6-1-1967

# Convergence in Topological Spaces

Richard A. Jensen

CONVERGENCE IN TOPOLOGICAL SPACES

by

Richard A. Jensen

B.S. in Mathematics, Augustana College 1965

A Thesis

Submitted to the Faculty

of the

University of North Dakota

in partial fulfillment of the requirements

for the Degree of

Master of Science

Grand Forks, North Dakota

June
1967

This thesis submitted by Richard A. Jensen in partial fulfillment of the requirements for the Degree of Master of Science in the University of North Dakota is hereby approved by the Committee under whom the work has been done.

Thomas J. Robinson

Edward Nelson

K L Hankerson

Dean of the Graduate School

## ACKNOWLEDGMENTS

I would like to express my sincere appreciation to Dr. Thomas J. Robinson for his guidance and encouragement in the writing of this thesis. I would also like to thank Professors Edward O. Nelson and Kenneth L. Hankerson for serving on my committee.

# TABLE OF CONTENTS

## ABSTRACT

In this thesis the writer has considered types of convergence in an arbitrary topological space. Three types of convergence are considered, convergence of sequences in the real numbers with the usual topology, convergence of Moore-Smith sequences in an arbitrary topological space, and convergence of filters in an arbitrary topological space.

In Chapter II, we show that sequences are inadequate to describe limit points of sets and hence the topology in an arbitrary topological space. The idea of a sequence is generalized in this chapter to Moore-Smith sequences and in Chapter III to filters.

In Chapter II we prove that convergence of Moore-Smith sequences is sufficient to describe limit points, closed sets, the closure of a set, open sets, and in fact the topology of an arbitrary space. Although the convergence of filters and Moore-Smith sequences differ greatly, we prove these same results using filters. However, in Chapter II we also prove an iterated limit theorem using generalized Cauchy sequences in a complete metric space

Chapter IV is in many respects the most important, although containing the fewest results. For

in Chapter IV we are able to relate convergence of filters to convergence of Moore-Smith sequences. In fact we prove that filters and Moore-Smith sequences are equivalent convergence theories.

# CHAPTER I

## INTRODUCTION

Throughout this paper we assume the reader is familiar with the fundamentals of set theory and its notation.

In this paper we are concerned with types of convergence in an arbitrary topological space. The set of real numbers can be given a topology such that convergence of sequences is adequate to describe the topology. However, sequences contain only a countable number of points and are inadequate to describe the topology of an arbitrary topological space.

In Chapter II we generalize the idea of a sequence allowing it to contain an uncountable number of points. This type of sequence is known as a Moore-Smith sequence. We also give some examples showing why we must generalize the idea of a sequence. Then, using the concept of a Moore-Smith sequence, we prove certain basic theorems relating to an arbitrary topological space. Although a Moore-Smith sequence is a generalized sequence, limits of Moore-Smith sequences are analogous to limits in calculus.

In Chapter III we consider a type of convergence involving the concept of a filter. This type of convergence involves a collection of sets in a space instead of a sequence of points. In Chapter IV we show that the concepts of Moore-Smith sequences and filters lead to equivalent convergence theories.

## CHAPTER II

## GENERALIZED SEQUENCES

To begin our discussion we give some basic definitions concerning a topological space.

Definition 2.1. A topology is a family $\tau$ of sets which satisfies the following conditions:

(i) the intersection of any two members of $\tau$ is a member of $\tau$, and

(ii) the union of members of $\tau$ is a member of $\tau$.

Definition 2.2. The pair $(X, \tau)$ is a topological space when X is a set and $\tau$ is a class of subsets of X satisfying Definition 2.1. In this paper we use space as an abbreviation for topological space.

Definition 2.3. A subset G of a space X is open if and only if G is a member of $\tau$. A set G is said to be a neighborhood of a point $x \epsilon X$ if and only if G is an open set containing x.

Definition 2.4. A subset H of a space X is said to be closed if and only if X-H is open.

Definition 2.5. A point p of a space X is a limit point of a subset A if and only if every neighborhood of p contains points of A-{p}.

Using the above definitions it can be shown that a subset G of a space S is open if and only if for each $p \in G$ there exists a neighborhood U of p contained in G. It can also be shown that a subset H of a space S is closed if and only if H contains all its limit points.

A special kind of topological space is a metric space.

Definition 2.6. A set S is said to be metric if and only if there is associated with S a mapping $d: S \times S \rightarrow R$ having the following properties for every x,y,z in S:

(i) $d(x,y) \geq 0$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) = 0$ if and only if $x = y$

(iv) triangle property: $d(x,z) \leq d(x,y) + d(y,z)$.

The mapping d is called the metric for the set S.

Definition 2.7. Let K be a metric set. Then with each point p of K and each real number $r > o$, we associate a subset $S_r(p)$ of K called a spherical neighborhood of p. A point q of K is in $S_r(p)$ if and only if $d(p,q) < r$.

Definition 2.8. A metric set S is said to be a metric space if and only if the topology of S is that which is generated by the collection of subsets of S consisting of all spherical neighborhoods in S. The topology of S is said to be induced by the metric d.

At this point it may be helpful to consider some examples of a topological space.

Example 2.1. Let X be the set consisting of the three elements a, b, and c. Let $\tau$ consist of the sets X,{a}, {a,b}, {b}, and $\emptyset$, where $\emptyset$ is used to designate the empty set. The pair $(X,\tau)$ is a topological space. The union of members of $\tau$ is a member of $\tau$ and the intersection of any two members of $\tau$ is a member of $\tau$. Also, the sets are open, since they are members of $\tau$. In the example c is a limit point of {a,b} since every neighborhood of c, the space itself, contains points of {a,b}. The closed sets in this space are {c}, {b,c}, {a,c}, X since it is the compliment $\emptyset$, and $\emptyset$ since it is the compliment of X.

Definition 2.9. The closure of a set A is the intersection of the family of closed sets containing A and is denoted by $\bar{A}$. It can be shown that the closure of a set A is the union of the set A with the set of all its limit points. By its definition the closure of a set is closed.

In our example the closure of the set {a,b} is X, while the closure of {c} is {c}, since {c} contains all its limit points.

A more familiar topological space is the set of real numbers R with the topology described below.

The usual topology of the real numbers is the collections of all sets G such that for each $p \varepsilon G$ there exists an open interval (a,b) such that $p \varepsilon (a,b) \subset G$. It can be shown that this collection of sets will satisfy the conditions for a topology. The usual metric for R is $d(x,y) = |x-y|$, for $x,y \varepsilon R$. This metric induces the usual topology for R.

Using the concept of sequences we can describe limit points, closed sets, the closure of a set, and in fact the usual topology for R.

Definition 2.10. A sequence is a set A indexed by the set of positive integers. The nth element of the sequence is the element a of A which is indexed by n, i.e. $a_n$. The sequence is denoted by $\{a_n\}$.

Definition 2.11. A sequence $\{a_n\}$ of real numbers converges to a real number A if and only if given any real number $\epsilon>0$ there exists an integer N such that for all $n>N$ $|a_n - A|<\epsilon$.

The following theorem is an immediate consequence of the definition of convergence of sequences and the structure of the usual topology.

Theorem 2.1. The sequence $\{a_n\}$ converges to A if and only if, for every neighborhood G of A, there exists an integer N such that $a_n \in G$ for all $n>N$.

From the idea of limit point of a sequence we can proceed to the idea of limit point of a set.

Definition 2.12. A point p is a limit point of a subset H of R if and only if there exists a sequence $\{p_n\}$ of distinct points of H converging to p.

Theorem 2.2. Let R be the space of real numbers. Then:

(a) A point x belongs to the closure of a subset A if and only if there is a sequence in A converging to x.

(b) A set H is closed if and only if no sequence of distinct points in H converges to a point of R-H.

(c) A set G is open if and only if, for every sequence of distinct points converging to a point of $x \varepsilon G$, there exists N such that for all $n>N$, $x_n \varepsilon G$.

Proof: (a) Suppose x belongs to the closure of A. If $x \varepsilon A$, let every element in the sequence be x, and if x is a limit point of A, by Definition 2.13 there is a sequence of distinct points of A converging to x.

Now suppose there is a sequence in A converging to x. Then, by Theorem 2.1 every neighborhood G of x contains points of A. Thus x is either a member of A or a limit point of A.

(b) Suppose H is closed and that there is a sequence of distinct points in H converging to a point x of R-H. This implies by Definition 2.12 that x is a limit point of H. But, H is closed and contains all its limit points. Thus a set H is not closed if a sequence of distinct points in H converges to a point of R-H.

