8-1-1933

Isosceles Trigonometry

Ethel C. Muggli

ISOSCELES TRIGONOMETRY

A Thesis

Submitted to the Graduate Faculty

of the

University of North Dakota

by

Ethel C. Muggli

In Partial Fulfillment of the Requirements

for the

Degree of

Master of Arts

August, 1933

University, North Dakota

August, 1933

This thesis, presented by Ethel C. Muggli in partial fulfillment of the requirements for the degree of Master of Arts, is hereby approved by the Committee on Instruction in charge of her work.

Committee on Instruction

J D Leith
Chairman

A. M. Rovelstad

A. V. Overn

J. V. Breitwieser
Director of the Graduate Division

ACKNOWLEDGMENT

The writer wishes to thank the members of her committee and particularly Professor John Douglas Leith, for their kindly criticisms and suggestions.

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CHAPTER I

INTRODUCTION

This thesis defines the limits of and develops independently a branch of mathematics which may be described as "isosceles trigonometry".

Hipparchus (c. 150 B.C.), asserted by some writers to be the "father of trigonometry", made use of chords of arcs in the graphic solution of spherical triangles. Three hundred years later Ptolemy extended the tables of chords but at the same time made use of the half-chords in some cases. The Greeks and the Arabs developed this half-chord trigonometry, based on the ratios of the sides of right triangles. This is the ordinary trigonometry of modern mathematics.

In the present thesis the full chord trigonometry, a trigonometry based on the isosceles triangle (and any triangle may be dissected into isosceles triangles) is developed by methods of modern mathematical analysis available only within comparatively recent times. This development is entirely independent of the ordinary trigonometry, and in this thesis it appears that the six functions and the structure of ordinary trigonometry could be replaced entirely in both the theoretical and practical aspects by the two functions here defined and developed in isosceles trigonometry.

As a background for isosceles trigonometry, a brief survey of the history of ordinary trigonometry is given.

The word "trigonometry" is derived from two Greek words; trigonon (triangle) and metrein (to measure), which gives us the definition "to measure a triangle." Accepting this literal translation as the definition of the word trigonometry, its origin would be placed in the second and probably in the third millenium B. C.

In the Ahmes Papyrus there are problems relating to the mensuration of pyramids, and four of these mention "seqt" (or "seqet") which apparently means "ratio number". In the early literature of China, references are made to shadow reckoning. The first recorded use of shadow reckoning appears in the story of Thales measuring the height of an Egyptian pyramid by taking the ratio of the height of a vertical staff to the length of its shadow equal to a similar ratio for the pyramid.

Aristarchus, (c. 270 B.C.) the astronomer, attempted to find the distance from the earth to the sun and moon, and also the diameters of these bodies, and in this attempt he used a ratio which is substantially the tangent of an angle. Hipparchus is called the "father of trigonometry" for the reason that he worked out a table of chords, which is the first known table of trigonometric functions. Ptolemy (c. 150 A.D.) summarized the trigonometry known to Hipparchus and extended the table of chords, using the half-chords in some cases. He also knew the equivalent of the law of sines but expressed it in chords, a form similar to the "bos law" derived in this thesis.

The Hindus used a table of half-chords which was apparently based on Ptolemy's work. Aryabhata (c. 500 A.D.) wrote the first oriental purely mathematical treatise which contained definite traces of the function of the angles and Bhaskara gave trigonometric formulae, including the equivalent of $d(\sin\theta) = \cos\theta \, d\theta$.

Both the Greeks and the Arabs used trigonometry in connection with astronomy. The Arabs introduced all the ordinary trigonometric functions and constructed tables of tangents and cotangents. Of the contributions of the Greeks and Arabs, David Eugene Smith says:(1)

> "Pythagoras seems to have been the first to affirm the sphericity of the earth, Erastosthenes computed the circumference as approximately 25,000 miles, Aristarchus forecast the Copernican system, and Menelaus solved the spherical triangle for four cases of a fair degree of difficulty; and these, together with the computation of tables, serve to establish a science which later developed into the trigonometry which we know today, - a science improved by the Arabs, but resting upon a Greek foundation."

The Romans made a more practical use of mathematics than the Greeks did developing it for its commercial applications. For the modern vocabulary of trigonometry, we are indebted to both Greece and Rome.

In the fifteenth century, the work of Regiomontanus had great influence in separating trigonometry from astronomy. Vieta, a little later, contributed to the analytic development of the science. Oughtred's trigonometry appeared in 1657. He tried to found a symbolic trigonometry but the idea was not generally accepted until Euler's influence was exerted in the 18th century.

Newton (c. 1680) published the most complete work on trigonometry up to his time and by expanding the function arc sin x in series and inverting deduced a series for the sin x. He also had the general formulas for sin nx and cos nx. Thomas Fantet de Lagny was the first to set forth in clear form the periodicity of functions. The word "goniometry" was first used by him.

---

(1) Smith, David Eugene: Mathematics, Our Debt to Greece and Rome, p. 131.

The use of the imaginary in trigonometry is due to several writers of the 18th century. Jean Bernoulli discovered the relation between the arc function and logarithm of a complex number, i.e.

$$i\theta = \log(\cos\theta + i\sin\theta),$$

and Euler gave the equivalent of the formula:

$$e^{\theta i} = \cos\theta + i\sin\theta,$$

where i is the square root of minus one. To DeMoivre is attributed the following theorem dealing with a complex number:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Lambert developed the theory of hyperbolic functions.

In Europe in the 17th century, trigonometry became an analytic science and so entered the field of higher mathematics.

CHAPTER II

FUNCTIONS OF ISOSCELES TRIGONOMETRY

Trigonometry concerns itself with the ratios of the sides of the triangle. These ratios are known as the functions of the angles. Definitions of a set of two ratios belonging to isosceles trigonometry are here followed by certain immediate consequences of those definitions.

Section 1

Definitions of the Functions.

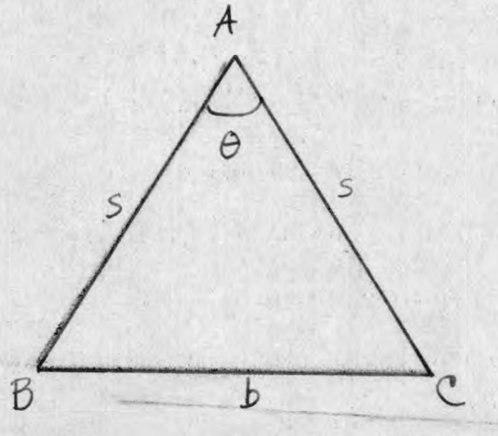


Figure 1.

In the isosceles triangle ABC (Figure 1) the ratio of the base(b) to one of the equal sides(s) is called <u>bos θ.</u>

This may be written

$$\frac{b}{s} = \text{bos}\,\theta$$

The reciprocal function is designated by the symbol cbs θ, so that

$$\text{cbs}\,\theta = \frac{s}{b}$$

From these two definitions we obtain the very evident relation

$$\text{bos}\,\theta \cdot \text{cbs}\,\theta = 1$$

The second principal function is indicated by the symbol sub θ and is defined by the relation

$$\boxed{\text{sub}\,\theta = \text{bos}\,(180 - \theta).}$$

There is of course a reciprocal function of sub θ corresponding to cbs θ. Since these reciprocal functions have no necessary use, they are referred to here merely for the sake of completeness.

To avoid abiguity of algebraic sign in the definition of the bos function, it is necessary to adopt the following arbitrary convention, a device which was first made use of in the mathematical theory of functions of a complex variable by Riemann[(1)] (c. 1850)

A two-sheeted Riemann surface is used on which to represent the variation of the angle and its function. One sheet is chosen as positive, the other as negative. A radius vector in the plane, starting from a definite initial position $\theta = 0$ turns about the origin in the positive sense and so describes the positive sheet of the surface. When this radius has returned to its initial position after one revolution, the surface thus formed has two borders lying adjacent to each other. But these two are not yet to be united with each other; the moving radius is allowed to pierce (in the line $\theta = 0$) the surface just generated and to make a revolution on the second or negative sheet of the Riemann surface. It is then connected with its initial position, having completed one cycle of two revolutions.

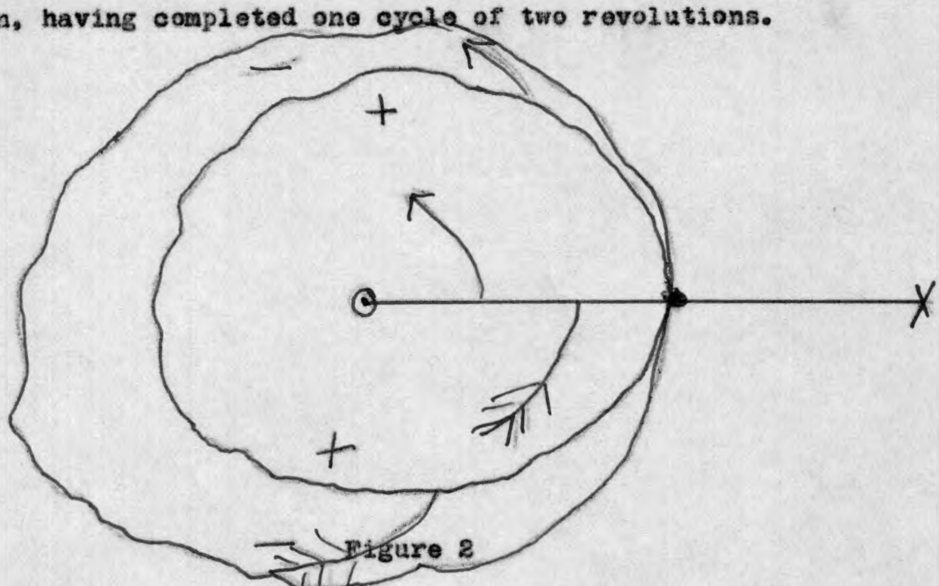


Figure 2

(1) See, for instance, Burkhard-Rasor: Theory of Functions of a Complex Variable.

Figure 2 is a schematic representation of the manner in which this connected surface may be traversed in the positive direction.

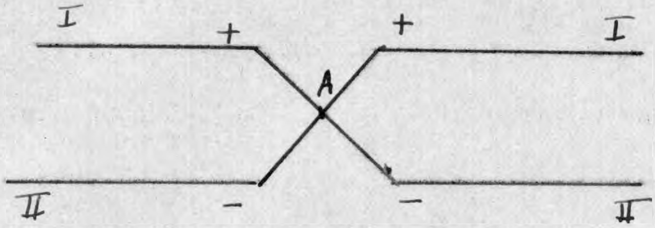


Figure 3

Figure 3 represents a cross-section of the Riemann surface looking from the positive end of the line $\theta = 0$ toward the vertex of the angle. A is the point corresponding to the line in which the sheets are regarded as piercing one another and at which we must change from one sheet to the other in making a circuit of the origin.

Instead of using this two-sheeted surface, we might agree to fix signs to the two sides of a single plane, calling the upper side positive, say, and the lower negative. Then allow the radius to make one revolution on the positive side, pierce the plane, make one revolution on the negative side, and return to its original position.

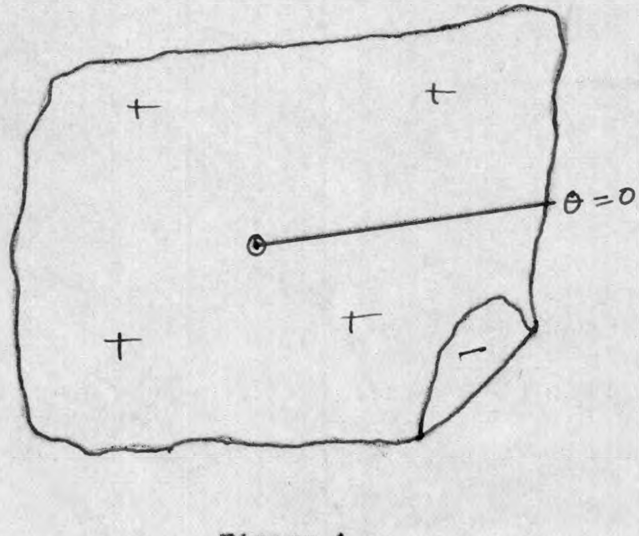
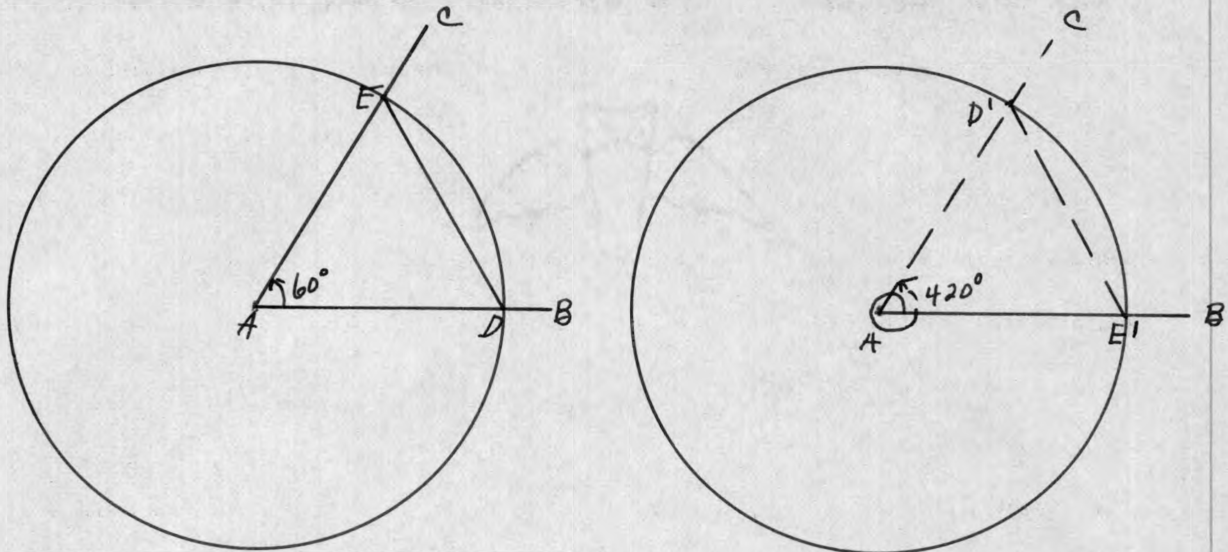


Figure 4

In either case, the radius vector must be rotated through an angle of 720 degrees to make a complete cycle.



(a) (b)

Figure 5

In the interest of clearness, we may examine and compare the geometrical representations of bos 60° and bos 420°, for example. In figure 5a, AB represents the line $\theta = 0$, AC indicate $\theta = 60°$. If a unit circle is taken on the plane, DE is the bos 60° and is equal to 1. Figure 5b represents an angle of 420°. The arrow indicates that one revolution of 360° on the positive side of the plane plus an angle of 60° on the negative side of the plane are required to form an angle of 420° . The broken-line is used to represent the parts of the construction on the negative side of the plane. Taking a unit circle, D'E'(= DE) represents the bos 420. Since the terminal arm of this angle is on the negative side of the plane, the functional value is negative and equal to 1.

For the general case of the negative angle, a direct consequence of the preceding definitions with their accompanying conventions, as may easily be verified, is the relation

$$\text{bos}(-\theta) = -\text{bos}\ \theta$$

which holds for all values of θ .

