial/\partial x^n = n^c \partial/\partial x^c$ along Σ. Thus along Σ, the L_{ab} of [1, 2] is the tensor whose component L_{ab} is given by

$$L_{ab} = -\frac{1}{2}\, n^c \frac{\partial}{\partial x^c}\, g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) \ .$$

For all integers $1 \leq i \leq n$, we now let ∂_i denote $\partial/\partial x^i$. Let L_{ab} be as above. Then by the Weingarten equations and elementary properties of the Levi-Civita connection,

$$L_{ab} = -\tfrac{1}{2} N g(\partial_a, \partial_b) = -\tfrac{1}{2} g(\nabla_{\partial_n} \partial_a, \partial_b) - \tfrac{1}{2} g(\partial_a, \nabla_{\partial_n} \partial_b) =$$
$$-\tfrac{1}{2} g(\nabla_{\partial_a} \partial_n, \partial_b) - \tfrac{1}{2} g(\partial_a, \nabla_{\partial_b} \partial_n) = -g(\nabla_{\partial_a} \partial_n, \partial_b) \ .$$

Now let X^a and Y^b be any vector fields which are tangent to Σ along Σ. Then with L as above,

$$L(X,Y) = X^a Y^b L_{ab} = -g(X^a \nabla_a \partial_n, Y^b \partial_b) = -g(\nabla_X N, Y)$$

along Σ.

Now instead let L denote *our* second fundamental form (the second fundamental form of [3]). Then

$$L(X,Y) = X^a Y^b \Pi_a{}^c \nabla_c n_b = (X^c \nabla_c n_b) Y^b = g_{ij}(\nabla_X N^i) Y^j = g(\nabla_X N, Y)$$

along Σ. Thus our second fundamental form differs from the second fundamental form of [1, 2] by a sign.

In our notation, Rc_{ab} denotes the Ricci tensor. Let Sc denote the scalar curvature, i.e. $\mathrm{Rc}_a{}^a$. Chang and Qing's Laplacian is $\Delta = -\nabla_a \nabla^a$, and their Yamabe operator, in their notation, is

$$\Delta + \frac{m-2}{4(m-1)} \tau \ .$$

Here m is the dimension of M, and τ is Chang and Qing's scalar curvature. Our Yamabe operator, in our notation, is

$$\Delta - \frac{n-2}{4(n-1)} \mathrm{Sc} \ .$$

Our Laplacian is $\Delta = \nabla_a \nabla^a$, so our Yamabe operator differs from the Yamabe operator of Chang and Qing by a sign. Thus Chang and Qing's scalar curvature τ is the same as our scalar curvature Sc.

Chang and Qing's Ricci tensor ρ_{ab} satisfies $\rho_a{}^a = \tau$, and in our notation, $\mathrm{Rc}_a{}^a =$ Sc. Thus Chang and Qing's Ricci tensor is the same as ours. In Chang and Qing's notation, R^i_{jkl} denotes the Riemannian curvature tensor, and $\rho_{ij} = R^k_{ikj}$. In our notation, $\mathrm{Rc}_{ij} = R^k{}_{ikj}$. Thus Chang and Qing's R^i_{jkl} is equal to our $R^i{}_{jkl}$.

Chang and Qing state that, in their notation, $H = L^a_a$. Thus Chang and Qing's H, in our notation, is

$$\begin{aligned} &g^{ab}(-L_{ab}) = -g^{ab} \Pi_a{}^c \nabla_c n_b = -g^{ab}(\delta_a{}^c - n_a n^c) \nabla_c n_b = -(g^{bc} - n^b n^c) \nabla_c n_b = \\ &\quad -g^{bc} \nabla_c n_b = -\nabla_b n^b = -(n-1)H = -3H \ . \end{aligned}$$

Here the last equality follows from the fact that $n = 4$ in this context. Thus Chang and Qing's H, in terms of our notation, is $-3H$.

As we suggested above, [2] uses the "framework" of [1]. On page 495 of [1], Branson and Gilkey say that tr L^3 is an abbreviation for $L^a{}_b L^b{}_c L^c{}_a$. The L_{ab} of [1] is the same as that of [2], and both of these differ from our L_{ab} by a sign. Thus Chang and Qing's tr L^3 is given, in terms of our notation, by $-L_a{}^b L_b{}^c L_c{}^a$.

8. An updated version of our article

In April of 2020, we developed an updated version of [3]. As of this writing, this updated version is under consideration for publication in a journal. This section discusses the verification of some of the symbolic formulae in this updated version. This section may be more meaningful to readers who have access to this updated version. If the journal publishes our updated version, then the final published version will likely be very similar to the version we discuss in this section.

We now consider some of the symbolic formulae in the April 2020 updated version.

- A symbolic formula for the Q-type curvature of our $\delta_{1,2}$ operator family appears near the end of the introduction of the April 2020 version of our article. A verification of this formula appears in the notebook NewQ3.nb.
- A symbolic formula for our δ_2 operator as acting on a density appears near the end of Section 2 of the April 2020 version. A verification of this formula appears near the beginning of the notebook OTwo.nb.
- Near the beginning of the examples section, there is a figure containing a symbolic formula for our operator $\delta_{1,2}$ as applied to a density. A verification of this symbolic formula appears near the beginning of the notebook OThreed.nb. This formula is is given by the identifier StepAAFive in that notebook.
- The examples section of the April 2020 version contains a symbolic formula for $Q_g(\delta_2)$. A verification of this formula appears near the beginning of the notebook NewQ2.nb.
- The examples section of the April 2020 updated version contains a displayed equation which relates our operator δ_2 to the intrinsic Yamabe operator and the trace-free second fundamental form. A verification of this equation appears in the notebook CQ2020B.nb.
- The examples section of the April 2020 version contains a displayed equation which expresses $\bar{D}^A \Pi_A{}^B D_B$ in terms of $\overline{\Box}$ and other operators. We obtained this equation directly from page 43 of [4]. (We also made some notational changes to the equation.)
- The examples section of the April 2020 version contains an equation which effectively gives a symbolic formula for $(n + w - 3)\delta_2 f + \bar{D}^A \Pi_A{}^B D_B f$. A verification of this equation appears in the notebook CQ2020B.nb.
- The examples section of the April 2020 updated version contains a displayed equation which relates the third-order boundary operator of Chang and Qing to our operator $\delta_{1,2}$. The notebook CQ2020B.nb contains a verification of this equation.
- The examples section of the April 2020 updated version contains a displayed equation involving Chang and Qing's T-curvature. The notebook CQ2019T.nb contains a verification of this equation.

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