On the other hand suppose no sequence of distinct points in H converges to a point of R-H and that H is not closed. Then there exists a point x of R-H that is a limit point of H. This implies there is a sequence of distinct points in H converging to x. Hence we have a contradiction and H is closed.

(c) If G is open, then it is a neighborhood of each of its points and, if $\{x_n\}$ is any sequence of distinct points converging to $x \varepsilon G$, by Theorem 2.1 there exists N such that for all $n>N$, $x_n \varepsilon G$.

Suppose now that for every sequence of distinct points $\{x_n\}$ converging to $x \varepsilon G$ there exists N such that for all $n>N$, $x_n \varepsilon G$. Suppose that G is not open. Then there is a sequence of distinct points in R-G converging to some $x \varepsilon G$. Thus for this sequence there will not exist N such that for all $n>N$, $x_n \varepsilon G$. Hence G must be open.

Thus we see that using sequences of real numbers we can describe the closure of a set, limit points of a set, closed sets and the usual topology for R.

Sequences in a topological space that is first countable behave the same way as sequence of real numbers. The space R is first countable. However, sequences in an arbitrary topological space will not behave the same way that they do in R. We now give an example to illustrate this.

<u>Example 2.2.</u> Let $S = \{x: 0 \leqq x \leqq 1\}$. Define a set U to be open if and only if U is empty, or S-U is countable. Then with this definition of open set S is a topological space.

Proof: We have to show that the intersection of any two open sets is open and that the union of any collection of open sets is open. Let U, V be open sets in S. If one of these is empty then $U \cap V$ is open. Now suppose U, V are non-empty. $U \cap V \neq \emptyset$, for if not an uncountable set would be a subset of a countable set, which is impossible. Now, $S - (U \cap V) = (S - U) \cup (S - V)$ which is countable. Thus $U \cap V$ is open. Let $\bigcup_{a \varepsilon A} U_a$ be a union of open sets. Then $S - \bigcup_{a \varepsilon A} U_a$ is countable, since for any open set $U_a \subset \bigcup_{a \varepsilon A} U_a$ implies $S - \bigcup_{a \varepsilon A} U_a \subset S - U_a$. Thus $S - \bigcup_{a \varepsilon A} U_a$ is countable since $S - U_a$ is countable. Thus S is a topological space.

Consider the subset K of S where $K = \{x : 0 \leq x \leq 1/2\}$. Let p be any point of S and U any open set containing p. Then $U \cap K \neq \emptyset$, for if not K would be contained in S-U which is impossible since S-U is countable. $K = (U \cap K) \cup ((S-U) \cap K)$ which implies $U \cap K$ is uncountable since $(S-U) \cap K$ is countable. Thus, we see that every point of S is a limit point of K. 1/4 is a limit point of K, yet no sequence $\{x_n\}$ of distinct points of K can converge to 1/4, since $S - \bigcup_{n \varepsilon I^+} x_n$ is an open set containing 1/4 and no points of $\{x_n\}$. In this topological space the only sequences which converge are constant sequences. Hence, using the concept of sequence we cannot describe limit points of sets, closed sets, or open sets.

This example shows that if we want sequences in an arbitrary topological space to behave the way they do on the real line, we have to generalize the idea of a sequence, allowing it to contain an uncountable number of points. This leads us to the notion of a Moore-Smith sequence.

<u>Definition 2.13.</u> A binary relation F from a set A to a set B assigns to each pair (a,b) in A x B exactly one

of the following statements:

(i) a is related to b, written aFb

(ii) a is not related to b, written a$\not F$b.

Definition 2.14. Let Q be a non-empty set on which a binary relation F is defined. Then Q is said to be a Moore-Smith set with respect to F if and only if the following conditions hold.

(i) Transitivity: Given $q_1$, $q_2$, $q_3 \varepsilon Q$ such that $q_1 F q_2$ and $q_2 F q_3$. Then $q_1 F q_3$.

(ii) Composition property: Given $q_1$, $q_2 \varepsilon Q$ there exists $q_3 \varepsilon Q$ such that $q_3 F q_1$ and $q_3 F q_2$. The symbol qfp is usually read q follows p.

Definition 2.15. A mapping f of a Moore-Smith set into a set S is known as a Moore-Smith sequence. Notationally, this is written as $\{x_q\}$ where $f(q) = x_q$.

Definition 2.16. Let S be a space and $x_0 \varepsilon S$. Then a Moore-Smith sequence $\{x_q\}$ of points of S is said to converge to $x_0$ if and only if given any neighborhood U of $x_0$ there exists an element $q_0 \varepsilon Q$ such that $qFq_0$ implies $x_q \varepsilon U$. "$qFq_0$ implies $x_q \varepsilon U$" can also be stated "$qFq_0$ implies the Moore-Smith sequence $\{x_q\}$ is eventually in U." In a metric space S a Moore-Smith sequence $\{x_q\}$ is said to converge to $x_0 \varepsilon S$ if and only if, given $\epsilon > 0$ there exists a $q_\epsilon \varepsilon Q$ such that $qFq_\epsilon$ implies $d(x_q, x) < \epsilon$. This is written as $\lim_q x_q = x_0$ or $\{x_q\} \to x_0$.

We now give some examples of Moore-Smith sequences from analysis.

<u>Example 2.3.</u> $\lim_{n\to\infty} 1/2^n = 0$. In this case we may interpret the Moore-Smith set Q, to be the set of positive integers and the Moore-Smith sequence to be $\{1/2^n\}$.

<u>Example 2.4.</u> $\lim_{x\to a} f(x) = L$ if and only if given $\epsilon>0$, there exists $\delta>0$ such that $|f(x) - L|<\epsilon$ whenever $0<|x - a|<\delta$. For $x,y\varepsilon(a-\delta, a+\delta)$, xFy if and only if $|x-a| < |y-a|$. Then $(a-\delta, a+\delta)$ is a Moore-Smith set.

Proof: For x, y, $z\varepsilon(a-\delta, a+\delta)$ such that xFy and yFz then clearly xFz. Let x, $y\varepsilon(a-\delta, a+\delta)$ and x, $y \neq a$. We want to show the existence of $z\varepsilon(a-\delta, a+\delta)$ such that zFx and zFy. Choose $z = a + \text{minimum} \left(\frac{|x-a|}{2}, \frac{|y-a|}{2}\right)$, then $|z - a| < |x - a|$ and $|z - a| < |y - a|$. Thus, $(a-\delta, a+\delta)$ is a Moore Smith set. The Moore-Smith sequence would be $\{f(x)\}$ for $x\varepsilon(a-\delta, a+\delta)$. Note that in this example the Moore-Smith set and Moore-Smith sequences have an uncountable number of points.

The next example is that of a Moore-Smith set from topology.

<u>Example 2.5.</u> Let Q be the topology for a space S. Define a binary relation F on Q as follows: for U, $V\varepsilon Q$, UFV if and only if $U\subset V$. Then with this binary relation Q is a Moore-Smith set.

Proof: If U, V, $W\varepsilon Q$ such that UFV and UFW then UFW. For U, $V\varepsilon Q$ then $U\cap V\varepsilon Q$. Thus $U\cap V$FU and $U\cap V$FV.

In example 2.2 we showed that in general sequences are inadequate to describe the topology of a space. However, using the concept of a Moore-Smith sequence, we can characterize the topology of a space S.

Theorem 2.3. Let S be a topological space. Then:

(a) A point s is a limit point of a subset A of S if and only if there is a Moore-Smith sequence in A-{s} converging to s.

(b) A point s belongs to the closure of a subset A of S if and only if there is a Moore-Smith sequence in A converging to s.

(c) A subset H of S is closed if and only if no Moore-Smith sequence in H converges to a point of S-H.

(d) A subset G of S is open if and only if for every Moore-Smith sequence converging to a point of G, there exists a $q_0 \varepsilon Q$ such that $qFq_0$ implies $x_q \varepsilon G$.

Proof: (a) Suppose s is a limit point of A. Then every neighborhood U of s contains points of A-{s}. Let $\mathcal{U}$ denote the neighborhood system of s. Define a binary relation F on $\mathcal{U}$ as follows: For U, $V \varepsilon \mathcal{U}$, UFV if and only if $U \subset V$. Clearly, $\mathcal{U}$ is a Moore-Smith set. Now from each neighborhood U of s choose a point $x_u$ where $x_u$ is contained in $(A-\{s\}) \cap U$. Then $\{x_u\}$ converges to s.