A simple additional argument gives the corresponding relation for the sub function. Since sub θ = bos (180 $-\theta$), the initial position for the radius vector may be taken at $\theta = 180^\circ$. The radius completes half a revolution going in the positive direction before it comes to the position $\theta = 360^\circ$ on the Riemann surface. Here it pierces the plane and makes a complete revolution on the negative sheet on the surface. Then it pierces the plane again, makes half a revolution on the positive side and joins its original position at $\theta = 180^\circ$. If the radius is rotated in the negative direction from the same starting position, it follows the identical path just described but in the opposite sense. Therefore,

$$\text{sub}(-\theta) = \text{sub}\ \theta .$$

Section 2

A Fundamental Relation between the Functions,

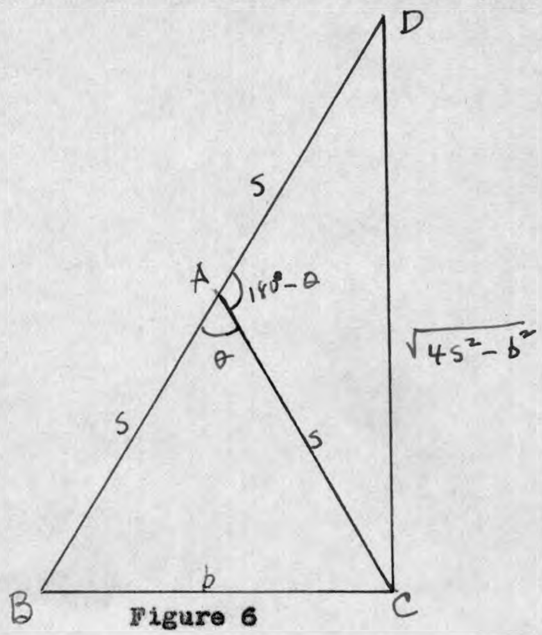


Figure 6

To obtain a fundamental relation connecting bosθ and sub θ, construct the isosceles triangle ABC, (Figure 6). Extend AB its own length to form the angle ($180^\circ - \theta$). From plane geometry, we know that triangle BCD is a right triangle and that side DC is $\sqrt{4s^2 - b^2}$.

By definition,

$$\text{sub}\,\theta = \frac{\sqrt{4s^2 - b^2}}{s}$$

Squaring both sides of the equation we have

$$\text{sub}^2\theta = \frac{4s^2 - b^2}{s^2} = 4 - \left(\frac{b}{s}\right)^2$$

But $\frac{b}{s} = \text{bos}\,\theta$.

Substituting this for $\frac{b}{s}$ in the previous equation, we obtain

$$\text{sub}^2\theta = 4 - \text{bos}^2\theta$$

or,

$$\boxed{\text{bos}^2\theta + \text{sub}^2\theta = 4.}$$

The original definitions together with the conventions established in this chapter thus supply the following relations fundamental in the isosceles trigonometry of a single angle:

$$\frac{b}{s} \equiv \text{bos}\,\theta \qquad\qquad \text{sub}\,\theta \equiv \text{bos}\,(180 - \theta)$$

$$\text{bos}\,(-\theta) = -\text{bos}\,\theta \qquad\qquad \text{sub}\,(-\theta) = \text{sub}\,\theta$$

$$\text{bos}^2\theta + \text{sub}^2\theta = 4.$$

CHAPTER III

GONIOMETRY OF ISOSCELES FUNCTIONS

Section 1

Line-values of the functions

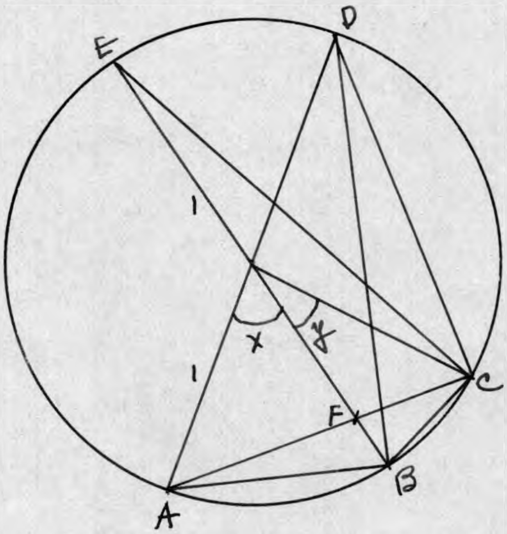


Figure 7

In a unit circle(Figure 7), these lines represent graphically the functions of isosceles trigonometry.

AB = bos x	BD = sub x
BC = bos y	CE = sub y
AC = bos (x + y)	DC = sub (x + y)

Section 2

Bos of the sum of two angles.

We proceed to a derivation of the fundamental addition theorems of isosceles trigonometry.

From the definitions in Section 1,

$$\text{bos}\,(x + y) = AC$$

and from Figure 7 we observe that

$$AC = AF + FC \qquad (1)$$

To find expressions for AF and FC in terms of the known quantities AB, BC, CE, and ED, it is necessary to make use of certain similar triangles. From the similar triangles AFB and EBC we obtain the following proportion:

$$\frac{AF}{2} = \frac{AB}{EC} \qquad \text{or} \qquad AF = \frac{2AB}{EC}$$

From the similar triangles ABF and DFC, we obtain

$$\frac{DF}{2} = \frac{FC}{BC} \qquad FC = \frac{DF \cdot BC}{2}$$

Substituting these values for FC and AF in equation (1), we have

$$AC = \frac{2AB}{EC} + \frac{DF \cdot BC}{2} \qquad (2)$$

DF is not a known quantity and therefore must be eliminated. But

$$DF = DB - BF$$

Using the similar triangles ABF and BCE, we have

$$\frac{AB}{EC} = \frac{BF}{BC}, \qquad BF = \frac{AB \cdot BC}{EC}$$

Therefore,

$$DF = \frac{DB \cdot EC - AB \cdot BC}{EC}$$

Substituting this value for DF in equation (2), we obtain

$$AC = \frac{2AB}{EC} + \frac{DB \cdot EC \cdot BC - AB.\overline{BC}^2}{2EC}$$

$$= \frac{4AB + DB.EC.BC - 4AB + AB.\overline{EC}^2}{2EC}$$

$$AC = \frac{DB.BC + AB.EC}{2}$$

Substituting the functional values for these lines, we obtain theorem

$$\text{bos}\ (x + y) = \tfrac{1}{2}(\text{bos}\ x \cdot \text{sub}\ y + \text{sub}\ x \cdot \text{bos}\ y).$$

Replacing y by -y gives

$$\text{bos}\ (x - y) = \tfrac{1}{2}(\text{bos}\ x \cdot \text{sub}\ y - \text{sub}x \cdot \text{bos}\ y).$$

Section 3

Sub of the sum of two angles

To develop a theorem concerning the sub $(x + y)$, use the similar triangles BEC and FDC.

$$DC = \frac{DF.EC}{2}$$

As proved in section V,

$$DF = \frac{DB.EC - AB.BC}{EC}.$$

Therefore,

$$DC = \frac{DB.EC - AB.BC}{\not{E}C} \cdot \frac{\not{E}C}{2}$$

$$= \tfrac{1}{2}(DB.EC - AB.BC).$$

Substituting the functional values,

$$\text{sub}\ (x + y) = \tfrac{1}{2}(\text{sub}\ x \cdot \text{sub}\ y - \text{bos}\ x \cdot \text{bos}\ y).$$

Replacing y by -y,

$$\text{sub}\ (x - y) = \tfrac{1}{2}(\text{sub}\ x \cdot \text{sub}\ y + \text{bos}\ x \cdot \text{bos}\ y).$$

The theorems derived independently in sections 2 and 3 for bos $(x + Y)$ and sub $(x + y)$ are the fundamental theorems of isosceles trigonometry. They constitute the foundation of its structure, resting on the bedrock of the definitions. These addition theorems are necessary for the derivation of the remaining expressions of this chapter, on which, in turn, the further developments depend. For instance, bos $(x+y)$ was used in obtaining the derivative of bos x. Again, in proving bos x a continuous function, both theorems were required. We might examine each section in detail and we would find that, either directly or indirectly, these theorems were used. These theorems could also be obtained by the application of Ptolemy's theorem that in a quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.

Section 4

Functions of any number of angles.

Using the addition theorem for bos $(x+y)$ repeatedly gives an expression for bos $(x+y+z+w+\ldots\ldots)$. For,

$$\text{bos } (x+y+z) = \text{bos } \left[(x+y)+z\right] = \tfrac{1}{2} \text{ bos } (x+y) \cdot \text{sub } z + \tfrac{1}{2} \text{ sub } (x+y) \cdot \text{bos } z .$$

Expanding the terms bos $(x+y)$ and sub $(x+y)$ we may write

$$\text{bos } (x+y+z) = \tfrac{1}{2}\left[(\tfrac{1}{2} \text{ bos } x \text{ sub } y + \tfrac{1}{2} \text{ sub } x \text{ bos } y) \text{ sub } z + (\tfrac{1}{2} \text{ sub } x \text{ sub } y - \tfrac{1}{2} \text{ bos } x \text{ bos } y) \text{ bos } z\right]$$

$$\therefore \text{bos } (x+y+z) = \tfrac{1}{4}\left[\text{bos } x \text{ bus } y \text{ sub } z + \text{sub } x \text{ bos } y \text{ sub } z + \text{sub } x \text{ sub } y \text{ bos } z - \text{bos } x \text{ bos } y \text{ bos } z\right] .$$

The extension to bos $(x+y+z+w+\ldots\ldots\ldots)$ is obvious.

Similarly, the repeated use of the addition theorem for sub $(x+Y)$ yields an expression for sub $(x+y+z+w+\ldots\ldots)$. For,

$$\text{sub } (x+y+z) = \text{sub } \left[(x+y)+z\right] = \tfrac{1}{2}\left[\text{sub } (x+y) \text{ sub } z - \text{bos } (x+y) \text{ bos } z\right]$$

$$= \tfrac{1}{2}\left\{\tfrac{1}{2}(\text{sub } x \text{ sub } y - \text{bos } x \text{ bos } y) \text{ sub } z - \tfrac{1}{2}(\text{bos } x \text{ sub } y - \text{sub } x \text{ bos } y) \text{ bos } z\right]$$

$$\therefore \text{sub } (x+y+z) = \tfrac{1}{4}\left[\text{sub } x \text{ sub } y \text{ subbz} - \text{bos } x \text{ bos } y \text{ sub } z - \text{bos } x \text{ sub } y \text{ bos } z - \text{sub } x \text{ bos } y \text{ bos } z\right] .$$

The extension to sub $(x+y+z+w+\ldots\ldots)$ is obtainable as before.

Section 5

Functions of multiple angles

Using the addition theorem for bos $(x+y)$ and letting angle y equal angle x, we obtain (1)

$$\text{bos } 2x = \tfrac{1}{2}(\text{bos } x \text{ sub } x + \text{bos } x \text{ sub } x)$$

$$\boxed{\text{bos } 2x = \text{bos } x \text{ sub } x}$$

Allow both angle y and angle z to equal angle x in the theorem for (2) bos (x y z)

$$\begin{aligned}\text{bos } 3x &= \tfrac{1}{4}\left[\text{bos } x \text{ sub}^2 x + \text{bos } x \text{ sub}^2 x + \text{bos } x \text{ sub}^2 x - \text{bos}^3 x\right] \\ &= \tfrac{1}{4}\left[3 \text{ bos } x \text{ sub}^2 x - \text{bos}^3 x\right] \\ &= \tfrac{1}{4}\left[3 \text{ bos } x (4 - \text{bos}^2 x) - \text{bos}^3 x\right] \\ &= \tfrac{1}{4}\left[12 \text{ bos } x - 4 \text{ bos}^3 x\right]\end{aligned}$$

$$\boxed{\text{bos } 3x = 3 \text{ bos } x - \text{bos}^3 x}$$

By making the same substitution as in (1) we obtain the following expression for sub 2x:

$$\boxed{\text{sub } 2x = \tfrac{1}{2}(\text{sub}^2 x - \text{bos}^2 x) = 2 - \text{bos}^2 x = \text{sub}^2 x - 2}$$

Observe that for the bos 2x there was one relationship, while for the sub 2x three relationships appear.

Sub 3x is obtained by using the theorem for sub $(x+y+z)$ and making the same substitution as in (2).

$$\boxed{\text{sub } 3x = \text{sub}^3 x - 3 \text{ sub } x}$$

Expansions for bos nx and sub nx $(n = 2, 3, 4, \ldots\ldots)$ may be obtained by repeated applications of the method just given, but a much neater method exists in the application of the analog of DeMoivre's theorem to be found in Chapter IV, Sections 1 and 2.

Section 6

Functions of half an Angle

Use the equation sub $2x = 2 -$ bos^2 x and let $x = \frac{A}{2}$.

Now,

$$\text{sub } A = 2 - \text{bos}^2 \frac{A}{2}, \qquad \text{or,} \qquad \text{bos}^2 \frac{A}{2} = 2 - \text{sub } A.$$

Therefore,

$$\boxed{\text{bos } \frac{A}{2} = \pm\sqrt{2 - \text{sub } A}}$$

By making the same substitution as above and using the equation sub $2x$ sub^2 $x - 2$, we obtain a formula for the sub functions of half an angle.

$$\text{sub } A = \text{sub}^2 \frac{A}{2} - 2 \qquad \text{or} \qquad \text{sub}^2 A = \text{sub } \frac{A}{2} + 2$$

Whence

$$\boxed{\text{sub } \frac{A}{2} = \pm\sqrt{\text{sub } A + 2}}$$

The algebraic sign in each case depends upon the measure of the Angle A.

Section 7

Sums and Differences of Functions

Adding the equations

$$\text{bos } (x+y) = \tfrac{1}{2}(\text{bos } x \text{ sub } y + \text{sub } x \text{ bos } y) \qquad (1)$$

$$\text{bos } (x-y) = \tfrac{1}{2}(\text{bos } x \text{ sub } y - \text{sub } x \text{ bos } y) \qquad (2)$$

we obtain

$$\text{bos } (x+y) + \text{bos}(x-y) = \text{bos } x \text{ sub } y.$$

Let $x+y = u$, and $x - y = v$, and therefore $x = \frac{u+v}{2}$, and $y = \frac{u-v}{2}$. (3)

Then

$$\boxed{\text{bos } u + \text{bos } v = \text{bos } \frac{u+v}{2} \cdot \text{sub } \frac{u-v}{2}.}$$

Subtracting equation (2) from equation (1), we obtain

$$\text{bos}\ (x+y) - \text{bos}\ (x-y) = \text{sub}\ x\ \text{bos}\ y.$$

Using the substitutions obtained in (3) above, we have (4)

$$\text{bos}\ u - \text{bos}\ v = \text{sub}\ \frac{u+v}{2}.\ \text{bos}\ \frac{u-v}{2}$$

By addition of the expressions for sub (x+y) and sub (x - y), we have

$$\text{sub}\ (x+y) + \text{sub}\ (x-y) = \text{sub}\ x\ \text{sub}\ y.$$

Using the same substitution as in (3) and (4) above, we obtain

$$\text{sub}\ u + \text{sub}\ v = \text{sub}\ \frac{u+v}{2}.\ \text{sub}\ \frac{u-v}{2}.$$

Subtracting the expressions for sub (x+y) and sub (x - y), we have

$$\text{sub}\ (x+y) - \text{sub}\ (x-y) = -\ \text{bos}\ x\ \text{bos}\ y.$$

Therefore,

$$\text{sub}\ u - \text{sub}\ v = -\ \text{bos}\ \frac{u+v}{2}.\ \text{bos}\ \frac{u-v}{2}.$$

CHAPTER IV

ANALYSIS AND GRAPHICAL REPRESENTATION OF THE FUNCTIONS

It is now possible to study some of the properties of the isosceles functions, e.g., continuity and periodicity. The derivatives and integrals of the functions will be obtained, and immediate use made of them in preparing graphs of the functions.

Section 1

Continuity of the Functions

In the theorem,

$$\text{bos } A - \text{bos } B = \text{sub } \frac{A+B}{2} \cdot \text{bos } \frac{A-B}{2}$$

set $A = \theta + \Delta\theta$, $B = \theta$.

Then $\text{bos}(\theta + \Delta\theta) - \text{bos }\theta = \text{sub }(\theta + \frac{\Delta\theta}{2}) \cdot \text{bos } \frac{\Delta\theta}{2}$.

Hence $\lim_{\Delta\theta \to 0} \left[\text{bos}(\theta + \Delta\theta) - \text{bos }\theta\right] = 0.$

Therefore bos θ is a continuous function for all finite values of θ.

Since the bos function is continuous, and since sub θ = bos $(180 - \theta)$, the sub function is also continuous.

Section 2

Derivative of bos u

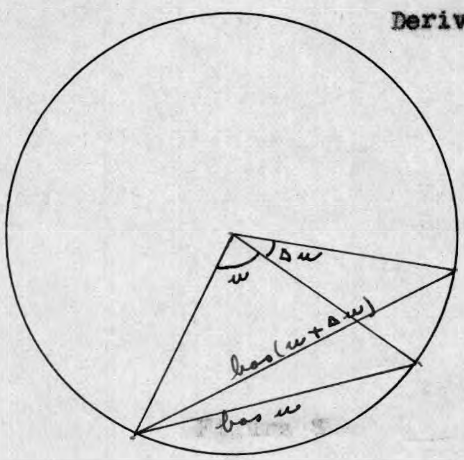


Figure 8

Let

$y = \text{bos } u$, where $u = f(\theta)$ (1)

Give u an increment Δu, and thereupon y takes an a corresponding increment Δy:

$$y + \Delta y = \text{bos}\,(u + \Delta u) \qquad (2)$$

Subtract equation (2) from (1) $\Delta y = \text{bos}\,(u + \Delta u) - \text{bos}\,u$

Using the theorem for the difference of the bos of two angles, we have

$$\Delta y = \text{sub}\,\frac{u + \Delta u + \Delta u}{2} \cdot \text{bos}\,\frac{u + \Delta u - u}{2}$$

$$= \text{sub}\,\frac{(u + \Delta u)}{2} \cdot \text{bos}\,\frac{\Delta u}{2}$$

Then

$$\frac{\Delta y}{\Delta u} = \frac{\text{sub}\left(u + \frac{\Delta u}{2}\right) \cdot \text{bos}\,\frac{\Delta u}{2}}{\Delta u}.$$

Hence

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \left[\text{sub}\left(u + \frac{\Delta u}{2}\right) \cdot \frac{\text{bos}\,\frac{\Delta u}{2}}{\Delta u} \right]$$

$$= \left[\lim_{\Delta u \to 0} \text{sub}\left(u + \frac{\Delta u}{2}\right) \right] \left[\lim_{\Delta u \to 0} \frac{\text{bos}\,\frac{\Delta u}{2}}{\Delta u} \right]$$

Evaluate each of these limits separately. For the first, at once

$$\lim_{\Delta u \to 0} \left[\text{sub}\left(u + \frac{\Delta u}{2}\right) \right] = \text{sub}\,u$$

For the second, modify the form by multiplying and dividing by $\frac{1}{2}$. Then

$$\lim_{\Delta u \to 0} \left[\frac{\text{bos}\,\frac{\Delta u}{2}}{\Delta u} \right] = \lim_{\Delta u \to 0} \frac{1}{2} \left[\frac{\text{bos}\,\frac{\Delta u}{2}}{\frac{\Delta u}{2}} \right].$$

But, from plane geometry, we know that for a small central angle A as A approaches 0, the chord of the central angle A approaches the arc, and, indeed,

$$\lim_{A \to 0} \left[\frac{\text{bos}\,A}{A} \right] = 1,$$ where A is measured in terms of radians.

Hence

$$\lim_{\Delta u \to 0} \left[\frac{\text{bos } \frac{\Delta u}{2}}{\frac{\Delta u}{2}} \right] = \frac{1}{2}$$

Therefore, combining these results,

$$\frac{dy}{du} = \frac{1}{2} \text{ sub } u \text{ .}$$

From the calculus, we know that

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} \text{ .}$$

Hence, finally, if $y = \text{bos } u$, where $u = f(\theta)$,

$$\boxed{\frac{dy}{d\theta} = \frac{1}{2} \text{ sub } u \cdot \frac{du}{d\theta} \text{.}}$$

Section 3

Derivative of Sub θ .

Let $y = \text{sub } u$, where $u = f(\theta)$.

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \frac{d(\text{sub } u)}{du} \cdot \frac{du}{d\theta} \text{ .}$$

By definition, we have sub u = bos (180 - u), and sub (180 - u) = bos u .

Then

$$\frac{d(\text{sub } u)}{du} = \frac{d\left[\text{bos } (180 - u)\right]}{du} = \frac{1}{2} \text{ sub } (180 - u).(-1)$$

Therefore

$$\boxed{\frac{dy}{d\theta} = -\frac{1}{2} \text{ sub } (180 - u) \cdot \frac{du}{d\theta} = -\frac{1}{2} \text{ bos } u \cdot \frac{du}{d\theta} \text{.}}$$

Section 4

The n^{th} Derivatives of the Functions

Let $f(x) = \text{bos } x$

Forming successive derivatives, as obtained in sections (2) and (3), we may observe that

$$f^{(n)}(x) = (-1)^{\frac{n}{2}} \cdot \frac{1}{2^n} \text{ bos } x \qquad (\text{ n even})$$

and

$$f^{(n)}(x) = (-1)^{\frac{n-1}{2}} \cdot \frac{1}{2^n} \text{ sub } x \ . \qquad (\text{n odd})$$

Similarly, if $f(x) = \text{sub } x$

Then $$f^{(n)}(x) = (-1)^{\frac{n}{2}} \cdot \frac{1}{2^n} \text{ sub } x \qquad (\text{ n even})$$

And $$f^{(n)}(x) = (-1)^{\frac{n-1}{2}} \cdot \frac{1}{2^n} \text{ bos } x \ . \qquad (\text{n odd})$$

Section 5

Integrals of the Functions

Using the preceding sections, it is easy to verify that

$$\text{bos } u \, du = 2 \text{ sub } u + C$$

$$\text{sub } u \, du = 2 \text{ bos } u + C.$$

Many definite integrals involving the isosceles functions are evaluated in Chapter VII in connection with the Derivation of the Analog of Foureer's Series.

Section 6

Periodicity of the Isosceles Functions

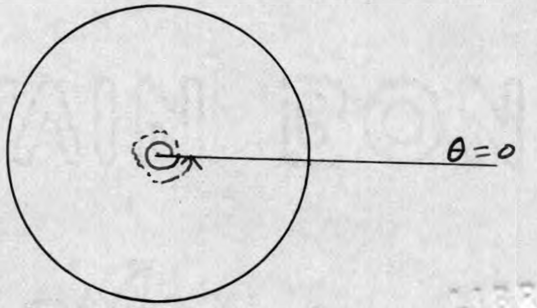


Figure 9

The nature of the periodicity of bos θ is determined from Figure 9 by the following arguments:

First, bos $\theta = 0$ for $\theta = 0, 360, 720, \ldots,$

i.e. for $\theta = 0 \pm k \cdot 360$ $(k = 1, 2, 3, \ldots\ldots)$

Second, considering the nature of the Riemann surface, we note that bos θ is positive when $0 > \theta > 360^\circ$ and negative for $360 > \theta > 720^\circ$.

To say that bos $\theta = 0$ for $\theta = 0 \pm k \cdot 360$ simply means that on the graph the curve representing the bos function would cross the θ axis at $\theta = 0 \pm k \cdot 360$, or every 360°, but from the construction of our Riemann surface, we know that it takes 720° to give a complete set of values for bos θ. Hence, we say bos θ is periodic with a period of 720°.

In like manner sub $\theta = 0$, for $\theta = 180^\circ$.

But sub θ = bos $(180 - \theta)$.

Now, bos $(180 - \theta) = 0$, whenever $(180 - \theta) = 0 \pm k \cdot 360$,

i.e., if $\theta = 180 \mp k \cdot 360$.

Hence sub $\theta = 0$, for $\theta = 180 \mp k \cdot 360$.

From the above argument, we see that the curve representing sub θ crosses the θ axis at $\theta = 180, 540, \ldots,$ or $180 \mp k \cdot 360$. Again, from the nature of the Riemann surface, sub θ also requires 720° to pass through a complete cycle of values because sub θ = bos$(180 - \theta)$, sub θ is positive for $0 > \theta > 180$, negative $180 > \theta > 540$, and positive $540 > \theta > 720$.

Hence, we find that each function is periodic with a period of 720°.

Section 7

Maxima and Minima.

If $y =$ bos x, then $\frac{dy}{dx} = \frac{1}{2}$ sub x.

Setting $\frac{1}{2}$ sub x = 0, we have critical values for x (i.e. values corresponding to the horizontal tangents) at

$$x = \pi \pm k \cdot 2\pi \qquad (k = 1, 2, 3, \ldots) .$$

Now $\frac{d^2y}{dx^2} = -\frac{1}{2}$ bos x, and $\left.\frac{d^2y}{dx^2}\right]_{x=\pi} = -1 > 0$

and therefore a maximum exists at $x = \pi$. Since y = bos x is a continuous periodic function, maxima and minima must alternate at equal intervals along the sequence of critical values.

The following table locates and gives the values of maxima (M) and Minima (m) for the bos x .

X rad	X degrees	bos x	$\frac{dy}{dx}$ $\frac{1}{2}$ sub x
0	0	0	
π	180	2 (M)	0
2π	360	0	
3π	540	-2 (m)	0
4π	720	0	
5π	900	2 (M)	0
⋮	⋮	⋮	

It is not necessary to determine maxima and minima for sub x by the above method. Sub x = bos (π - x), hence the maxima and minima for sub x are displaced by an amount π from those of bos x; so sub x has a maximum at $\pi - \pi$ or 0° .

Section 8

Graphs of the Functions

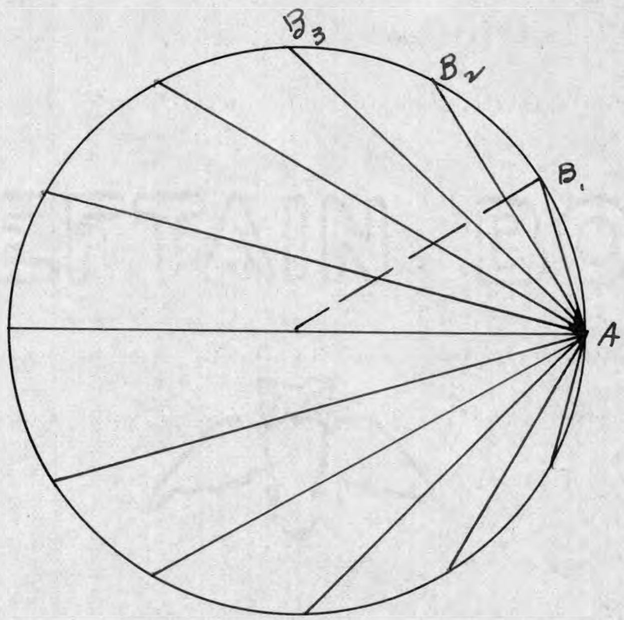


Figure 10

The line-values of the functions together with the properties just described now furnish a means for constructing as many points as desired on the graphs of bos x and sub x .

Let us take a unit circle, as in Figure 10, and, beginning at A, locate along the circumference points B_1 , B_2 , B_3 ,....., for convenience at equal intervals, say 30 degrees. Then connect each of these points with A. The lines AB_1 , AB_2 , AB_3 ,....represent the bos functions of the central angles subtended by the corresponding arcs. Starting at O, Figure 11, lay off the points B_1 , B_2 , B_3 .....corresponding to the points taken along the circumference of the circle, Figure 10, using a 1:2 scale for convenience. Erect perpendiculars, equal respectively to AB_1 , AB_2 , AB_3 ,.....The locus of points so obtained is the graph of bos x.

For the sub curve, let us go back to the definition of the sub function, namely sub x = bos (180 - x). Because of this definition, the graph of function is identically like that of the bos function displaced on the x - axis by an amount corresponding to an angle of 180^o. Figure 12 gives the graph of sub x, drawn to the same scale as Figure 11.

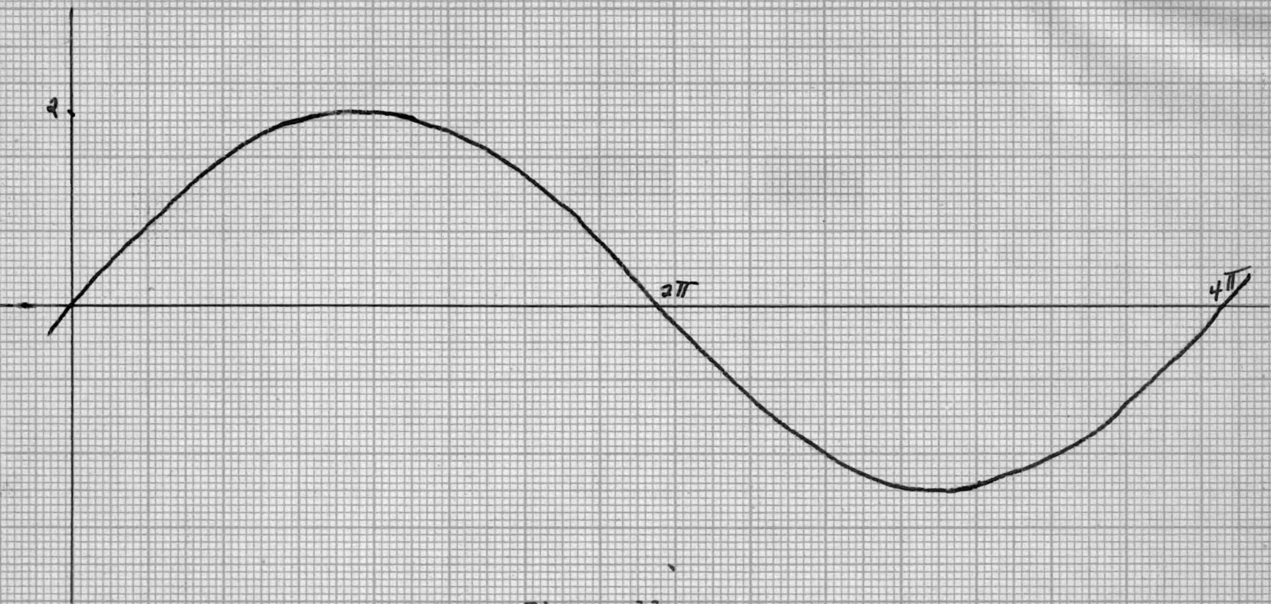


Figure 11

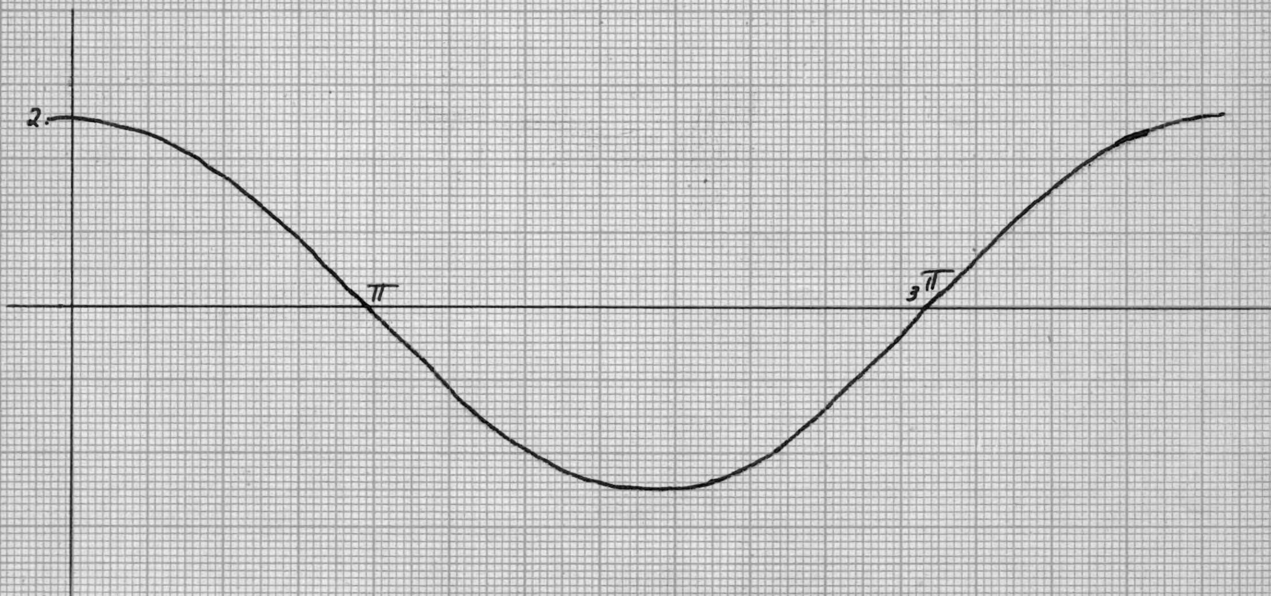


Figure 12

CHAPTER V

POWER SERIES EXPANSIONS OF ISOSCELES TRIGONOMETRY

Section 1

Analog of DeMoivre's Theorem

Let us take a complex number in the isosceles trigonometric form $(\text{sub}\,\theta + i \text{ bos}\,\theta)$, where $i = \sqrt{-1}$. Now multiply two such numbers together, as follows:

$$(\text{sub}\,\theta_1 + i \text{ bos}\,\theta_1)(\text{sub}\,\theta_2 + i \text{ bos}\,\theta_2) = \text{sub}\,\theta_1 \text{sub}\,\theta_2 - \text{bos}\,\theta_1 \text{ bos}\,\theta_2$$

$$+ i \text{ bos}\,\theta_1 \text{ sub}\,\theta_2 + i \text{ sub}\,\theta_1 \text{ bos}\,\theta_2$$

$$= 2 \text{ sub}\,(\theta_1 + \theta_2) + 2i \text{ bos}\,(\theta_1 + \theta_2)$$

$$= 2 \left[\text{sub}\,(\theta_1 + \theta_2) + i \text{ bos}\,(\theta_1 + \theta_2)\right].$$

This is the same form as either of the original expressions, except that it is multiplied by 2.