To conclude the proof of (a) suppose there is a Moore-Smith sequence in A-{s} converging to s. Then every neighborhood U of s contains points of A-{s}. Thus s is a limit point of A.

(b) Suppose s belongs to $\bar{A}$. If s is a member of A let each element of the Moore-Smith sequence be s. Then this Moore-Smith sequence will converge to s. If s is a limit point of A then by (a) there is a Moore-Smith sequence in A converging to s.

Suppose on the other hand there is a Moore-Smith sequence in A converging to s. Then every neighborhood U of s intersects A. Thus, s is either a member of A or a limit point of A and hence s belongs to the closure of A.

(c) Suppose a subset H of S is closed and suppose there exists a Moore-Smith sequence in H converging to a point x of S-H. This would imply that x is a limit point of H. But, H is closed and contains all its limit points, therefore no Moore-Smith sequence in H can converge to a point in S-H.

Next, suppose no Moore-Smith sequence in H converges to a point of S-H and assume that H is not closed. Then there exists a point $x \varepsilon S\text{-}H$ that is a limit point of H. Part (a) then, implies there is a Moore-Smith sequence in H converging to x. Thus, if H is not closed there is a Moore-Smith sequence in H converging to a point of S-H.

(d) Suppose G is open and $\{x_q\}$ is any Moore-Smith sequence converging to $x \varepsilon G$. Then since G is open, G is a neighborhood of x and hence there exists $q_0 \varepsilon Q$ such that $qFq_0$ implies $x_q \varepsilon G$.

Next, suppose that the condition is satisfied and that G is not open. Then since G is not open, there is a point $x \varepsilon G$ such that $x$ is a limit point of S-G. Then, by (a), there exists a Moore-Smith sequence $\{x_q\}$ in S-G such that $\{x_q\}$ converges to x. Then for this Moore-Smith sequence there does not exist $q_0 \varepsilon Q$ such that $qFq_0$ implies $x_q \varepsilon G$. Thus, we have a contradiction and G is open.

The results of this theorem show that given any topological space S, using the concept of a Moore-Smith sequence we can describe limit points, points of closure, closed sets, and the topology of the space.

Definition 2.17. A space S is Hausdorff if and only if given any two distinct points x and y of S there exist disjoint open sets U and V such that $x \varepsilon U$ and $y \varepsilon V$.

The concept of a generalized sequence also lends itself to theorems relating to Hausdorff spaces. It can be shown that if a space S is Hausdorff then every convergent sequence has a unique limit. (See [1] p. 100) However, if every convergent sequence has a unique limit, the space is not necessarily Hausdorff.

Example 2.6. Consider the space S of Example 2.2. We showed that the intersection of any two open sets in S could not be empty, so given two distinct points x and y in S, every open set that contains x has a non-empty intersection with every open set containing y. The only sequences which converge in this space are constant sequences. Thus we have a space S such that every convergent sequence has a unique limit, but the space is not Hausdorff.

<u>Theorem 2.4.</u> A space S is Hausdorff if and only if every convergent Moore-Smith sequence in S has a unique limit point in S.

Proof: First suppose that S is Hausdorff and let $\{x_q\}$ be a convergent Moore-Smith sequence of S. Let $\lim_q x_q = a$ and $b \varepsilon S$ where $a \neq b$. By Definition 2.17 there exist disjoint open sets U and V such that $a \varepsilon U$ and $b \varepsilon V$. Since $\lim_q x_q = a$ there exists an element $q_0 \varepsilon Q$ such that $qFq_0$ implies $x_q \varepsilon U$. Thus, since $U \cap V = \emptyset$ there does not exist $q_1 \varepsilon Q$ such that $qFq_1$ implies $x_q \varepsilon V$ and hence $\{x_q\}$ cannot converge to b. Therefore the limit of a convergent Moore-Smith sequence in a Hausdorff space is unique.

To conclude the proof we must show that if every convergent Moore-Smith sequence in S has a unique limit point, then the space S is Hausdorff. To do this we assume the space is not Hausdorff. Then there exist, $x, y \varepsilon S$, $x \neq y$, such that every neighborhood U of x intersects every neighborhood V of y. Let $\mathcal{U}_x$ be the family of neighborhoods for x and $\mathcal{V}_y$ be the family of neighborhoods for y. Define a binary relation F on the family of neighborhoods as follows: For $U, U' \varepsilon \mathcal{U}_x$ $UFU'$ if and only if $U \subset U'$ and similarly for $V, V' \varepsilon \mathcal{V}_y$. Now consider the cartesian product $\mathcal{U}_x \times \mathcal{V}_y$. For the product we define a binary relation R in the following manner: For (U,V) and (U',V'), let (U,V)R(U',V') if and only if UFU' and VFV'. It follows readily that the product, with the binary relation R, is a Moore-Smith set. Now for (U,V) in the product $U \cap V \neq \emptyset$, and from this intersection choose a point $x_{u,v}$. There exists a pair $(U',V')$ such that $(U',V')R(U,V)$ and $U' \cap V' \neq \emptyset$. From this intersection choose a point $x_{u',v'}$. We also have $U' \cap V' \subset U \cap V \subset U$ and $U' \cap V' \subset U \cap V \subset V$. By continuing this process we obtain a Moore-Smith sequence $\{x_{u,v}\}$ that converges to both x and y. For, given any neighborhood G of x and any neighborhood H of y there exists a pair (U,V) in $\mathcal{U}_x x \mathcal{V}_y$ such that $(U',V')R(U,V)$ imply $x_{u',v'} \varepsilon G$ and $x_{u',v'} \varepsilon H$. Thus if S is not Hausdorff, not every convergent Moore-Smith sequence has a unique limit point.

Let $\{x_q\}$ be a Moore-Smith sequence of a space S. Define a binary relation R on $\{x_q\}$ as follows: $x_q R x_p$ if and only if qFp where F is the binary relation for the Moore-Smith set Q. Clearly, with the above binary relation, $\{x_q\}$ is a Moore-Smith set. Thus if $\{x_q\}$ is a Moore-Smith sequence of a space S and if f is a mapping of S into another space T, and $\{x_q\}$ has the binary relation R defined on it $\{f(x_q)\}$ is a Moore-Smith sequence of T. Using the fact that the image of a Moore-Smith sequence is a Moore-Smith sequence and that the limit point of a convergent Moore-Smith sequence in a Hausdorff space is unique we prove a theorem involving continuity of a mapping $f:S \rightarrow T$, where S and T are Hausdorff spaces.

Definition 2.18. Let S and T be spaces and $f:S \rightarrow T$ a mapping. Then f is said to be continuous at the point s of S if and only if given any open set G of T such that $s \varepsilon f^{-1}(G)$, there exists an open set V of S such that $s \varepsilon V \subset f^{-1}(G)$. f is continuous on S if and only if it is continuous at s for all s in S.

<u>Theorem 2.5.</u> Let S and T be Hausdorff spaces and $f:S\to T$ a mapping. Then f is continuous on S if and only if given any $x\varepsilon S$ and any Moore-Smith sequence $\{x_q\}$ of S that converges to x, the Moore-Smith sequence $\{f(x_q)\}$ converges to f(x).

Proof: Suppose first that f is continuous on S and $\{x_q\}$ is a Moore-Smith sequence in S such that $\{x_q\}\to x\varepsilon S$. By the previous theorem we know x is unique. Since f is continuous on S it is continuous at every point of S. This implies given any open set G of T such that $f(x)\varepsilon G$, there exists an open set V of S such that $x\varepsilon V\subset f^{-1}(G)$. Since $\{x_q\}\to x$, there exists $q_0\varepsilon Q$ such that $qFq_0$ implies $x_q\varepsilon V$, but this implies $f(x_q)\varepsilon G$ and thus $\{f(x_q)\}$ converges to f(x). Since T is Hausdorff, f(x) is unique.

Suppose now that the condition holds. We know that x and f(x) are unique. To prove that f is continuous we use the fact that if, for every subset A of S, $f(\overline{A})\subset\overline{f(A)}$, then f is continuous (See [1] p. 72). Let A be any set in S and $y\varepsilon f(\overline{A})$. Then there exists an $x\varepsilon \overline{A}$ such that $f(x) = y$. By Theorem 2.3(b), there exists a Moore-Smith sequence of A converging to x. Now $\{f(x_q)\}\to f(x)$, and since $f(x_q)\varepsilon f(A)$ for all $q\varepsilon Q$, then $f(x)\varepsilon\overline{f(A)}$, thus since y was any point of $f(\overline{A})$, we have $f(\overline{A})\subset\overline{f(A)}$ and f is continuous.