Forming the product of three such expressions, we have

$$(\text{sub}\,\theta_1 + i \text{ bos}\,\theta_1)(\text{sub}\,\theta_2 + i \text{ bos}\,\theta_2)(\text{sub}\,\theta_3 + i \text{ bos}\,\theta_3)$$

$$= 2 \left[\text{sub}(\theta_1 + \theta_2) + i \text{ bos}\,(\theta_1 + \theta_2)\right] \left[\text{sub}\,\theta_3 + i \text{ bos}\,\theta_3\right].$$

Let $\phi = \theta_1 + \theta_2$; then

$$(\text{sub}\,\theta_1 + i \text{ bos}\,\theta_1)(\text{sub}\,\theta_2 + i \text{ bos}\,\theta_2)(\text{sub}\,\theta_3 + i \text{ bos}\,\theta_3)$$

$$= 2(\text{sub}\,\phi + i \text{ bos}\,\phi)(\text{sub}\,\theta_3 + i \text{ bos}\,\theta_3)$$

and by the obtained theorem, this is equal to

$$2 \left[2 \text{ sub}(\phi + \theta_3) + 2 i \text{ bos}(\phi + \theta_3)\right]$$

$$= 2^2 \left[\text{sub}(\theta_1 + \theta_2 + \theta_3) + i \text{ bos}\,(\theta_1 + \theta_2 + \theta_3)\right].$$

Finally, on successive application of the original product relation, $(\text{sub}\,\theta_1 + i \text{ bos}\,\theta_1)(\text{sub}\,\theta_2 + i \text{ bos}\,\theta_2)(\text{sub}\,\theta_3 + i \text{ bos}\,\theta_3)\ldots\ldots$ $(\text{sub}\,\theta_n + i \text{ bos}\,\theta_n)$

$$= 2^{n-1}\left[\text{sub}(\theta_1+\theta_2+\theta_3+\cdots+\theta_n) + i\ \text{bos}(\theta_1+\theta_2+\theta_3+\cdots+\theta_n)\right.$$

If we let $\theta_1, \theta_2, \theta_3 \cdots \theta_n$ each equal θ, then

$$(\text{sub}\,\theta_1 + i\ \text{bos}\,\theta_1)(\text{sub}\,\theta_2 + i\ \text{bos}\,\theta_2)\text{.......ten factors}$$

$$= 2^{n-1}\left[\text{sub}(\theta+\theta+\cdots+\theta) + i\ \text{bos}(\theta+\theta+\cdots+\theta)\right]$$

$$= 2^{n-1}\left[\text{sub}\,n\,\theta + i\ \text{bos}\,n\,\theta\right].$$

Therefore,

$$\boxed{(\text{sub}\,\theta + i\ \text{bos}\,\theta)^n = 2^{n-1}(\text{sub}\,n\,\theta + i\ \text{bos}\,n\,\theta)}$$

This analog of DeMoivre's theorem could also be proved by mathematical induction.

Section 2

General Multiple Angle Formulas

By the analog of DeMoivre's theorem just established,

$$(\text{sub}\,\theta + i\ \text{bos}\,\theta)^n = 2^{n-1}(\text{sub}\,n\,\theta + i\ \text{bos}\,n\,\theta). \qquad (1)$$

But by the binomial theorem, applied to the left member of (1)

$$(\text{sub}\,\theta + i\ \text{bos}\,\theta)^n = \text{sub}^n\theta + n\,\text{sub}^{n-1}\theta\,(i\ \text{bos}\,\theta)$$
$$+ \frac{n(n-1)}{\lfloor 2}\,\text{sub}^{n-2}\theta\,(i\ \text{bos}\,\theta)^2 + \frac{n(n-1)(n-2)}{\lfloor 3}\,\text{sub}^{n-3}\theta\,(i\ \text{bos}\,\theta)^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\,\text{sub}^{n-4}\theta\,(i\ \text{bos}\,\theta)^4 + \cdots\cdots \qquad (2)$$

Now $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,....; hence the first, third, fifth....terms of (2) do not contain i, while the even numbered terms do contain $\pm i$. In fact,

$$(\text{sub}\,\theta + i\ \text{bos}\,\theta)^n = \text{sub}^n\theta + n\,\text{sub}^{n-1}\theta\ \text{bos}\,\theta$$
$$- \frac{n(n-1)}{\lfloor 2}\,\text{sub}^{n-2}\theta\ \text{bos}^2\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\,\text{sub}^{n-3}\theta\ \text{bos}^3\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\,\text{sub}^{n-4}\theta\ \text{bos}^4\theta + \cdots \qquad (3)$$

Equating the imaginary parts of equations (1) and (3), we have

$$2^{n-1} \text{ bos } n\theta = n \text{ sub}^{n-1}\theta \text{ bos}\theta - \frac{n(n-1)(n-2)}{\lfloor 3} \text{ sub}^{n-3}\theta \text{ bos } 3\theta$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5} \text{ sub}^{n-5}\theta \text{ bos}^5\theta \ldots\ldots\ldots \quad (4)$$

From this by setting, $n = 2, 3, 4, \ldots\ldots$ the multiple angle relations for bos nx are obtained.

$2 \text{ bos } 2\theta = 2 \text{ sub } \theta \text{ bos } \theta$, or $\boxed{\text{bos } 2\theta = \text{sub } \theta \text{ bos } \theta .}$

$2^2 \text{ bos } 3\theta = 3 \text{ sub}^2\theta \text{ bos}\theta - \text{bos}^3\theta$

$= 3 \text{ bos } \theta \ (4 - \text{bos}^2\theta) - \text{bos}^3\theta$

$= 12 \text{ bos } \theta - 4 \text{ bos}^3\theta$, or $\boxed{\text{bos } 3\theta = 3 \text{ bos}\theta - \text{bos}^3\theta .}$

$2^3 \text{ bos } 4\theta = 4 \text{ sub}^3\theta \text{ bos } \theta - 4 \text{ sub}\theta \text{ bos}^3\theta$

$= \text{sub}\theta \text{ bos}\theta \ (4 \text{ sub}^2\theta - \text{bos}^2\theta)$

$= \text{sub } \theta \text{ bos}\theta \ (5 \text{ sub}^2\theta - 4)$

or $\boxed{\text{bos } 4\theta = \frac{5}{8} \text{ bos } \theta \text{ sub}^3\theta - \tfrac{1}{2} \text{ bos } \theta \text{ sub } \theta .}$

Equating the real parts of equations (1) and (3) in section 2, we have

$$2^{n-1} \text{ sub } n\theta = \text{sub}^n\theta - \frac{n(n-1)}{\lfloor 2} . \text{ sub}^{n-2}\theta \text{ bos}^2\theta$$

$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} . \text{ sub}^{n-4}\theta \text{ bos}^4\theta \ldots\ldots\ldots \quad (5)$$

From this, by setting $n = 2, 3, 4, \ldots$ the multiple angle formulas for sub n θ are obtained.

$2 \text{ sub } 2\theta = \text{sub}^2\theta - \text{bos}^2\theta$, or $\boxed{\text{sub } 2\theta = \tfrac{1}{2}(\text{sub}^2\theta - \text{bos}^2\theta) .}$

$2^2 \text{ sub } 3\theta = \text{sub}^3\theta - 3 \text{ sub } \theta \text{ bos}^2\theta$

$= \text{sub}^3\theta - 3 \text{ sub } \theta (4 - \text{sub}^2\theta)$

$= 4 \text{ sub}^3\theta - 12 \text{ sub } \theta$, or $\boxed{\text{sub } 3\theta = \text{sub}^3\theta - 4 \text{ sub } \theta .}$

$2^3 \text{ sub } 4\theta = \text{sub}^4\theta - 6 \text{ sub}^2\theta \text{ bos}^2\theta + \text{bos}^4\theta$

$= \text{sub}^4\theta - 6 \text{ sub}^2\theta \ (4 - \text{sub}^2\theta) + (4 - \text{sub}^2\theta)^2$

$= \text{sub}^4\theta - 24 \text{ sub}^2\theta + 6 \text{ sub}^4\theta + 16 - 8 \text{ sub}^2\theta + \text{sub}^4\theta$

$$= 8 \text{ sub}^4\theta - 32 \text{ sub}^2\theta + 16$$

or $$\boxed{\text{sub } 4\theta + \text{sub}^4\theta - 4 \text{ sub}^2\theta + 2 .}$$

Section 3

Expansions in Power Series

By the Analog DeMoivre's Theorem

To obtain the power series expansion for bos x, let $n\theta = x$ in relation (4) of the preceding section.

Then $n = \frac{x}{\theta}$ and

$$2^{n-1} \text{ bos } x = x \text{ sub}^{n-1}\theta \text{ bos }\theta - \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)\left(\frac{x}{\theta} - 2\right)}{\lfloor 3} \cdot \text{sub}^{n-3}\theta \text{ bos}^3\theta$$

$$+ \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)\left(\frac{x}{\theta} - 2\right)\left(\frac{x}{\theta} - 3\right)\left(\frac{x}{\theta} - 4\right)}{\lfloor 5} \cdot \text{sub}^{n-5}\theta \text{ bos}^5\theta \ldots\ldots$$

or

$$2^{n-1} \text{ bos } x = x \text{ sub}^{n-1}\theta \text{ bos }\theta \frac{-x(x-\theta)(x-2\theta)}{\lfloor 3} \text{ sub}^{n-3}\theta\left(\frac{\text{bos }\theta}{\theta}\right)^3$$

$$+ \frac{x(x-\theta)(x-2\theta)(x-3\theta)(x-4\theta)}{\lfloor 5} \cdot \text{sub}^{n-5}\theta\left(\frac{\text{bos }\theta}{\theta}\right)^5$$

But $\lim_{\theta \to 0} \left(\frac{\text{bos }\theta}{\theta}\right) = 1$, where θ is measured in terms of radians, and therefore

$$2^{n-1} \text{ bos } x = x \cdot 2^{n-1} - \frac{x^3}{\lfloor 3} \cdot 2^{n-3} + \frac{x^5}{\lfloor 5} 2^{n-5} - \ldots$$

Dividing by the coefficient of bos x, 2^{n-1}, we have

$$\text{bos } x = x \frac{2^{n-1}}{2^{n-1}} \frac{-x^3}{\lfloor 3} \cdot \frac{2^{n-3}}{2^{n-1}} + \frac{x^5}{\lfloor 5} \cdot \frac{2^{n-5}}{2^{n-1}} - \cdots$$

$$= x - \frac{x^3}{\lfloor 3} \cdot 2^{-2} + \frac{x^5}{\lfloor 5} \cdot 2^{-4} - \ldots$$

and finally the power series

$$\text{bos } x = x - \frac{x^3}{\lfloor 3} \cdot \frac{1}{2} + \frac{x^5}{\lfloor 5} \cdot \frac{1}{2^4} - \frac{x^7}{\lfloor 7} \cdot \frac{1}{2^6} + \cdots$$

To obtain the power series expansion for sub x, let $n\theta = x$, in relation (5) of the preceding section.

Then $n = \frac{x}{\theta}$ and

$$2^{n-1} \text{ sub } x = \text{sub}^n \theta - \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)}{\lfloor 2} \cdot \text{sub}^{n-2}\theta \text{ bos}^2\theta$$

$$+ \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)\left(\frac{x}{\theta} - 2\right)\left(\frac{x}{\theta} - 3\right)}{\lfloor 4} \cdot \text{sub}^{n-4}\theta \text{ bos}^4\theta - \cdots\cdots$$

$$= \text{sub}^n\theta - \frac{x(x - \theta)}{\lfloor 2} \cdot \text{sub}^{n-2}\theta \left(\frac{\text{bos } \theta}{\theta}\right)^2$$

$$+ \frac{x(x - \theta)(x - 2\theta)(x - 3\theta)}{\lfloor 4} \cdot \text{sub}^{n-4}\theta \left(\frac{\text{bos } \theta}{\theta}\right)^4 - \cdots$$

Hence, as before,

$$2^{n-1} \text{ sub } x = 2^n - \frac{x(x)}{\lfloor 2} \cdot 2^{n-2} + \frac{x(x)(x)(x)}{\lfloor 4} \cdot 2^{n-4} - \cdots$$

$$= 2^n - \frac{x^2}{\lfloor 2} \cdot 2^{n-2} + \frac{x^4}{\lfloor 4} \cdot 2^{n-4} - \cdots$$

Dividing by 2^{n-1}, we obtain and simplifying, we have the infinite series

$$\text{sub } x = 2 - \frac{x^2}{\lfloor 2} \cdot \frac{1}{2} + \frac{x^4}{\lfloor 4} \cdot \frac{1}{2^3} - \frac{x^6}{\lfloor 6} \cdot \frac{1}{2^5} + \cdots\cdots$$

Section 4

The Power Series As Obtained By

Maclaurin's Expansion

Let $f(x) = \text{bos } x$, perform successive differentiations on this function, and then evaluate $f(0)$, $f'(0)$, $f''(0)$,......

$f(x) = \text{bos } x \qquad\qquad f(0) = 0$

$f'(x) = \frac{1}{2} \text{ sub } x \qquad\qquad f'(0) = 1$

$f''(x) = -\frac{1}{2^2} \text{ bos } x \qquad\qquad f''(0) = 0$

$f'''(x) = \frac{-1}{2^3} \text{ sub } x \qquad\qquad f'''(0) = \frac{-1}{2^2}$

$f^{IV}(x) = \frac{1}{2^4} \text{ bos } x \qquad\qquad f^{IV}(0) = 0$

$f^{V}(x) = \frac{1}{2^5} \text{ sub } x \qquad\qquad f^{V}(0) = \frac{1}{2^4}$

$\vdots \qquad\qquad \vdots$

The values of even numbered derivatives are zero, odd numbered ones are equal to $\frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}}$.

According to Maclaurin's expansion, $f(x) = f(0) + f'(0) + \frac{f''(0)}{\lfloor 2} \cdot x^2 + \frac{f'''(0)}{\lfloor 3} \cdot x^3 + \ldots\ldots$ and

Therefore,

$$\text{bos } x = x - \frac{\frac{1}{2^2}}{\lfloor 3} \cdot x^3 + \frac{\frac{1}{2^4}}{\lfloor 4} \cdot x^5 - \ldots\ldots$$

or finally

$$\boxed{\text{bos } x = x - \frac{x^3}{\lfloor 3} \cdot \frac{1}{2^2} + \frac{x^5}{\lfloor 5} \cdot \frac{1}{2^4} - \ldots\ldots}$$

identical with the series obtained in section 3.

Similarly, for sub x,

$f(x) = \text{sub } x \qquad\qquad f(0) = 2$

$f'(x) = -\frac{1}{2} \text{ bos } x \qquad\qquad f'(0) = 0$

$f''(x) = -\frac{1}{2^2} \text{ sub } x \qquad\qquad f''(0) = -\frac{1}{2}$

$f'''(x) = \frac{1}{2^3} \text{ bos } x \qquad\qquad f'''(0) = 0$

$f^{IV}(x) = \frac{1}{2^4} \text{ sub } x \qquad\qquad f^{IV}(0) = \frac{1}{2^3}$

$\vdots \qquad\qquad \vdots$

Whence

$$\text{sub } x = 2 - \frac{x^2}{\underline{|2}} \cdot \frac{1}{2} + \frac{x^4}{\underline{|4}} \cdot \frac{1}{8} - \cdots\cdots ,$$

the series previously obtained in Section 3.

Section 5

Convergence of the Power Series

By examining the series,

$$\text{bos } x = x - \frac{x^3}{\underline{|3}} \cdot \frac{1}{2^2} + \frac{x^5}{\underline{|5}} \cdot \frac{1}{2^4} - \frac{x^7}{\underline{|7}} \cdot \frac{1}{2^6} + \cdots\cdots$$

we find the general terms

$$U_n = \frac{x^{2n-1}}{\underline{|2n-1}} \cdot \frac{1}{2^{2n-2}}$$

and

$$U_{n+1} = \frac{x^{2n+1}}{\underline{|2n+1}} \cdot \frac{1}{2^{2n}} .$$

Forming Cauchy's test ratio,

$$\frac{U_{n+1}}{U_n} = \frac{\frac{x^{2n+1}}{\underline{|2n+1}} \cdot \frac{1}{2^{2n}}}{\frac{x^{(2n-1)}}{\underline{|2n-1}} \cdot 2^{\left(\frac{1}{2n-2}\right)}} = \frac{x^2 \cdot 2^{-2}}{2n(2n+1)}$$

$$= \frac{x^2}{2n(2n+1) \cdot 2^2} .$$

Then, by Cauchy's test for convergence, since

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \lim_{n \to \infty} \left[\frac{x^2}{2n(2n+1) \cdot 2^2} \right] = 0 \text{, and } |0| < 1 ,$$

the series for bos x converges for all values of x.

Similarly in the series

$$\text{sub } x = 2 - \frac{x^2}{\lfloor 2} \cdot \frac{1}{2} + \frac{x^4}{\lfloor 4} \cdot \frac{1}{2^3} - \frac{x^6}{\lfloor 6} \cdot \frac{1}{2^5} + \cdots\cdots$$

$$U_n = \frac{x^{2n-2}}{\lfloor 2n-2} \cdot \frac{1}{2^{2n-3}}$$

and

$$U_{n+1} = \frac{x^{2n}}{\lfloor 2n} \cdot \frac{1}{2^{2n-1}} .$$

Then

$$\frac{U_{n+1}}{U_n} = \frac{\dfrac{x^{2n}}{\lfloor 2n} \cdot \dfrac{1}{2^{2n-1}}}{\dfrac{x^{2n-2}}{\lfloor 2n-2} \cdot \dfrac{1}{2^{2n-3}}}$$

$$= \frac{x^2}{2n(2n-1) \cdot 2^2} .$$

Applying Cauchy's ratio test,

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \lim_{n \to \infty} \frac{x^2}{2n(2n-1)\, 2^2} = 0,$$

whereupon the series for sub x converges for all values of x.

These series for bos x and sub x derived in sections 2 and 3, and other readily obtainable Taylor's series expansions may among other uses, be employed to compute complete tables of numerical values for the functions of isosceles trigonometry.

Section 6

Analogs of Euler's Formulas.

Multiplying bos by i,

$$i \text{ bos } \theta = i\theta - \frac{\theta^3}{\lfloor 3} \cdot \frac{1}{2^2} + \frac{i\,\theta^5}{\lfloor 5} \cdot \frac{1}{2^4} - \cdots\cdots\cdots\cdots$$

and adding

$$\text{sub}\,\theta = 2 - \frac{\theta^2}{\lfloor 2} \cdot \frac{1}{2} + \frac{\theta^4}{\lfloor 4} \cdot \frac{1}{2^3} - \cdots\cdots\cdots$$

we have

$$\text{sub}\,\theta + i\,\text{bos}\,\theta = 2 + i\theta - \frac{\theta^2}{\lfloor 2} \cdot \frac{1}{2} - \frac{i\,\theta^3}{\lfloor 3} \cdot \frac{1}{2^2}$$

$$+ \frac{\theta^4}{\lfloor 4} \cdot \frac{1}{2^3} + \frac{i\,\theta^5}{\lfloor 5} \cdot \frac{1}{2^4} - \cdots\cdots$$

Dividing by 2, we obtain

$$\frac{\text{sub}\,\theta + i\,\text{bos}\,\theta}{2} = 1 + \frac{i\,\theta}{2} - \frac{\theta^2}{\lfloor 2} \cdot \frac{1}{2^2} - \frac{i\,\theta^3}{\lfloor 3} \cdot \frac{1}{2^3}$$

$$\frac{\theta^4}{\lfloor 4} \cdot \frac{1}{2^4} + \frac{i\theta^5}{\lfloor 5} \cdot \frac{1}{2^5} - \cdots\cdots\cdots \quad (1)$$

Examining the standard power series for we find

$$e^{i\frac{\theta}{2}} = 1 + \frac{i\,\theta}{2} - \frac{\theta^2}{\lfloor 2} \cdot \frac{1}{2^2} - \frac{i\,\theta^3}{\lfloor 3} \cdot \frac{1}{2^3} + \frac{\theta^4}{\lfloor 4} \cdot \frac{1}{2^4}$$

$$+ \frac{i\,\theta^5}{\lfloor 5} \cdot \frac{1}{2^5} - \cdots\cdots$$

Observe that the series in (1) is identically the right member just written.

Therefore

$$2e^{i\frac{\theta}{2}} = \text{sub}\,\theta + i\ \text{bos}\,\theta. \qquad (2)$$

Replacing θ by $-\theta$, we obtain

$$2e^{-i\frac{\theta}{2}} = \text{sub}\,\theta - i\,\text{bos}\,\theta. \qquad (3)$$

Subtracting g equation (3) from (2), we have

$$2\ i\ \text{bos}\,\theta = 2(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}})$$

or

$$\boxed{\text{bos}\,\theta = \frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{i}.}$$

Adding (2) and (3), we have

$$2 \text{ sub}\,\theta = 2(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}})$$

or

$$\boxed{\text{sub}\,\theta = e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}.}$$

CHAPTER VI

INVERSE ISOSCELES FUNCTIONS

The inverses of the isosceles functions are defined as follows:

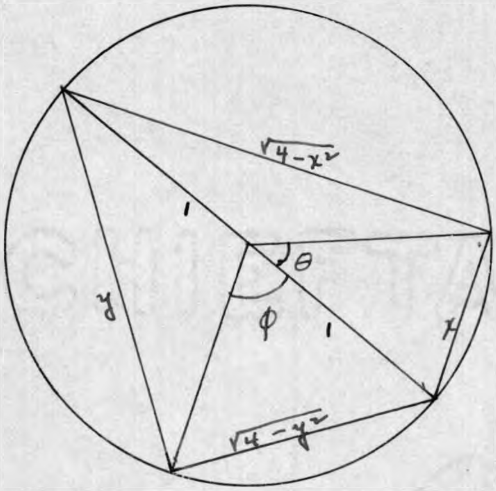


Figure 13.

From Figure 13, if $x = \text{bos}\,\theta$,

$\theta = \text{arc bos } x = \text{arc sub } \sqrt{4 - x^2}$

and also if $y = \text{sub}\,\phi$,

$\phi = \text{arc sub } y = \text{arc bos} \sqrt{4 - y^2}$.

There is an important difference between the isosceles functions and the inverse isosceles functions. When an angle is given, its functions are completely determined; but when one of the functions is given, the angle may have any one of several values. Thus, if bos x = 1, x may be 60°, 300°, 780°, ...

In order to obtain the graph of the function $y = \text{arc bos } x$, reflect the graph of $y = \text{bos } x$ across the line $y = x$. (Figure 14)

This graph of arc bos x represents a multiple-valued function because the line $x = x'$, where $-2 < x' < 2$, cuts the graph in more than one point; in fact in an infinite number of points. Of the ordinates of this infinite number of points, we choose one, whose value is called the principal value of the function. For the arc bos function the most convenient range to select for x is the one between $-\pi$ and π. Using these limits, y arc bos x becomes a single-valued function.

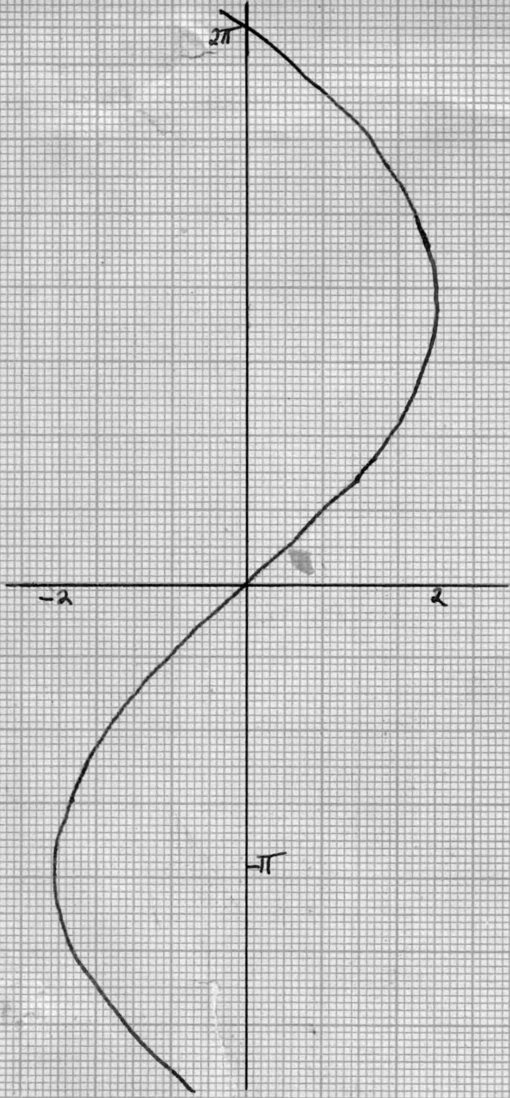
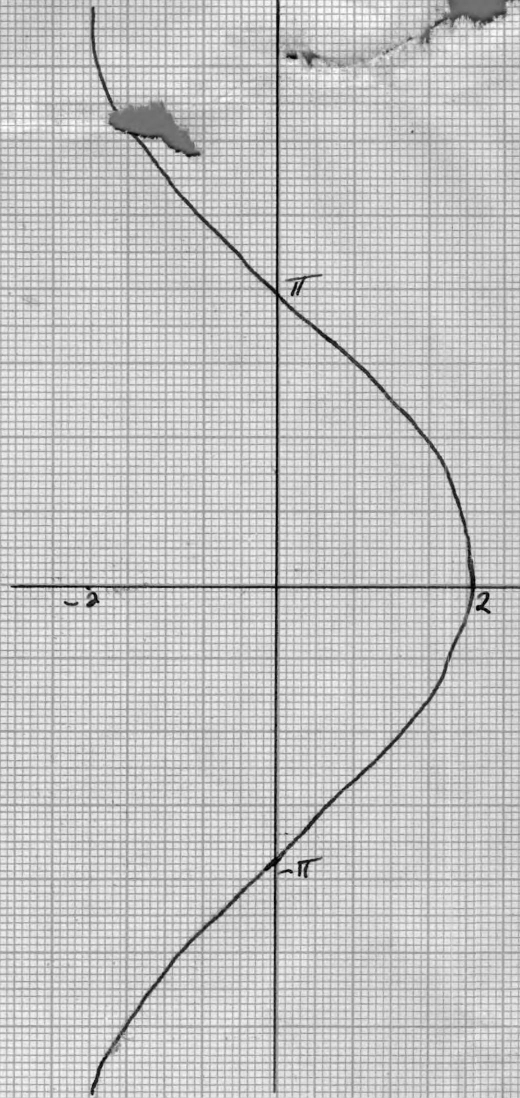

2π
−2
2
−π

π
−2
2
−π

Figure 14

Figure 15

Figure 15 is the graph for $y = \text{arc sub } x$. The explanation is similar to that given for $y = \text{arc bos } x$. Arc sub x becomes a single-valued functions by selecting the limits 0 and 2π, and if x is positive the principal value lies between 0 and π. If x is neagative the principal value lies between π and 2π.

CHAPTER VII

SOLUTION OF TRIANGLES IN GENERAL

We shall now develop a number of laws or theorems which are used in the solution of general plane triangles, as one of the applications of the theory of isosceles trigonometry.

Section 1

The Bos Law

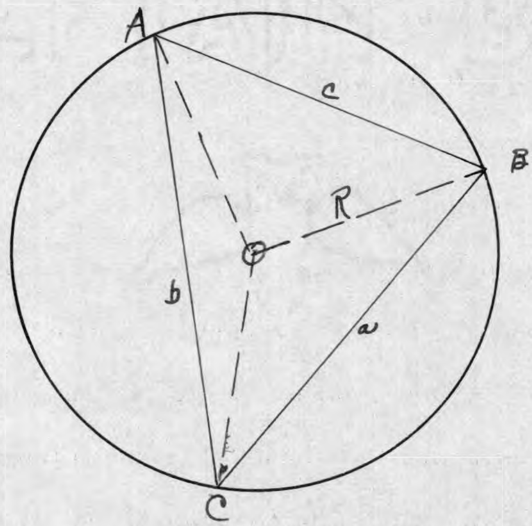


Figure 16

Let ABC be any triangle, with sides a, b, and c. Inscribe ABC in a circle, and connect the vertices A, B, and C with the center of circle.

By definition, bos AOB $= \frac{c}{R}$.

Since the angles in a circle are measured by their arcs, angle AOB $=$ 2 angle C.

Therefore,

$$\text{bos } 2C = \frac{c}{R}, \text{ or } \quad R = \frac{c}{\text{bos } 2C}.$$

Similarly,

$$\text{bos } 2A = \frac{a}{R} \quad \text{or} \quad R = \frac{a}{\text{bos } 2A}$$

and

$$\text{bos } 2B = \frac{b}{R} \quad \text{or} \quad R = \frac{b}{\text{bos } 2B}.$$

Therefore,

$$\boxed{\frac{a}{\text{bos } 2A} = \frac{b}{\text{bos } 2B} = \frac{c}{\text{bos } 2C}.}$$

Hence the theorem: In any triangle the sides are proportional to the bos functions of twice the opposite angles.

Section 2

Expressing the Bos of an Angle in Terms Of The Sides of the Triangle.