Since the real numbers with the usual topology is a Hausdorff space and a sequence of real numbers is a

Moore-Smith sequence, this theorem is sometimes given as the definition of continuity of a mapping $f:R\to R$, where Moore-Smith sequence is replaced by sequence. However, this condition with sequences does not hold in an arbitrary topological space. For an example of this the reader is referred to [1] p. 100.

Since we have generalized the idea of a sequence in a space S, it would seem natural to generalize the concept of a Cauchy sequence in a metric space.

Definition 2.19. A sequence $\{a_n\}$ in a metric space is said to be Cauchy if and only if given $\epsilon>0$ there exists N such that for $n,m>N$ $d(a_n,a_m)<\epsilon$.

Definition 2.20. A metric space S is complete if and only if every Cauchy sequence in S converges to a point in S.

Definition 2.21. A Moore-Smith sequence in a metric space is said to be a generalized Cauchy sequence if and only if, given $\epsilon>0$ there is a $q_\epsilon \varepsilon Q$ such that $q_1 F q_\epsilon$ and $q_2 F q_\epsilon$ imply $d(x_{q_1},x_{q_2})<\epsilon$.

Theorem 2.6. Let S be a complete metric space and $\{x_q\}$ a Moore-Smith sequence in S. Then $\{x_q\}$ converges in S if and only if $\{x_q\}$ is a generalized Cauchy sequence.

Proof: First suppose that $\{x_q\}$ converges to $x \varepsilon S$ and let $\epsilon>0$ be given. Then there exists $q_\epsilon \varepsilon Q$ such that $qFq_\epsilon$ implies $d(x_q, x)<\epsilon/2$. In particular there is a $q_1 \varepsilon Q$ such that $q_1 F q_\epsilon$. Now, by Definition 2.14(ii), $q_1, q_\epsilon$ imply there exists $q_2 \varepsilon Q$ such that $q_2 F q_\epsilon$ and $q_2 F q_1$. By the triangle inequality

$$d(x_{q_1}, x_{q_2}) \leqq d(x_{q_1}, x_q) + d(x_q, x_{q_2}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

thus $\{x_q\}$ is a generalized Cauchy sequence.

Suppose that $\{x_q\}$ is a generalized Cauchy sequence. Let $\epsilon_1 = 1/2$. Then there exists $q_1 \epsilon Q$ such that $qFq_1$ and $pFq_1$ imply $d(x_p,x_q) < 1/2$. Define $Q_1 = \{p\epsilon Q : pFq_1\}$. Let $\epsilon_2 = 1/4$. Then there exists $q_0 \epsilon Q$ such that $pFq_0$ and $qFq_0$ imply $d(x_p,x_q) < 1/4$. Now $q_0$, $q_1 \epsilon Q$ imply, by Definition 2.14(ii), that there exists an element $q_2 \epsilon Q$ such that $q_2Fq_0$ and $q_2Fq_1$. Since $q_2Fq_0$ for any $q\epsilon Q$ such that $qFq_2$, $d(x_q,x_{q_2}) < 1/4$. Define $Q_2 = \{p\epsilon Q : pFq_2\}$. Now since $q_2Fq_1$ we have $Q_2 \subset Q_1$. Assume that $\epsilon_{n-1}$, $q_{n-1}$ and $Q_{n-1}$ have been defined and let $\epsilon_n = 1/2^n$. Then there exists $q_0' \epsilon Q$ such that $pFq_0'$ and $qFq_0'$ imply $d(x_q,x_p) < 1/2^n$. Now $q_0'$, $q_{n-1} \epsilon Q$ imply, by Definition 2.14(ii), that there exists $q_n \epsilon Q$ such that $q_nFq_{n-1}$ and $q_nFq_0'$. Since $q_nFq_0'$, for any $p\epsilon Q$ such that $pFq_n$ $d(x_p,x_{q_n}) < 1/2^n$. Define $Q_n = \{p\epsilon Q : pFq_n\}$. Since $q_nFq_{n-1}$ we have $Q_n \subset Q_{n-1}$ continuing this process inductively, we will obtain a sequence $\{x_{q_n}\}$ which is a subset of $\{x_q\}$. We will also have $\ldots Q_n \subset Q_{n-1} \subset \cdots \subset Q_1$. The sequence $\{x_{q_n}\}$ is a Cauchy sequence.

In order to prove this we must show that given $\epsilon > 0$ there exists N such that for all $n,m > N$, $d(x_{q_n}, x_{q_m}) < \epsilon$.

Let $\epsilon > 0$ be given. Then there exists N such that $1/2^N < \epsilon/2$. By the construction, for n,m N, $q_n F q_N$ and $q_m F q_N$. Now $q_n F q_N$ implies $d(x_{q_N}, x_{q_n}) < 1/2^N$ and $q_m F q_N$ implies $d(x_{q_m}, x_{q_N}) < 1/2^N$. Using the triangle inequality,

$$d(x_{q_n}, x_{q_m}) \leq d(x_{q_n}, x_{q_N}) + d(x_{q_N}, x_{q_m})$$

$$< 1/2^N + 1/2^N < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\{x_{q_n}\}$ is a Cauchy sequence. Since S is a complete metric space $\{x_{q_n}\}$ converges to $x \varepsilon S$. We want to show that $\{x_q\}$ converges to x.

Let $\epsilon > 0$ be given. Since $\{x_{q_n}\}$ converges to x there exists $N_1$ such that $d(x_{q_N}, x) < \epsilon/2$. There exists N such that $1/2^{N_2} < \epsilon/2$. Choose N = maximum $(N_1, N_2)$. For $q \varepsilon Q_N$, $qFq_N$ implies $d(x_q, x_{q_N}) < 1/2^N < \epsilon/2$. By the triangle inequality,

$$d(x_q, x) \leq d(x_q, x_{q_N}) + d(x_{q_N}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

thus $\{x_q\}$ converges to x.

Note that in the first part of the proof we did not use the fact that S is a complete metric space. Thus, we have the following corollary.

<u>Corollary 2.1.</u> Let S be a metric space and $\{x_q\}$ a convergent Moore-Smith sequence in S. Then $\{x_q\}$ is a generalized Cauchy sequence.

The concept of a generalized Cauchy sequence in a complete metric space lends itself to a theorem relating to iterated limits. Before we can state and prove this theorem we need the following definition and lemma.

In the proof of Theorem 2.4, we showed that if P and Q are Moore-Smith sets, then PxQ can be made into a Moore-Smith set in a natural manner using the ordering on P and Q.

<u>Definition 2.22.</u> Let P, Q be Moore-Smith sets, S a metric space $f:P\times Q\rightarrow S$, $g:Q\rightarrow S$ mapping, and suppose that the $\lim_p f(p,q) = g(q)$. Then $\lim_p f(p,q)$ is said to be uniform in q if and only if given any real number $\epsilon>0$ there exists an element $p_\epsilon$ of P such that $p\epsilon P$ and $pFp_\epsilon$ implies $d(f(p,q),g(q)) < \epsilon$. (That is, $p_\epsilon$ does not depend on q).

<u>Lemma 2.1.</u> Let P, Q be Moore-Smith sets, S a complete metric space, $f:P\times Q\rightarrow S$, $g:Q\rightarrow S$, and $h:P\rightarrow S$ mappings, and suppose that $\lim_p f(p,q) = g(q)$ uniformly in q and $\lim_q f(p,q) = h(p)$ for each $p\epsilon P$. Then,

(a) $\lim_q(\lim_p f(p,q))$ exists;

(b) $\lim_p(\lim_q f(p,q))$ exists; and

(c) $\lim_{p,q} f(p,q)$ exists.

Proof: (a) Since $\lim_p f(p,q) = g(q)$ uniformly in q, we must show that the $\lim_q g(q)$ exists. To do this we show that $\{g(q)\}$ is a generalized Cauchy sequence. Let $\epsilon>0$ be given. By the uniformity of $\lim_p f(p,q)$ there exists $p_\epsilon \epsilon P$, depending only on   such that $pFp_\epsilon$ implies

$$\text{(i)}\quad d(f(p,q),g(q)) < \epsilon/3.$$

Since $\lim_p f(p,q)$ is uniform in q, $pFp_\epsilon$ implies

$d(f(p,q_1),g(q_1)) < \epsilon/3$ and $d(f(p,q_2),g(q_2)) < \epsilon/3$ for arbitrary $q_1$, $q_2 \epsilon Q$. To conclude the proof we are concerned only with those $p \epsilon P$ such that $pFp_\epsilon$. Since $\lim_q f(p,q) = h(p)$, there exists $q_{\epsilon,p} \epsilon Q$ such that $qFq_{\epsilon,p}$ implies $d(f(p,q),h(p)) < \epsilon/6$. There is a $q_3 \epsilon Q$ such that $q_3 Fq_{\epsilon,p}$, and this implies $d(f(p,q_3),h(p) < \epsilon/6$. $q_3, q_{\epsilon,p} \epsilon Q$ imply, by Definition 2.14(ii), that there exists $q_4 \epsilon Q$ such that $q_4 Fq_3$ and $q_4 Fq_{\epsilon,p}$. $q_4 Fq_{\epsilon,p}$ implies $d(f(p,q_4),g(q_4)) < \epsilon/6$. Now by the triangle inequality

$$d(f(p,q_3),f(p,q_4)) \leqq d(f(p,q_3),h(p)) + d(f(p,q_4),h(p))$$
$$< \epsilon/6 + \epsilon/6 = \epsilon/3.$$

By (i) we know that $d(f(p,q_3),g(q_3)) < \epsilon/3$ and $d(f(p,q_4),g(q_4)) < \epsilon/3$. Using the triangle inequality again we have

$$d(g(q_3),g(q_4)) \leqq d(g(q_3),f(p,q_3)) + d(f(p,q_3),g(q_4))$$
$$\leqq d(g(q_3),f(p,q_3)) + d(f(p,q_3),f(p,q_4))$$
$$+ d(f(p,q_4),g(q_4))$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus $\{g(q)\}$ is a generalized Cauchy sequence, and by Theorem 2.6, $\{g(q)\}$ converges to a point $x_0 \epsilon S$.

(b) To show that $\lim_p(\lim_q f(p,q))$ exists, we must show that $\lim_p h(p)$ exists by again applying Theorem 2.6.

Let $\epsilon > 0$ be given. Since $\lim_p f(p,q) = g(q)$ uniformly in q, there exists $p_\epsilon \epsilon P$ such that $pFp_\epsilon$ implies $d(f(p,q),g(q)) < \epsilon/6$ for all $q \epsilon Q$. In particular, there exists $p_1 \epsilon P$ such that $p_1 Fp_\epsilon$, which implies

(ii) $d(f(p_1,q),g(q)) < \epsilon/6$.

Now, by Definition 2.14(ii), there exists $p_2 \varepsilon P$ such that $p_2 F p_\epsilon$, which implies

(iii) $d(f(p_2,q),g(q)) < \epsilon/6$.

Then, using the triangle inequality, we have

$$d(f(p_1,q),f(p_2,q)) \leq d(f(p_1,q),g(q)) + d(f(p_2,q),g(q)) < \epsilon/6 + \epsilon/6 = \epsilon/3.$$

Since $\lim_q f(p_1,q) = h(p_1)$ there exists $q_{\epsilon,p_1} \varepsilon Q$ such that $qFq_{\epsilon,p}$ implies $d(f(p_1,q),h(p_1)) < \epsilon/3$. Similarly there exists $q_{\epsilon,p_2} \varepsilon Q$ such that $qFq_{\epsilon,p_2}$ implies $d(f(p_2,q),h(p_2)) < \epsilon/3$. Now $q_{\epsilon,p_1}, q_{\epsilon,p_2} \varepsilon Q$ imply, by Definition 2.14(ii), that there exists $q \varepsilon Q$ such that $qFq_{\epsilon,p_1}$ and $qFq_{\epsilon,p_2}$. Then for those $q \varepsilon Q$ that follow $q_{\epsilon,p_1}$ and $q_{\epsilon,p_2}$, $d(f(p_1,q),g(q)) < \epsilon/3$ and $d(f(p_2,q),g(q)) < \epsilon/3$. Now, since $\lim_p f(p,q)$ is uniform in q, the inequalities (ii) and (iii) are true for those $q \varepsilon Q$ such that $qFq_{\epsilon,p_1}$ and $qFq_{\epsilon,p_2}$. Then using the same type of inequality that we used in the final part of the proof of (a), we have $d(h(p_1),h(p_2)) < \epsilon$. Hence, $\{h(p)\}$ is a generalized Cauchy sequence and by Theorem 2.6 converges to a point $x_1 \varepsilon S$.

(c) Recall that in $P \times Q$, $(p_2,q_2)F(p_1,q_1)$ if, and only if, $p_2 F p_1$ and $q_2 F q_1$.

Let $\epsilon > 0$ be given. Then since $\{g(q)\}$ is a generalized Cauchy sequence there is a $q_\epsilon \varepsilon Q$ such that $q_1 F q_\epsilon$ and $q_2 F q_\epsilon$ imply $d(g(q_1),g(q_2)) < \epsilon/3$. Since $\lim_p f(p,q) = g(q)$ uniformly in q there exists an element $p_\epsilon \varepsilon P$ such that $pFp_\epsilon$ implies $d(f(p,q_1)g(q_1)) < \epsilon/3$ and $d(f(p,q_2),g(q_2)) < \epsilon/3$.

There is a $p_1 \varepsilon P$ such that $p_1 F p_\epsilon$. Then $p_1, p_\epsilon \varepsilon P$ imply, by Definition 2.14(ii), that there exists $p_2 \varepsilon P$ such that $p_2 F p_1$ and $p_2 F p_\epsilon$. It then follows that $d(f(p_1,q_1),g(q_1)) < \epsilon/3$ and $d(f(p_2,q_2),g(q_2)) < \epsilon/3$. Thus,

$$d(f(p_1,q_1),f(p_2,q_2))$$
$$\leqq d(f(p_1,q_1),g(q_1)) + d(f(p_2,q_2),g(q_1))$$
$$\leqq d(f(p_1,q_1),g(q_1)) + d(f(p_2,q_2),g(q_2)) + d(g(q_1),g(q_2))$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

and we have found a pair $(p_\epsilon, q_\epsilon) \varepsilon P x Q$ such that if $(p_1,q_1)F(p_\epsilon,q_\epsilon)$ and $(p_2,q_2)F(p_\epsilon,q_\epsilon)$ then $d(f(p_1,q_1),f(p_2,q_2)) < \epsilon$. Hence $\{f(p,q)\}$ is a generalized Cauchy sequence and by Theorem 2.6 converges to a point $x_2 \varepsilon S$.

We are now ready to state and prove a theorem relating to iterated limits in a complete metric space.

Theorem 2.7. Let P, Q be Moore-Smith sets, S a complete metric space and $f:PxQ \rightarrow S, g:Q \rightarrow S$, and $h:P \rightarrow S$ mappings, and suppose that $\lim_q f(p,q) = g(q)$ uniformly in q and $\lim_q f(p,q) = h(p)$ for every $p \varepsilon P$. Then $\lim_q(\lim_p f(p,q))$, $\lim_p(\lim_q f(p,q))$, and $\lim_{p,q} f(p,q)$ each exist and are all equal.

Proof: By Lemma 2.1 we know that $\lim_q(\lim_p f(p,q)$, $\lim_q(\lim_p f(p,q))$, and $\lim_{p,q} f(p,q)$ each exist. Let $x_0 = \lim_q(\lim_p f(p,q))$, $x_1 = \lim_p(\lim_q f(p,q)$ and $x_2 = \lim_{p,q} f(p,q)$. Assume that $x_0 \neq x_1 \neq x_2 \neq x_0$. Let $d(x_0,x_1) = r_1$, $d(x_1,x_2) = r_2$ and $d(x_0,x_2) = r_3$. Choose $\epsilon$ to be the minimum of $r_1/2$, $r_2/2$, and $r_3/2$. Then the spherical neighborhoods

$S_\epsilon(x_i)$ and $S_\epsilon(x_j)$ are disjoint for $i,j = 0, 1, 2$ and $i \neq j$. We list the following statements since we need to refer to them later on in the proof.

(i) $\lim_q g(q) = x_0$. Then there exists $q_0 \varepsilon Q$ such that $qFq_0$ implies $g(q) \varepsilon S_\epsilon(x_0)$.

(ii) $\lim_p h(p) = x_1$. There exists $p_0 \varepsilon P$ such that $pFp_0$ implies $h(p) \varepsilon S_\epsilon(x_1)$.

(iii) $\lim_{p,q} f(p,q) = x_2$. There exists $(p',q') \varepsilon PxQ$ such that $(p,q)F(p',q')$ imply $f(p,q) \varepsilon S_\epsilon(x_2)$.