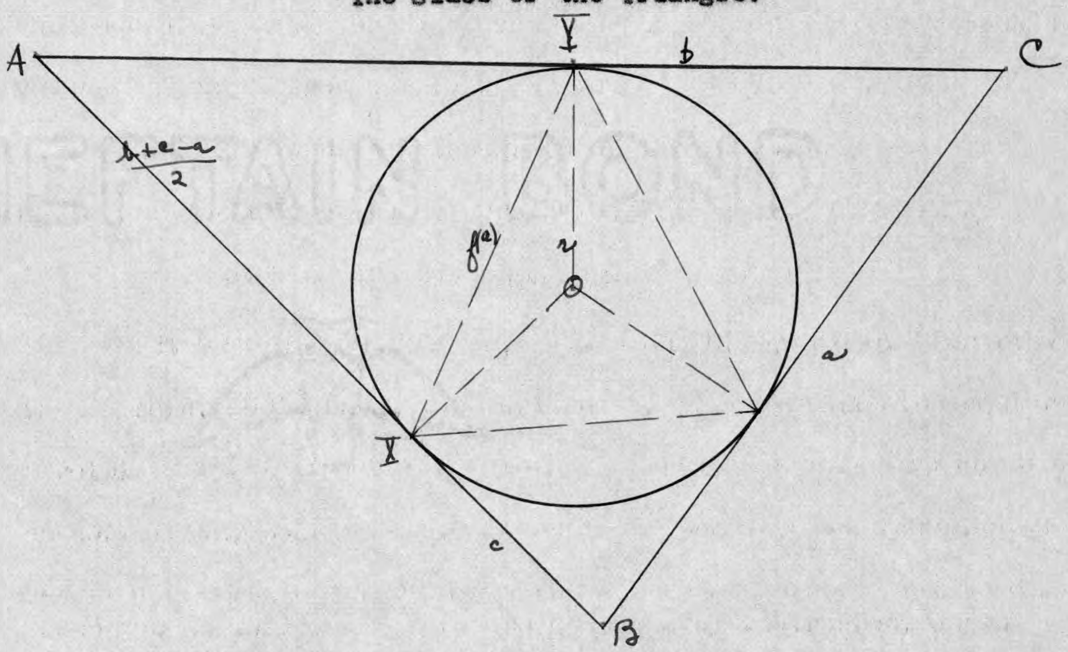


Figure 17.

Inscribe a circle within any triangle ABC, Figure 17. Draw the radii to the points of tangency and also connect these points of tangency with one another.

Since the three sides of the triangle are tangent to the circle, AX is easily shown to be $\frac{b+c-a}{2}$ and therefore

$$\text{bos } A = \frac{XY}{AX} = \frac{XY}{\frac{b+c-a}{2}}. \qquad (1)$$

Radii of the circle form right angles at the points of contact of the sides of the triangle; hence angle XOY is a supplement of angle A

and

$$\text{sub } A = \frac{XY}{r}.$$

Whence

$$XY = r \text{ sub } A.$$

Substituting in equation (1), we have

$$\text{bos } A = \frac{r \text{ sub } A}{\frac{b+c-a}{2}}$$

From plane geometry, r is shown to be $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, where a, b, and c are the sides of the triangle and $s = \frac{a+b+c}{2}$.

Therefore

$$\text{bos } A = \frac{\text{sub } A\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}}{\frac{b+c-a}{2}}.$$

However $\frac{b+c-a}{2} = s-a$, and so

$$\text{bos } A = \text{sub } A \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)}} \qquad (2)$$

But sub $A = \sqrt{4 - \text{bos}^2 A}$ and

$$\text{bos } A = \sqrt{4 - \text{bos}^2 A}\ \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Squaring both sides of this equation, we have

$$\text{bos}^2A = (4 - \text{bos}^2A) \cdot \frac{(s - b)(s - c)}{s(s - a)}.$$

Expanding and transposing the term containing bos^2A, we obtain

$$\left(1 + \frac{(s - b)(s - c)}{s(s - a)}\right)\text{bos}^2A = 4\,\frac{(s - b)(s - c)}{s(s - a)}$$

or

$$\frac{s(s - a) + (s - b)(s - c)}{s(s-a)} \cdot \text{bos}^2A = 4\,\frac{(s \ominus b)(s - c)}{s(s - a)}.$$

Dividing by the coefficient of bos^2A, we have

$$\text{bos}^2A = \frac{4(s - b)(s - c)}{s(s - a)} \cdot \frac{s(s - a)}{s(s - a) + (s - b)(s - c)}.$$

Using the definition $s = \frac{a + b + c}{2}$, we may simplify the denominator as

follows:

$$s(s - a) + (s - b)(s - c) = \frac{a + b + c}{2} \cdot \frac{b + c - a}{2} + \frac{a + c - b}{2} \cdot \frac{a + b - c}{2}$$

$$= \frac{b^2 + 2bc + c^2 - a^2 + a^2 + 2bc - b^2 - c^2}{4}$$

$$s(s - a) + (s - b)(s - c) = \frac{4\,bc}{4} = bc. \qquad (3)$$

Therefore

$$\text{bos}^2A = 4\frac{(s - b)(s - c)}{bc}$$

and

$$\boxed{\text{bos } A = \pm 2\sqrt{\frac{(s - b)(s - c)}{bc}}}$$

Section 3

The Sub of an Angle in Terms of the Sides

of the Triangle.

From equation (2) in section 2, we know

$$\text{bos } A = \text{sub } A \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

But bos $A = \sqrt{4 - \text{sub}^2 A}$ and therefore

$$\sqrt{4 - \text{sub}^2 A} = \text{sub } A \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}\,.$$

Squaring both sides of the equation, we have

$$4 - \text{sub}^2 A = \text{sub}^2 A \left[\frac{(s - b)(s - c)}{s(s - a)}\right]$$

or,

$$\text{sub}^2 A \left[\frac{(s - b)(s - c) + s(s - a)}{s(s - a)}\right] = 4$$

Then

$$\text{sub}^2 A = \frac{4s(s - a)}{(s - b)(s - c) + s(s - a)}\,.$$

From equation (3) in section 2, $(s - b)(s - c) + s(s - a) = bc$.

Therefore,

$$\text{sub}^2 A = 4\,\frac{s(s - a)}{bc}$$

and

$$\boxed{\text{sub } A = \pm 2 \sqrt{\frac{s(s - a)}{bc}}\,.}$$

Section 4

The Sub Law.

Using the equation bos $A = \pm 2\sqrt{\frac{(s - b)(s - c)}{bc}}$ obtained in section 2 and squaring both sides, we have

$$\frac{bc}{4}\,\text{bos}^2 A = (s - b)(s - c)$$

Expanding the terms, we obtain

$$\frac{bc}{4}\,\text{bos}^2 A = s^2 - (b + c)\,s + bc\,.$$

Transposing the terms to one side of the equation,

$$s^2 - (b + c)s + \frac{4bc}{4} - \frac{bc}{4}\,\text{bos}^2 A = 0$$

or

$$s^2 - (b+c)s + \frac{bc}{4}(4 - \text{bos}^2 A) = 0$$

But $4 - \text{bos}^2 A = \text{sub}^2 A$.

$$s^2 - (b+c)s + \frac{bc}{4}\,\text{sub}^2 A = 0$$

Solving for s by the quadratic formula, we obtain

$$s = \frac{(b+c) \pm \sqrt{b^2 + 2bc + c^2 - bc\,\text{sub}^2 A}}{2}$$

Substituting the value $\frac{a+b+c}{2}$ for s and simplifying, we have

$$a = \sqrt{(b+c)^2 - bc\,\text{sub}^2 A}$$

or

$$a^2 = (b+c)^2 - bc\,\text{sub}^2 A .$$

Hence the sub law: In a triangle the square of any side equals the square of the sum of the other two sides diminished by their product into the sub squared of the included angle.

If the triangle is isosceles, with A the vertex angle and $b = c$, then

$$a^2 = 4b^2 - b^2\,\text{sub}^2 A = b^2\,\text{bos}^2 A$$

or $\quad a = b\,\text{bos}\,A,$

which is, of course, the definition $\text{bos}\,A = \frac{a}{b}$.

And in a more special case of some interest, if $A = 180^\circ$ then the sub law reduces to $a^2 = (b+c)^2$, i.e., the triangle collapses to area zero.

Section 5

Application of the Laws of

Isosceles Trigonometry

The theorems and laws derived in this chapter together with the equation $A+B+C=180^\circ$ are sufficient for solving any triangle. The three parts that determine the triangle may be:

Case I: One side and two angles;

Case II; Two sides and the angle opposite one of those sides;

Case III; Two sides and the included angle;

Case IV; Three sides

Case I

One side a and two angles A and B, (Figure 18)

$$C = 180^\circ - (A+B).$$

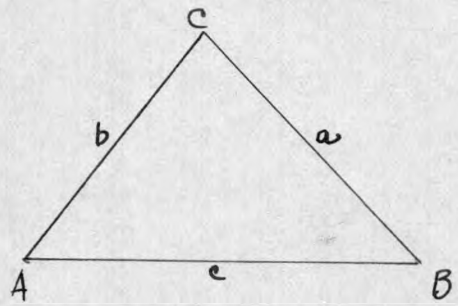


Figure 18.

Apply the bos law to find b and c, thus;

$$\frac{a}{\text{bos } 2A} = \frac{b}{\text{bos } 2B}, \quad b = \frac{a \text{ bos } 2B}{\text{bos } 2A}$$

$$\frac{a}{\text{bos } 2A} = \frac{c}{\text{bos } 2C}, \quad c = \frac{a \text{ bos } 2C}{\text{bos } 2A}.$$

Case II

Two sides a and b and the angle A opposite side.

This is also solved by the bos law.

$$\frac{a}{\text{bos } 2A} = \frac{b}{\text{bos } 2B}, \quad \text{bos } 2B = \frac{b \text{ bos } 2A}{a}$$

and

$$B = \frac{\text{arc bos}\left(\frac{b \text{ bos } 2A}{a}\right)}{2}.$$

$$C = 180^\circ - (A+B).$$

Side c is then found by the bos law as in Case I.

When an angle in a triangle is determined by its bos it admits of two values which are supplements of each other. From the graph, it is easily seen that in the interval between 0° and 360°, there are two values of θ corresponding to each value of bos θ. Either value of θ may be taken unless excluded by the conditions of the problem.

This problem must be considered at some length, for it may have one solution, two solutions, or no solution.

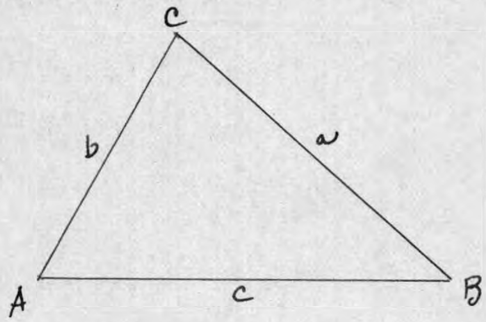


Figure 19.

If $a > b$, (Figure 19) by geometry $A > B$ and B must be acute whatever the value of A may be. Therefore, there is only one triangle which will satisfy these conditions.

If $a = b$, $A = B$, and the triangle is isosceles, and so there is only one solution for this condition.

If $a < b$, by geometry $A < B$ and A must be an acute angle to make this triangle possible.

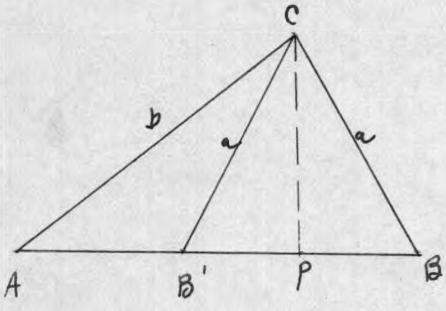


Figure 20.

In figure 20, if a equals the perpendicular PC, APC is a right triangle and there is only one solution. Since PC is the shortest line from C to AB, any line shorter than PC will not form a triangle with the given conditions. So, therefore, there would be no solution.

However, if $b > c > CP$, the two triangles ABC and AB'C are formed as can easily be proved by means of plane geometry. Angle CB'A is a supplement to angle CBA. From this we see that there are two possible solutions. By applying the bos law one solution is obtained directly; the other is found by subtracting the known angle from 180°.

Case III

Two sides a and b and the included angle C. (Figure 21)

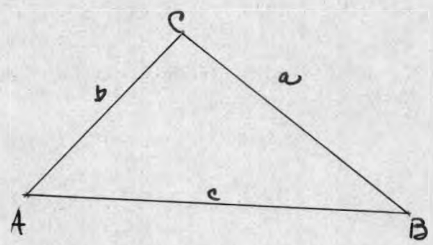


Figure 21.

Side c is found by using the sub law;

i.e. $c = \pm\sqrt{(a+b)^2 - ab \text{ sub}^2 C}$

By using the bos law, we obtain the value for A.

$$\text{bos } 2A = \frac{a \text{ bos } 2C}{c}$$

Therefore

$$A = \frac{\text{arc bos}\left(\frac{a \text{ bos } 2C}{c}\right)}{2}$$

Then $B = 180^\circ - (A + C)$.

Case IV.

Three sides a, b, and c.

The angles are found by using either the bos or the sub of an angle in terms of the sides of the triangle. i.e.

$$\text{bos } A = \pm 2\sqrt{\frac{(S - b)(s - c)}{bc}}, \qquad A = \text{arc bos}\left(\pm 2\sqrt{\frac{(s - b)(s - c)}{bc}}\right)$$

or

$$\text{sub } A = \pm 2\sqrt{\frac{(s(s - a)}{bc}}; \qquad A = \text{arc sub}\left(\pm 2\sqrt{\frac{s(s - a)}{bc}}.\right)$$

Section 6

Area of a Triangle

Case I

Two sides and the included angle.

The area of triangle ABC (Figure 22)

$= \frac{1}{2}$ ch.

Using the bos law, we obtain a val-

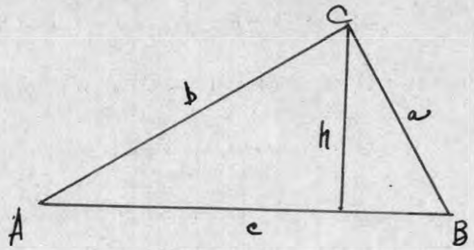


Figure 22.

ue for h in terms of the known quantities, thus:

$$\frac{h}{\text{bos } 2A} = \frac{b}{\text{bos } 180} \text{ or } h, = \frac{b \text{ bos } 2A}{2}$$

Therefore

$$\boxed{\text{Area of } \triangle \text{ ABC} = \tfrac{1}{4} \text{ bc bos } 2A} \ .$$

Case II

One side and two adjacent angles

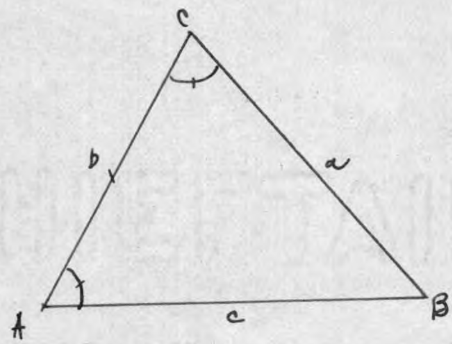


Figure 23.

In Figure 23, the given quantities are A, C, and b

$$B = 180 - (A + C) \ .$$

Find c by means of the bos law, as follows:

$$\frac{b}{\text{bos } 2B} = \frac{c}{\text{bos } 2C}, \quad c = \frac{b \text{ bos } 2C}{\text{bos } 2B}$$

But bos 2B = sub 2(A + C) and therefore

$$c = \frac{b \text{ bos } 2C}{\text{bos } 2(A + C)} \ .$$

Substituting this value for c in the area formula obtained under Case I, we have

$$\boxed{\text{Area} = \frac{B^2}{4} \cdot \frac{\text{bos } 2A \text{ bos } 2C}{\text{bos } 2(A + C)} = \frac{b^2}{4} \cdot \frac{\text{bos } 2A \text{ bos } 2C}{\text{bos } 2B}}$$

Case III

Three sides . Use the Standard

Formula from Plane Geometry

$$\boxed{A = \sqrt{s(s - a)(s - b)(s - c)}} \text{ where } s = \frac{a + b + c}{2}$$

Two other area formulas involving the radii of the circumscribed and inscribed circles respectively, are included as a matter of interest.