(iv) $\lim_p f(p,q) = g(q)$. There exists $p_1 \varepsilon P$ such that $pFp_1$ implies $f(p,q) \varepsilon S_\epsilon(x_0)$ where also $qFq_0$.

(v) $\lim_q f(p,q) = h(p)$. There exists $q_1 \varepsilon Q$ such that $qFq_1$ implies $f(p,q) \varepsilon S_\epsilon(x_1)$ where also $pFp_0$.

Now, by Definition 2.14(ii), for $q_0$, $q_1 \varepsilon Q$ there is a $q_2 \varepsilon Q$ such that $q_2Fq_0$ and $q_2Fq_1$. Similarly, there is a $p_2 \varepsilon P$ such that $p_2Fp_0$ and $p_2Fp_1$. Now $q_2Fq_0$, by (i), implies that $g(q_2) \varepsilon S_\epsilon(x_0)$ and $p_2Fp_0$, by (ii), implies that $h(p_2) \varepsilon S_\epsilon(x_1)$. By (iv) we have $f(p_2,q_2) \varepsilon S_\epsilon(x_0)$, since $p_2Fp_1$ and $q_2Fq_0$. (v) yields $f(p_2,q_2) \varepsilon S_\epsilon(x_1)$, since $q_2Fq_1$ and $p_2Fp_0$. Thus $S_\epsilon(x_0) \cap S_\epsilon(x_1) \neq \emptyset$ and $x_0 = x_1$. Now $(p',q')$ and $(p_2,q_2) \varepsilon PxQ$ imply, by Definition 2.14(ii), that there is a pair $(p_3,q_3) \varepsilon PxQ$ such that $(p_3,q_3)F(p',q')$ and $(p_3,q_3)F(p_2,q_2)$. By (iii) $(p_3,q_3)F(p',q')$ yields $f(p_3,q_3) \varepsilon S_\epsilon(x_2)$. From $(p_3,q_3)F(p_2,q_2)$, we have $p_3Fp_2$ and $q_3Fq_2$. Then since $p_2Fp_0$ and $q_2Fq_1$, $q_3Fq_1$ and $p_3Fp_0$, by Definition 2.14(i). By (v) we have $f(p_3,q_3) \varepsilon S_\epsilon(x_1)$, since $q_3Fq_1$ and $p_3Fp_0$. Thus, $S_\epsilon(x_1) \cap S_\epsilon(x_2) \neq \emptyset$ and $x_1$ must equal

$x_2$. Hence $x_0 = x_1 = x_2$. This completes the proof.

## CHAPTER III

## FILTERS

In this chapter we shall consider a type of convergence involving the concept of a filter. A filter is a more general form of convergence than a Moore-Smith sequence. A Moore-Smith sequence involves points in a space indexed by a set Q, whereas a filter involves a collection of sets in a space satisfying certain conditions.

Definition 3.1. A filter $F$ in a set $X$ is a collection of non-empty subsets of $X$ satisfying the conditions:

(i) The intersection of any two members of $F$ is a member of $F$, and

(ii) if $A \in F$ and $A \subset B \subset X$, then $B \in F$.

Suppose we consider the neighborhood system of a point $x$ in a space. Although this system satisfies the first condition of Definition 3.1, the second condition is not satisfied since any set containing a neighborhood of a point is not necessarily a neighborhood of the point. However, this difficulty is easily overcome by the introduction of a K-neighborhood of a point.

Definition 3.2. A subset $U$ of a space $S$ is called a K-neighborhood of a point $x$ if and only if $U$ contains an open set containing $x$.

From this definition it follows readily that the K-neighborhood system of a point satisfies the condition for a filter.

Definition 3.3. A filter $F$ converges to a point x in a space S if and only if $F$ contains the K-neighborhood system of x. (That is, the K-neighborhood system of x is a subfamily of $F$).

From the preceding definition and discussion the following theorem is immediate and we state it without proof.

Theorem 3.1. Let $U$ denote the K-neighborhood system of a point x in a space S. Then $U$ is a filter converging to x.

Theorem 3.2. Let S be a topological space. Then,

(a) if $F$ is a filter converging to x and $G$ is a filter which contains $F$ then $G$ converges to x.

(b) If $F_x$ is the collection of all filters which converge to a point x, then $\bigcap\{F : F \varepsilon F_x\}$ is the K-neighborhood system of x.

Proof: (a) Since $F$ converges to x, $F$ contains every K-neighborhood of x. Then since $G$ contains $F$, $G$ contains every K-neighborhood of x and hence by Definition 3.3 converges to x.

(b) Let $\{G_a\}_{a\varepsilon A}$ denote the K-neighborhood system of x and let $G_{a_1}$ be any set in $\{G_a\}_{a\varepsilon A}$. Then $G_{a_1} \varepsilon \bigcap\{F : F\varepsilon F_x\}$ since $G_{a_1}$ belongs to every filter converging to x. Let G be any set in $\bigcap\{F : F\varepsilon F_x\}$. Then $G \varepsilon \{G_a\}_{a\varepsilon A}$, for in particular, the neighborhood system of x is a member of $F_x$. Thus if $G \varepsilon \bigcap\{F: F \varepsilon F_x\}$ then G must belong to $\{G_a\}_{a \varepsilon A}$.

In Chapter Two we showed that using Moore-Smith sequences we are able to describe limit points, closure of a set, closed sets and open sets in a topological space. We can also do the same thing with filters.

Theorem 3. 3. A point x is a limit point of a set A if and only if A-{x} belongs to some filter which converges to x.

Proof: First suppose that A-{x} belongs to some filter F which converges to x. Since F converges to x each K-neighborhood of x is a member of F. Then since every neighborhood of x is a K-neighborhood of x, every neighborhood of x is a member of F. Let G be any neighborhood of x. Then since A-{x}, $G \varepsilon F$, $(A-\{x\}) \cap G \neq \emptyset$. This implies G contains points of A-{x} and hence x is a limit point of A.

Next suppose that x is a limit point of A. We wish to show that A-{x} belongs to some filter which converges to x.

Let $\{G_c\}_{c \varepsilon C}$ be the K-neighborhood system of x. Since x is a limit point of A, $(A-\{x\}) \cap G_c \neq \emptyset$, for every $c \varepsilon C$. Let $\{B_q\}_{q \varepsilon Q}$ denote the collection consisting of all subsets $B_q$ of S such that $(A-\{x\}) \cap G_c \subset B_q$ for some $q \varepsilon Q$ and some $c \varepsilon C$. Let F be the collection of sets consisting of $\{G_c\}_{c \varepsilon C}$, $\{(A-\{x\}) \cap G_c\}_{c \varepsilon C}$, and $\{B_q\}_{q \varepsilon Q}$. Now we wish to show that the collection is a filter. Clearly all the sets are non-empty. Now we show that the intersection of any two members of $F$ is a member of $F$.

Since the K-neighborhood system satisfies the conditions for a filter, the intersection of any two K-neighborhoods is a member of the K-neighborhood system and hence a member of $F$. The intersection of any two members of $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$ is a member of $F$, since $((A-\{x\})\cap G_{c_1})\cap(A-\{x\})\cap G_{c_2} = (G_{c_1}\cap G_{c_2})\cap(A-\{x\})$ and $G_{c_1}\cap G_{c_2}\varepsilon\{G_c\}_{c\varepsilon C}$, hence $(G_{c_1}\cap G_{c_2})\cap(A-\{x\}) \neq \emptyset$ and belongs to $\{A-\{x\}\cap G_c\}_{c\varepsilon C}$ and also $F$. The intersection of two members of $\{B_q\}_{q\varepsilon Q}$ is also a member of $F$. For if $B_{q_1}$ and $B_{q_2}$ belong to $\{B_q\}_{q\varepsilon Q}$ then there exist sets $(A-\{x\})\cap G_{c_1}$ and $(A-\{x\})\cap G_{c_2}$ such that $(A-\{x\})\cap G_{c_1}\subset B_{q_1}$ and $(A-\{x\})\cap G_{c_2}\subset B_{q_2}$. Then since $((A-\{x\})\cap G_{c_1})\cap((A-\{x\})\cap G_{c_2})$ is non-empty and is in $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$, $B_{q_1}\cap B_{q_2} \neq \emptyset$ and belongs to $\{B_q\}_{q\varepsilon Q}$ and hence to $F$. Now $((A-\{x\})\cap G_c)\cap B_{q_1}$ is a member of $F$ since $B_{q_1}\supset(A-\{x\})\cap G_{c_1}$ and $((A-\{x\})\cap G_c)\cap((A-\{x\})\cap G_{c_1}) \neq \emptyset$ and is contained in $B_{q_1}$ and $(A-\{x\})\cap G_c$. Thus $((A-\{x\})\cap G_c)\cap B_{q_1} \neq \emptyset$ and contains a member of $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$, so it belongs to $\{B_q\}_{q\varepsilon Q}$ and to $F$. Clearly the intersection of any K-neighborhood with a member of $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$ is a member of $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$ and of $F$. Finally, we show that the intersection of a member of $\{B_q\}_{q\varepsilon Q}$ and a member of $\{G_c\}_{c\varepsilon C}$ is non-empty and a member of $F$. Let $B_{q_1}$ be an arbitrary member of $\{B_q\}_{q\varepsilon Q}$

and $G_{c_1}$ be any member of $\{G_c\}_{c\varepsilon C}$. Then there exists a set $(A-\{x\})\cap G_{c_2}$ such that $(A-\{x\})\cap G_{c_2}\subset B_{q_1}$ and we know that $G_{c_1}\cap((A-\{x\})\cap G_{c_2})\varepsilon\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$. Thus $B_{q_1}\cap G_{c_1}\neq\emptyset$ and contains a member of $\{(A-\{x\})\cap G_c\}_{c\varepsilon C}$ and hence $B_{q_1}\cap G_{c_1}$ belongs to $\{B_q\}_{q\varepsilon Q}$ and $F$. By the way the filter was constructed, the collection of sets will satisfy condition (ii) of Definition 3.1. This collection of sets, $F$, is then a filter converging to x. Then since $(A-\{x\})\cap G_c \varepsilon F$ and $(A-\{x\})\cap G_c\subset A-\{x\}\subset S$, $A-\{x\}\varepsilon F$, and we have shown the existence of a filter $F$ converging to x such that $A-\{x\}\varepsilon F$.

<u>Theorem 3.4</u>. Let S be a space. Then a subset U of S is open if and only if U belongs to every filter converging to a point of U.

Proof: Suppose U is open. Let $F$ be any filter which converges to $x\varepsilon U$. Since U is an open set containing x it is a K-neighborhood of each of its points, hence $U\varepsilon F$, since $F$ contains every K-neighborhood of x. Then since $F$ is any filter converging to $x\varepsilon U$, U belongs to every filter converging to a point of U.

Suppose now that U belongs to every filter converging to $x\varepsilon U$. Then, since the K-neighborhood system of x is a filter converging to x, U is a K-neighborhood of x. Hence for each $x\varepsilon U$, U contains an open set containing x. Thus, U can be written as the union of open sets and U is open.

Theorem 3.5. Let S be a space. Then

(a) A point x belongs to the closure of a subset A of S if and only if A belongs to some filter converging to x.

(b) A set H is closed if and only if H belongs to no filter converging to a point of S-H.

Proof: (a) If x is a limit point of A, then by Theorem 3.3, A-{x} belongs to some filter F converging to x. Then since A-{x}$\subset$A, A belongs to F. If x is a member of A and not a limit point of A, then the collection of all sets U such that x$\varepsilon$U clearly form a filter converging to x, and A is a member of this filter.

Next, suppose that A belongs to some filter converging to x. Then since every neighborhood of x is a K-neighborhood of x, every neighborhood of x intersected with A is non-empty and hence x is either a point of A or a limit point of A.

(b) Suppose H is closed. Then S-H is open and, by Theorem 3.4, belongs to every filter converging to a point of S-H. Since (S-H)$\cap$H = Ø, H cannot belong to any filter converging to a point of S-H.

On the other hand, suppose H belongs to no filter converging to a point of S-H and assume that H is not closed. Then there exists a point x$\varepsilon$S-H that is a limit point of H. But this implies, by Theorem 3.3, that H-{x} belongs to some filter F converging to x. Then H-{x}$\subset$H, implies H$\varepsilon$F. Thus, if H is not closed, H belongs to some filter converging to a point of S-H. Therefore H is closed.

A filter converging to a limit point of a set A is analogous to a Moore-Smith sequence, or a sequence of distinct points of the set A converging to the limit point. Also, in the proof of Theorem 3.5(a), the collection of all sets containing a point is similar to the notion of a constant sequence or a constant Moore-Smith sequence.

Using the concept of a filter we are able to prove Theorems 2.4 and 2.5. We will replace Moore-Smith sequences with filters.

Theorem 3.6. A space S is Hausdorff if and only if every convergent filter has a unique limit.

Proof: Suppose first that S is Hausdorff, $F$ is a filter converging to x, and $y \varepsilon S$ with $y \neq x$. Then since $F$ converges to x, $F$ contains every K-neighborhood of x. But S is Hausdorff, which implies that given any two distinct points $x, y \varepsilon S$ then are disjoint open sets U, V such that $x \varepsilon U$ and $y \varepsilon V$. Now $U \varepsilon F$ and $U \cap V = \emptyset$ so $V \notin F$. Therefore $F$ cannot converge to y and so $F$ converges to a unique limit.

Suppose now that every convergent filter of S has a unique limit, then we must show that S is Hausdorff. To do this we assume that S is not Hausdorff. This implies that there are two distinct points $x, y \varepsilon S$ such that every K-neighborhood $U_x$ of x intersects every K-neighborhood $V_y$ of y. Let $\mathcal{U}_x$ denote the K-neighborhood system of x and $\mathcal{V}_y$ denote the K-neighborhood system of y. Since S is not Hausdorff $U_x \cap V_y \neq \emptyset$ for any $U_x \varepsilon \mathcal{U}_x$ and any $V_y \varepsilon \mathcal{V}_y$. Let $G = \{U_x \cap V_y : U_x \varepsilon \mathcal{U}_x \text{ and } V_y \varepsilon \mathcal{V}_y\}$ and let $\{W_a\}_{a \varepsilon A}$ be the collection of all sets $W_a \subset S$ such that $U_x \cap V_y \subset W_a$ for some $a \varepsilon A$ and for some $U_x \cap V_y \varepsilon G$. The collection of sets $F$, consisting of $\mathcal{U}_x$, $\mathcal{V}_y$, G and $\{W_a\}_{a \varepsilon A}$ is a filter converging to x and y. We must show that $F$ satisfies Definitions 3.1 and 3.3. Clearly all the sets in $F$ are non-empty. Next we show that $F$ satisfies condition (i) of Definition 3.1. Since the K-neighborhood system of a point satisfies the conditions for a filter the intersection of any two members of $\mathcal{U}_x$ is a member of $\mathcal{U}_x$ and the intersection of any two members of $\mathcal{V}_y$ is a member of $\mathcal{V}_y$. From this it follows that the intersection of any two members of G is a member of G. The intersection of any $U_x \varepsilon \mathcal{U}_x$ and any $U'_x \cap V'_y \varepsilon G$ since $U_x \cap (U'_x \cap V'_y) = (U_x \cap U'_x) \cap V'_y$ and $U'_x \cap U_x \varepsilon \mathcal{U}_x$. Similarly the intersection of an arbitrary $V_y \varepsilon \mathcal{V}_y$ with a member of G is a member of G. The intersection of a member of $\mathcal{U}_x$ and a member of $\{W_a\}_{a \varepsilon A}$ is a member of $\{W_a\}_{a \varepsilon A}$ since for $W_{a_1} \varepsilon \{W_a\}_{a \varepsilon A}$ there exists $U'_x \cap V'_y \varepsilon G$ such that $U'_x \cap V'_y \subset W_{a_1}$ and for $U_x \varepsilon \mathcal{U}_x$, $U_x \cap (U'_x \cap V'_y) \neq \emptyset$ and a member of G so $U_x \cap W_{a_1} \neq \emptyset$ and a member of $\{W_a\}_{a \varepsilon A}$. In the same manner we can show the same thing true for $V_y \varepsilon \mathcal{V}_y$ and $W_{a_1} \varepsilon \{W_a\}_{a \varepsilon A}$. Now we show that the intersection of two members of $\{W_a\}_{a \varepsilon A}$ is a member of $\{W_a\}_{a \varepsilon A}$. For $W_{a_1}, W_{a_2} \varepsilon \{W_a\}_{a \varepsilon A}$ there exist sets $U'_x \cap V'_y$ and $U_x \cap V_y$ in G such that $U'_x \cap V'_y \subset W_{a_1}$ and $U_x \cap V_y \subset W_{a_2}$. Then $(U_x \cap V_y) \cap (U'_x \cap V'_y)$ is a member of G and hence $W_{a_1} \cap W_{a_2} \varepsilon \{W_a\}_{a \varepsilon A}$. By the way we have defined the collection

$\mathcal{F}$, given any set A in S such that A contains a member of $\mathcal{F}$, $A \varepsilon \mathcal{F}$. Now since $\mathcal{F}$ contains the K-neighborhood system of both x and y, $\mathcal{F}$ is a filter which converges to x and y. But x is different from y, so if S is not Hausdorff, not every convergent filter has a unique limit. Thus, by the condition given, S must be Hausdorff.