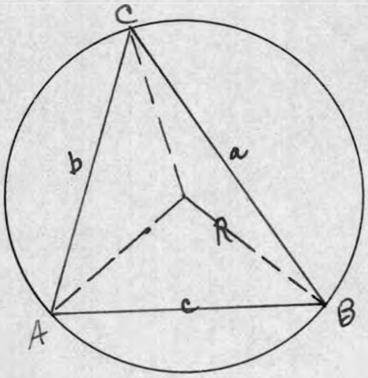


Figure 24.

We have the formula

$$\text{Area} = \tfrac{1}{2}\, bc \text{ bos } 2A \qquad (1)$$

But $\text{bos } 2A = \frac{a}{R}$.

Using this value in equation (1), have $\text{Area} = \frac{Abc}{4R}$.

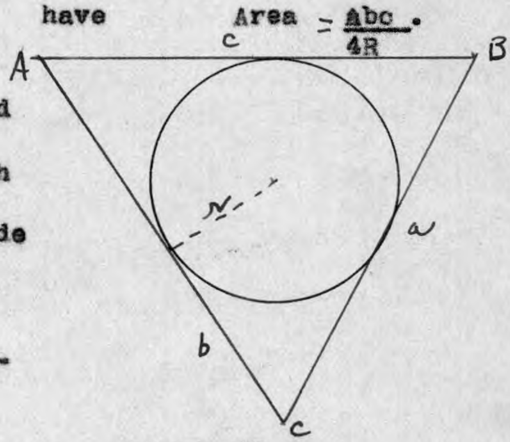


Figure 25.

If r is the radius of the inscribed circle, it is perpendicular to each of the sides, i.e., r is an altitude of triangles ACB, BCC, and ACC, leading at once to the standard relation

$$\text{Area} = \frac{r(a + b + c)}{2} = rs$$

CHAPTER VIII

OTHER APPLICATIONS OF ISOSCELES FUNCTIONS

Section 1

The Analog of the Fourier Series

Let us assume the possibility of an expansion for a function f(x) in the form

$$f(x) = a + \sum_{m=1}^{\infty}(a \text{ sub } nx + b_m \text{ bos } mx).$$

That is,

$$f(x) \equiv a_0 + a_1 \text{ sub } x + a_2 \text{ sub } 2x + \dots\dots + b_1 \text{ bos } x + b_2 \text{ bos } 2x + \dots$$

To integrate this expression, it is necessary to integrate it term by term:

$$\int_{-2\pi}^{2\pi} f(x)\, dx = a_0 \int_{-2\pi}^{2\pi} dx + a_1 \int_{-2\pi}^{2\pi} \text{sub } x\, dx + a_2 \int_{-2\pi}^{2\pi} \text{sub } 2x\, dx + \dots$$

$$+ b_1 \int_{-2\pi}^{2\pi} \text{bos } x\, dx + b_2 \int_{-2\pi}^{2\pi} \text{bos } 2x\, dx + \dots\dots$$

Now, $a_0 \int_{-2\pi}^{2\pi} dx = a_0 x \Big]_{-2\pi}^{2\pi} = 4\pi a_0$

Again, a_1 sub x dx 2a_1 bos x 0

Further, as may easily be shown by considering the graph of y = sub mx, (for m = 1, 2, 3, ...)

$$a_m \int_{-2\pi}^{2\pi} \text{sub } mx\, dx \equiv 0$$

In like manner

$$b_1 \int_{-2\pi}^{2\pi} \text{bos } x\, dx = -2b_1 \text{ sub } x \Big]_{-2\pi}^{2\pi} = 0$$

from which it follows as in the preceding paragraph that

$$b_m \int_{-2\pi}^{2\pi} \text{bos } mx\, dx = 0$$

Therefore

$$\int_{-2\pi}^{2\pi} f(x)\, dx = 4\pi a_0,$$

whence,

$$\boxed{a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(x)\, dx .}$$

This is an expression for the constant term in the expansion when ever $f(x)$ is integrable over the range indicated. We proceed to the determination of corresponding expressions for A_m and B_m $(M = 1, 2, 3, ..)$

To obtain an expression for A_m, we begin by multiplying equation (1) by sub mx, we have then

$$\int_{-2\pi}^{2\pi} f(x) \text{ sub } mx\, dx = a_0 \int_{-2\pi}^{2\pi} \text{sub } mx\, dx + a_1 \int_{-2\pi}^{2\pi} \text{sub } x \text{ sub } mx\, dx + a_2 \int_{-2\pi}^{2\pi} \text{sub } 2x \text{ sub } mx\, dx + \ldots,$$

$$+ a_m \int_{-2\pi}^{2\pi} \text{sub}^2\, mx\, dx + \ldots\ldots + b_1 \int_{-2\pi}^{2\pi} \text{bos } x \text{ sub } mx\, dx \quad (2)$$

$$+ b_2 \int_{-2\pi}^{2\pi} \text{bos } 2x \text{ sub } mx\, dx + \ldots\ldots\ldots$$

Integrate equation (2) term by term. Now,

$$a_0 \int_{-2\pi}^{2\pi} \text{sub } mx\, dx = 0$$

and $\quad a_1 \int_{-2\pi}^{2\pi} \text{sub } nx \text{ sub } mx\, dx = 0$, for $n \neq m$,

since sub $(A + B)$ + sub $(A - B)$ = sub A sub B. Let $A = nx$, and $B = mx$.

$$a_1 \int_{-2\pi}^{2\pi} \text{sub } nx \text{ sub } mx\, dx = \int_{2\pi}^{2\pi} \left[\text{sub } (n+m)x + \text{sub } (n-m)x\right] dx$$

$$= \frac{2 \text{ bos}}{n+m}(n+m)x + \frac{2}{n-m} \text{ bos } (n-m)x \Big]_{-2\pi}^{2\pi} = 0$$

To find $a_m \int_{-2\pi}^{2\pi} \text{sub}^2\, mx\, dx$, let us use the half angle formula of Chapter 2, section 6, $\text{sub}^2 \frac{A}{2} = \text{sub } A + 2)$.

In the above formula, substitute 2B for A, giving

$$\text{sub}^2 B = \text{sub } 2B + 2 .$$

$$sub^2 mx = sub\ 2mx + 2.$$

Substituting, we have

$$sub^2 mx\ dx = \int (sub\ 2mx + 2) dx$$
$$= \int sub\ 2mx\ dx + 2\int dx$$
$$= \frac{1}{2m} \cdot 2\ bos\ 2mx + 2x$$
$$= \frac{1}{m} \cdot bos\ mx\ sub\ mx + 2x$$

and therefore

$$a_m \int_{-2\pi}^{2\pi} sub^2 mx\ dx = a_m \left[\frac{1}{m}\ bos\ mx\ sub\ m + 2x\right]_{-2\pi}^{2\pi} = 8\pi a_m$$

Then

$$b_n \int_{-2\pi}^{2\pi} bos\ nx\ sub\ mx = 0$$

Since bos (A + B) + bos (A - B) = bos A sub B, let A = nx and B = mx.

$$b_m \int_{-2\pi}^{2\pi} bos\ nx\ sub\ mx\ dx = b_n \int_{-2\pi}^{2\pi} \left[bos\ (n+m)x + bos\ (n-m)x\right] dx$$
$$= -2b_n \left[sub\ (n+m)x + sub(n-m)x\right]_{-2\pi}^{2\pi} = 0$$

Therefore

$$\int_{-2\pi}^{2\pi} f(x)\ sub\ mx\ dx = 8\pi a_m$$

and finally

$$\boxed{a_m = \frac{1}{8\pi} \int_{-2\pi}^{2\pi} f(x)\ sub\ mx\ dx\ .}$$

To obtain a form for the determination of the coefficients b_m, begin by multiplying equation (1) by bos mx. We obtain

$$\int_{-2\pi}^{2\pi} f(x)\ bos\ mx\ dx = a_0 \int_{-2\pi}^{2\pi} bos\ mx\ dx + a_1 \int_{-2\pi}^{2\pi} sub\ x\ bos\ mx\ dx + a_2 \int_{-2\pi}^{2\pi} sub\ 2x\ bos\ mx\ dx$$
$$+ \cdots\cdots\cdots + b_1 \int_{-2\pi}^{2\pi} bos\ x\ bos\ mx\ dx + b_2 \int_{-2\pi}^{2\pi} bos\ 2x\ bos\ mx\ dx$$
$$+ \cdots\cdots\cdots + b_m \int_{-2\pi}^{2\pi} bos^2 mx\ dx. \qquad (4)$$

In integrating equation (4), we have to note that

$$a_0 \int_{-2\pi}^{2\pi} \text{bos } mx\, dx = 0,$$

and that $a_1 \int_{-2\pi}^{2\pi} \text{sub } nx \text{ } mx\, dx = 0$, For since bos (A + B) + bos (A - B)

= bos A sub B,

we may let A = mx and B = nx, Then

$$a_n \int_{-2\pi}^{2\pi} \text{sub } nx \text{ bos } mx\, dx = a_n \int_{-2\pi}^{2\pi} \left[\text{bos}(m+n)x + \text{bos } (m-n)x\right] dx$$

$$= a_n \left[\frac{-2}{m+n} \text{ sub } (m+n)x - \frac{2}{m-n} \text{ sub } (m-n)x\right]_{-2\pi}^{2\pi} = 0.$$

To evaluate $b_n \int_{-2\pi}^{2\pi}$ bos nx bos mx dx, let us return to the definitions relating the isosceles functions, namely bos nx = sub (180 - nx) and bos mx = sub(180 - mx).

Therefore

$$b_n \int_{-2\pi}^{2\pi} \text{bos } nx \text{ bos } mx\, dx = b_n \int_{-2\pi}^{2\pi} \text{sub } (180 - nx) \text{ sub } (180 - mx)\, dx.$$

Since sub (A + B) + sub (A - B) = sub A sub B, let A = 180 - nx and

B = 180 - mx

Then

$$b_n \int_{-2\pi}^{2\pi} \text{bos } nx \text{ bos } mx\, dx = b_n \int_{-2\pi}^{2\pi} \left[\text{sub } (360 - nx - mx) + \text{sub } (mx - nx)\right] dx$$

$$= b_n \left[2\text{bos } (360 - nx - mx) + 2 \text{ bos } (mx - nx)\right]_{-2\pi}^{2\pi} = 0.$$

Therefore

$$b_n \int_{-2\pi}^{2\pi} \text{bos } nx \text{ bos } mx\, dx = 0$$

To find $b_m \int_{-2\pi}^{2\pi} \text{bos}^2 mx\, dx$, use the half angle formula of Chapter 2, Section 6, $\text{bos}^2 \frac{A}{2} = 2 - \text{sub } A$. In this formula let A = 2B.

Then

$$\text{bos}^2 B = 2 - \text{sub } 2B, \qquad \text{or} \qquad \text{bos}^2 mx = 2 - \text{sub } 2\, mx.$$

Evaluating, we have

$$b_m \int_{-2\pi}^{2\pi} \text{bos}^2 mx \, dx = b_m \int_{-2\pi}^{2\pi} (2 - \text{sub } 2mx) \, dx$$

$$= b_m \left[2 \int_{-2\pi}^{2\pi} dx - \frac{1}{2m} + \int_{-2\pi}^{2\pi} \text{sub } 2mx \; 2m dx \right]$$

$$= b_m \left[2x - \frac{1}{2m} \cdot 2 \text{ bos } 2mx \right]_{-2\pi}^{2\pi}$$

$$= b_m \left[2x - \frac{1}{m} \text{ bos } mx \text{ sub } mx \right]_{-2\pi}^{2\pi} = 8\pi b_m.$$

Finally, therefore,

$$\int_{-2\pi}^{2\pi} f(x) \text{ bos } mx \, dx = 8\pi b_m$$

or

$$\boxed{b_m = \frac{1}{8\pi} \int_{-2\pi}^{2\pi} f(x) \text{ bos } mx \, dx.}$$

Using the expressions just obtained, i.e.,

$$a_0 = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} f(x) \, dx, \qquad a_m = \frac{1}{8\pi} \int_{-2\pi}^{2\pi} f(x) \text{ sub } mx \, dx,$$

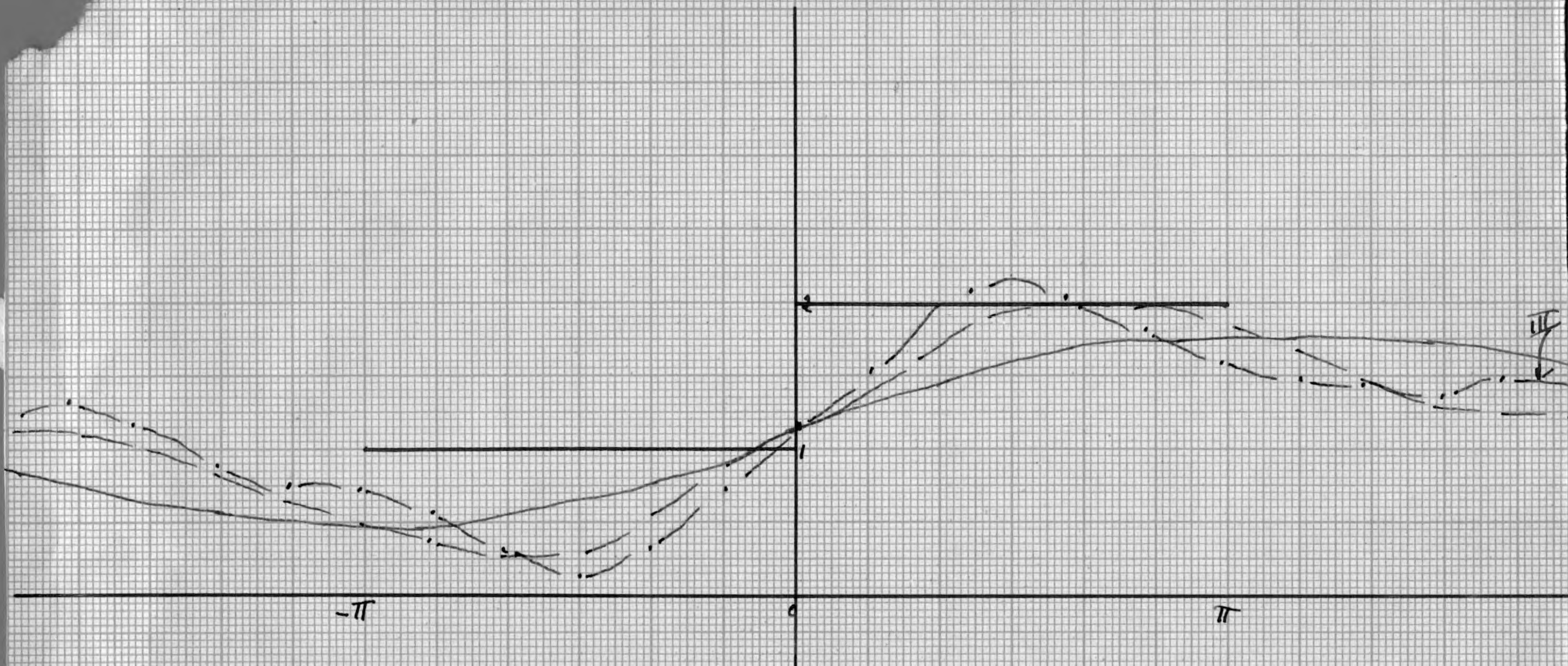
and $b_m = \frac{1}{8\pi} \int_{-2\pi}^{2\pi} f(x) \text{ bos } mx \, dx$ for the determination of the coefficients, a series, which is the analog of a Fourier series, may be obtained in the form desired.