Before we can prove the theorem pertaining to the continuity of a mapping f from a space S into a space T, where S and T are Hausdorff spaces we need the following lemma.

Lemma 3.1. Let S and T be spaces and f a mapping of S onto T. If $\mathcal{F}$ is a filter in S, then $f(\mathcal{F})$ is a filter in T.

Proof: Clearly every set in $f(\mathcal{F})$ is non-empty. Let $U, V \varepsilon f(\mathcal{F})$. Then $f^{-1}(U), f^{-1}(V) \varepsilon \mathcal{F}$ and $f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$ and $f^{-1}(U) \cap f^{-1}(V)$ is a member of $\mathcal{F}$. Then, since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$, $U \cap V$ is non-empty and a member of $f(\mathcal{F})$. Now let $A \varepsilon f(\mathcal{F})$ and B any set in T such that $A \subset B \subset T$. $f^{-1}(A) \varepsilon \mathcal{F}$ and since $f^{-1}(A) \subset f^{-1}(B)$, $f^{-1}(B) \varepsilon \mathcal{F}$ and hence B is a member of $f(\mathcal{F})$. Thus $f(\mathcal{F})$ is a filter.

By using this lemma and the fact that the limit of a convergent filter in a Hausdorff space is unique, we can prove the following theorem.

Theorem 3.7. Let S, T be Hausdorff spaces and f a mapping of S onto T. Then f is continuous on S if and only if given any $x \varepsilon S$ and any filter $\mathcal{F}$ which converges to x, then the filter $f(\mathcal{F})$ converges to f(x).

Proof: Let $x \varepsilon S$ be arbitrary and $F$ the K-neighborhood system of x and suppose the condition is satisfied. Then we wish to show f is continuous. Let G be any open set in T containing f(x). Then $G \varepsilon f(F)$ and hence $f^{-1}(G) \varepsilon F$. This implies $f^{-1}(G)$ is a member of the K-neighborhood system of x and that $f^{-1}(G)$ contains an open set V such that $x \varepsilon V \subset f^{-1}(G)$. Thus, by Definition 2.18, f is continuous at x, but since $x \varepsilon S$ is arbitrary f is continuous on S.

Suppose on the other hand that f is continuous and $F$ is any filter converging to $x \varepsilon S$. We must show that $f(F)$ contains the K-neighborhood system of f(x). Let V be any member of the K-neighborhood system of f(x). Then V contains an open set G such that $f(x) \varepsilon G \subset V$. Now since f is continuous $f^{-1}(G)$ contains an open set W such that $x \varepsilon W \subset f^{-1}(G)$, and W is a member of the K-neighborhood system of x, so $W \varepsilon F$. Now $W \subset f^{-1}(G) \subset f^{-1}(V)$ and hence $f^{-1}(G)$, $f^{-1}(V) \varepsilon F$. Thus $f(F)$ contains the K-neighborhood system of f(x) and since $f(F)$ is a filter, $f(F)$ converges to f(x).

# CHAPTER IV

## EQUIVALENCE OF MOORE-SMITH SEQUENCES AND FILTERS

In this chapter we show that filters and Moore-Smith sequences lead to equivalent convergence theories.

Theorem 4.1. If $\{x_q\}$ is a Moore-Smith sequence in S, then the family of all sets A, such that for each A there is a $q_A \varepsilon Q$ such that $qFq_A$ implies $x_q \varepsilon A$, is a filter in S.

Proof: Let $\mathcal{A}$ denote the family of sets A such that for each $A \varepsilon \mathcal{A}$ there is a $q_A \varepsilon Q$ such that $qFq_A$ implies $x_q \varepsilon A$. We must show that $\mathcal{A}$ satisfies Definition 3.1. Clearly all members of $\mathcal{A}$ are non-empty. If B is any set in S such that B contains A, where $A \varepsilon \mathcal{A}$, then $B \varepsilon \mathcal{A}$ since there exists $q_A \varepsilon Q$ such that $qFq_A$ implies $x_q \varepsilon A$ and $x_q \varepsilon B$ if $A \subset B$. Let $A_1, A_2 \varepsilon \mathcal{A}$. Then there exists $q_{A_1} \varepsilon Q$ such that $qFq_{A_1}$ implies $x_q \varepsilon A_1$ and there is a $q_{A_2} \varepsilon Q$ such that $qFq_{A_2}$ implies $x_q \varepsilon A_2$. $q_{A_1}, q_{A_2} \varepsilon Q$ imply, by Definition 2.14(ii), that there is a $q' \varepsilon Q$ such that $q'Fq_{A_1}$ and $q'Fq_{A_2}$. This implies $x_{q'} \varepsilon A_1$ and $x_{q'} \varepsilon A_2$. Hence for $q \varepsilon Q$ such that $qFq'$, $x_q \varepsilon A_1 \cap A_2$. Thus $A_1 \cap A_2 \neq \emptyset$ and is a member of $\mathcal{A}$. Thus $\mathcal{A}$ is a filter in S.

Lemma 4.1. Let $\mathcal{F}$ be a filter in S and define Q to be the set of all pairs (x,F) such that $x \varepsilon F$ and $F \varepsilon \mathcal{F}$. Define a binary relation R on Q as follows: For $(y,G),(x,F) \varepsilon Q$

$(y,G)F(x,F)$ if and only if $G \subset F$. Then Q is a Moore-Smith set.

Proof: Let $(y,G)$, $(x,F)$, $(h,H) \varepsilon Q$ such that $(y,G)R(x,F)$ and $(x,F)R(h,H)$. Then since $G \subset F$ and $F \subset H$, $G \subset H$ and $(y,G)R(h,H)$. Let $(y,G)$ and $(x,F)$ be members of Q. Then, since G, $F \varepsilon \mathcal{F}$, $G \cap F$ is a member of $\mathcal{F}$, and $G \cap F \subset G$ and $G \cap F \subset F$. Thus $(g, G \cap F)R(y,G)$ and $(g, G \cap F)R(x,F)$, and we have shown that Q is a Moore-Smith set.

<u>Theorem 4.2.</u> Let $\mathcal{F}$ be a filter in S and let Q be the set of all pairs $(x,F)$ such that $x \varepsilon F$ and $F \varepsilon \mathcal{F}$. Let $f(x,F) = x$ for all $x \varepsilon F$. Then $\mathcal{F}$ is precisely the family of all sets A such that for each A there is a pair $(x,F) \varepsilon Q$ such that $(y,G)R(x,F)$ implies $f(y,G) \varepsilon A$.

Proof: First, we show that the conclusion is true for every set in $\mathcal{F}$. Let $K \varepsilon \mathcal{F}$ be arbitrary. Since $\mathcal{F}$ is a filter any set in $\mathcal{F}$ intersected with K is non-empty and a member of $\mathcal{F}$. So we can find a pair $(y,H) \varepsilon Q$ such that $(y,H)R(x,K)$. $f(y,H) = y \varepsilon H \subset K$ which implies the conclusion is true for K, and hence every set in $\mathcal{F}$ since K is arbitrary.

Let A be any set in S such that there exists a pair $(y,G) \varepsilon Q$ such that $(x,H)R(y,G)$ implies $f(x,H) \varepsilon A$. But $f(x,H) = x$ for all $x \varepsilon H$, so $H \subset A$ which implies $A \varepsilon \mathcal{F}$ since $H \varepsilon \mathcal{F}$. This then completes the proof.

It follows then from these two theorems and the lemma that for each Moore-Smith sequence in a space S which converges to a point of S, we can always obtain a filter converging to that point. Conversely, given any filter in a space which converges to a point, we can construct a Moore-Smith sequence converging to the point.

## BIBLIOGRAPHY

1. Hall, D. W., and Spencer, G. L., II. Elementary Topology. New York, N. Y.: John Wiley and Sons, Inc., 1955.

2. Kelley, John L. General Topology. Princeton, N. J.: D. Van Nostrand Company, Inc., 1955.