Section 2

An Example of the Analog of a Fourier Series

Let $f(x)$ have the value 1 when $-2\pi < x < 0$ and the value 2 when $0 < x < 2\pi$. Expressing this function as a Fourier series, we have

$$a_0 = \frac{1}{4\pi} \int_{-2\pi}^{0} dx + \frac{1}{4\pi} \int_{0}^{2\pi} 2 \, dx = \frac{1}{4\pi} \Big[x \Big]_{-2\pi}^{0} + \frac{1}{4\pi} \Big[2x \Big]_{0}^{2\pi}$$



This figure gives a first (I) approximation, a second (II) approximation and a third (III) approximation for the Fourier series obtained in Section 2.

$$= \frac{1}{2\pi} + 1 = \frac{1 + 2\pi}{2\pi},$$

$$a_m = \frac{1}{8\pi}\left[\int_{-2\pi}^{0} \text{sub } mx\, dx + 2\int_{0}^{2\pi} \text{sub } mx\, dx\right]$$

$$= \frac{1}{8\pi}\left\{\left[\frac{2}{m} \text{ bos } mx\right]_{-2\pi}^{0} + \left[\frac{4}{m} \text{ bos } mx\right]_{0}^{2\pi}\right\}$$

$$= \frac{1}{8\pi}\quad (0 + 0) = 0,$$

and

$$b_m = \frac{1}{8\pi}\left[\int_{-2\pi}^{0} \text{bos } mx\, dx + \int_{0}^{2\pi} 2 \text{ bos } mx\, dx\right] = \frac{1}{8\pi}\left\{\left[\frac{-2}{m}\text{sub } mx\right]_{-2\pi}^{0}\right.$$

$$\left. + 2\left[\frac{-2}{m} \text{ sub } mx\right]_{0}^{2\pi}\right\}$$

$$= \frac{1}{8\pi}\left[\frac{-2}{m} \text{ sub } mx\right]_{0}^{2\pi} = \frac{1}{8\pi} \cdot \frac{-2}{m} \cdot -4 = \frac{1}{\pi m} \cdot$$

Hence the series

$$f(x) = \frac{1 + 2\pi}{2\pi} + \frac{1}{\pi} \text{ bos } x + \frac{1}{2\pi} \text{bos } 2x + \frac{1}{3\pi} \text{ bos } 3x + \ldots\ldots\ldots$$

Section 3

Isosceles Functions in Differential Equations

In the study of differential equations, one important appearance of the isosceles functions would be in the right members of linear differential equations with constant coefficients, i.e. in differential equations of the form

$$\frac{d^{(n)}y}{dx^{(n)}} + P_1 \frac{d^{(n-1)}y}{dx^{(n-1)}} + \ldots\ldots\ldots P_n\, y = X$$

where P_i $(i = 1, 2, \ldots, n)$ is a constant and X is a term of form bos ax or sub x. Use of the differential operator D[1] gives, as the particular integral of the equation $\left[\phi(D)\right] y = X$, the symbolic form $y = \frac{1}{\phi(d)} X$.

For the special case here considered, the following treatment suffices:

Successive differentiation of bos ax gives

$$D \text{ bos } ax = \frac{a}{2} \text{ sub } ax$$

$$D^2 \text{ bos } ax = \frac{-a^2}{4} \text{ bos } ax$$

$$D^3 \text{ bos } ax = \frac{-a}{8} \text{ sub } ax$$

$$D^4 \text{ bos } ax = \frac{a}{16} \text{ bos } ax$$

$$\vdots$$

and, in general

$$(D^2)^n \text{ bos } ax = \left(\frac{-a^2}{4}\right)^n \text{ bos } ax$$

Therefore, if $\phi(D^2)$ is a rational integral of D^2,

$$\phi(D^2) \text{ bos } ax = \phi\left(\frac{-a^2}{4}\right) \text{ bos } ax$$

Dividing this equation by $\phi(D^2) \cdot \phi\left(\frac{-a^2}{4}\right)$, we have

$$\frac{1}{\phi\left(\frac{-a}{4}\right)} \text{ bos } ax = \frac{1}{\phi(D^2)} \text{ bos } ax .$$

More generally

$$\frac{1}{\phi(D^2)} \text{ bos } (ax + \alpha) = \frac{1}{\phi\left(\frac{-a^2}{4}\right)} \text{ bos } (ax + \alpha) .$$

In like manner, by successive differentiation of sub ax, it can

---

(1) See Murray, R. H., or any other standard work on differential equations.

easily be shown that

$$\frac{1}{\phi\,(D^2)} \text{ sub } ax = \frac{1}{\phi\left(\frac{-a^2}{4}\right)} \text{ sub } ax$$

and more generally, that

$$\frac{1}{\phi\,(D^2)} \text{ sub } (ax+\alpha) = \frac{1}{\phi\left(\frac{-a^2}{4}\right)} \text{ sub } (ax+\alpha) .$$

CHAPTER IX

A COMPARISON OF ISOSCELES AND RIGHT TRIANGLE TRIGONOMETRY.

The structure of isosceles trigonometry parallels that of right triangle trigonometry. For the analytical expressions appearing in ordinary trigonometry, with very few exceptions, corresponding relations were obtained in this thesis on isosceles trigonometry.

In ordinary or right triangle trigonometry, six functions (sine, cosine, tangent, cotangent, secant and cosecant) are defined. Of these six functions two (sine and cosine) and possibly a third (tangent) are the only ones which are of much practical importance for computational purposes. Due to the nature of the isosceles triangle, it is necessary to define only four functions, bos, sub, and their reciprocals; and of these functions, the bos and sub are the only two which are of practical importance. While in ordinary trigonometry it is convenient to study the complement of an angle, in isosceles trigonometry it is natural to work with the supplement of an angle; hence the sub function.

A relationship between the isosceles functions and the ordinary functions may be determined from Figure 26. If a perpendicular is dropped from A to BC in the isosceles triangle ABC, it bisects angle $\theta \equiv A$ and it also bisects BC. Therefore $\sin \frac{\theta}{2} = \frac{b}{2s}$. $\text{Bos}\,\theta = \frac{b}{s}$ and dividing by 2, we have $\frac{1}{2}\,\text{bos}\,\theta = \sin \frac{\theta}{2}$.

Similarly, it can be shown that

$$\frac{1}{2}\,\text{sub}\,\theta = \cos \frac{\theta}{2}.$$

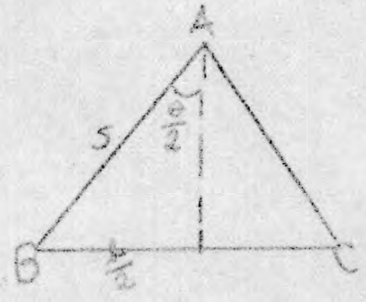


Figure 26.

In regard to signs, ordinary trigonometry found it necessary to adopt the convention of calling sineθ positive whenever, in rectangular coordinates the ordinate was positive, or above the x-axis, i. e. for $0 > \theta > 180^\circ$, and negative when the ordinate was negative or below the x-amis. Since the $\cos\theta = \sin(90 - \theta)$, the values of $\cos\theta$ are displaced by 90° from the values of $\sin\theta$ and therefore $\cos\theta$ is positive to the right of the y-axis and negative to the left of the y-axis. Thus in ordinary trigonometry we find a complete set of values in the interval between 0° and 360° . However, in isosceles trigonometry, the Riemann or two-sheeted surface was necessary, and the cycle is complete in the interval between 0° and 720°.

Due to the number of functions in ordinary trigonometry, there are a number of fundamental relations. Let us compare some of these relations.

Ordinary trigonometry	Isosceles trigonometry
$\sin^2\theta + \cos^2 = 1$	$\text{bos}^2\theta + \text{sub}^2\theta = 4$
$\cos\theta = \sin(90 - \theta)$	$\text{sub}\,\theta = \text{bos}\,(180 - \theta)$
$\sin(-\theta) = -\sin\theta$	$\text{bos}\,(-\theta) = -\text{bos}\,\theta$
$\cos\,(-\theta) = \cos\theta$	$\text{sub}\,(-\theta) = \text{sub}\,\theta$.

In the goniometry of right triangle trigonometry and of isosceles trigonometry the similarity between the expressions is obvious by direct comparisons. In the following, an expression from ordinary trigonometry is given first in each case, followed by the corresponding relation from isosceles trigonometry. The comparative simplicity of the relations of isosceles trigonometry in a majority of the cases is to be noted.

$$\sin\,(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\text{bos}\,(x \pm y) = \tfrac{1}{2}\,(\text{bos}\,x\;\text{sub}\,y \pm \text{sub}\,x\;\text{bos}\,y)$$

$\cos(x \pm y) = \cos x \cos y \mp \sin s \sin y$

$\text{bos}(x \pm y) = \frac{1}{2}(\text{sub } x \text{ sub } y \mp \text{bos } x \text{ bos } y)$

$\sin 2x = 2 \sin x \cos x$

$\text{bos } 2x = \text{bos } x \text{ sub } x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\text{sub } 2x = \frac{1}{2}(\text{sub}^2 x - \text{bos}^2 x) = 2 - \text{bos}^2 x = \text{sub } 2x - 2.$

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\text{bos } 3x = \text{sub}^3 x - \text{bos}^3 x$

$\cos 3x = 4 \cos^3 x - 3 \cos x$

$\text{sub } 3x = \text{sub}^3 x - 3 \text{ sub } x$

$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$

$\text{bos } \frac{x}{2} = \pm\sqrt{2 - \text{sub } x}$

$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$

$\text{sub } \frac{x}{2} = \pm\sqrt{\text{sub } x + 2}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cdot \text{bos } \frac{x-y}{2}$

$\text{bos } x + \text{bos } y = \text{bos } \frac{x+y}{2} \cdot \text{sub } \frac{x-y}{2}$

$\sin x - \sin y = 2 \cos \frac{x+y}{2} \cdot \text{bos } \frac{x-y}{2}$

$\text{bos } x - \text{bos } y = \text{sub } \frac{x+y}{2} \cdot \text{bos } \frac{x-y}{2}$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cdot \cos \frac{x-y}{2}$

$\text{sub } x + \text{sub } y = \text{sub } \frac{x+y}{2} \cdot \text{sub } \frac{x-y}{2}$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \cdot \sin \frac{x-y}{2}$

$\text{sub } x - \text{sub } y = -\text{bos } \frac{x+y}{2} \cdot \text{bos } \frac{x-y}{2}$

In a comparison of the derivative formulas, we observe that

$\frac{d(\sin u)}{d\theta} = \cos u \cdot \frac{du}{d\theta}$, while $\frac{d(\text{bos } u)}{d\theta} = \frac{1}{2} \text{ sub } u \cdot \frac{du}{d\theta}$

and

$\frac{d(\cos u)}{d\theta} = -\sin u \cdot \frac{du}{d\theta}$, while $\frac{d(\text{sub } u)}{d\theta} = -\frac{1}{2} \text{ bos } u \cdot \frac{du}{d\theta}$

DeMoivre's theorem $\left[(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta\right]$ is identical in form with the isosceles analog of DeMoivre's theorem $\left[(\text{sub}\,\theta + i\,\text{bos}\,\theta)^n = 2^{n-1}(\text{sub}\,n\theta + i\,\text{bos}\,n\theta)\right]$ excepting for the factor (2^{n-1}).

The differences in the Maclaurin's series may be observed

$$\sin x = x - \frac{x^3}{|\underline{3}} + \frac{x^5}{|\underline{5}} - \frac{x^7}{|\underline{7}} + \cdots\cdots$$

$$\text{bos}\, x = x - \frac{x^3}{|\underline{3}} \cdot \frac{1}{2^2} + \frac{x^5}{|\underline{5}} \cdot \frac{1}{2^4} \; \frac{-x^7}{|\underline{7}} \cdot \frac{1}{2^6} \cdots\cdots$$

$$\cos x = 1 - \frac{x^2}{|\underline{2}} + \frac{x^4}{|\underline{4}} - \frac{x^6}{|\underline{6}} + \cdots\cdots\cdots\cdots$$

$$\text{sub}\, x = 2 - \frac{x^2}{|\underline{2}} \cdot \frac{1}{2} + \frac{x^4}{|\underline{4}} \cdot \frac{1}{2^3} - \frac{x^6}{|\underline{6}} \cdot \frac{1}{2^5} + \cdots\cdots\cdots$$

These series are convergent for all values of x.

Euler has defined the trigonometric functions in terms of the exponential functions as follows:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Bos and sub have also been expressed in terms of the exponential functions; thus,

$$\text{bos}\,\theta = \frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{i} \quad \text{and} \quad \text{sub}\,\theta = e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}.$$

These expressions could, of course, be used as analytical definitions.

In the application of right triangle trigonometry to the solution of triangles, the sine law, cosine law, theorem of tangents and the half angle relations are used. In isosceles trigonometry, the bos law $\left(\frac{a}{\text{bos}\,2A} = \frac{b}{\text{bos}\,2B} = \frac{c}{\text{bos}\,2C}\right)$ is as convenient for the solution of triangles as the sine law $\left(\frac{a}{\sin A} = \frac{b}{\text{bos}\,B} = \frac{c}{\sin C}\right)$. In computation the sub law $\left[a^2 = (b + c)^2 - bc \operatorname{sub}^2 A\right]$ is more suitable for computational purposes than the cosine law ($a^2 = b^2 + c^2 - 2bc \cos A$). Isosceles trigonometry lacks a theorem corresponding to the very convenient tangent law $\left[\frac{b+c}{b-c} = \frac{\tan \frac{B+C}{2}}{\tan \frac{B-C}{2}}\right]$; but triangles which are solved by the tangent law in right triangle trigonometry, are solved by using the bos and sub laws in isosceles trigonometry. The expression for the bos of an angle in terms of the sides of the triangle $\left[\operatorname{bos} A = \pm 2\sqrt{\frac{(s-b)(s-c)}{bc}}\right]$ seems to be more easily adapted to the solution of triangles than the form for the $\sin \frac{A}{2}$ $\left[\sin \frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}\right]$ since the angle is obtained directly from the bos expression instead of half the angle from the sine theorem. There are the corresponding expressions for $\cos \frac{A}{2}$ $\left[\text{i.e. } \cos \frac{A}{2} = \pm\sqrt{\frac{s(s-a)}{bc}}\right]$ and for sub A $\left[\operatorname{sub} A = \pm 2\sqrt{\frac{s(s-a)}{bc}}\right]$.

The characteristic area formulas are compared, thus:

Ordinary trigonometry	Isosceles trigonometry
Area $= \frac{1}{2} bc \sin A$	Area $= \frac{1}{4} bc \operatorname{bos} 2A$
" $= \frac{c^2 \sin A \sin B}{2 \sin C}$	" $= \frac{c^2 \operatorname{bos} 2A \operatorname{bos} 2B}{4 \operatorname{bos} 2C}$
" $= \sqrt{s(s-a)(s-b)(s-c)}$	" $= \sqrt{s(s-a)(s-b)(s-c)}$

For computational purposes there is very little difference between the use of isosceles trigonometry and right triangle trigonometry. We find that right triangle trigonometry is more convenient for the solution of the special case of the right triangle, while isosceles trigonometry has a neater solution for the special case of the isosceles triangle. For practical purposes, right triangle trigonometry is perhaps the better one to use, since right triangles constitute a larger group than isosceles triangles in applied problems, as regards the theoretical structure and uses in higher mathematical analysis, the ordinary trigonometry is in general not superior to isosceles trigonometry.

TABLE OF NATURAL FUNCTIONS OF ISOSCELES TRIGONOMETRY

(This table gives the numberical values for the function regardless of signs)

degrees	bos	sub	degrees
0	.00000	2.00000	360
1	.01746	1.99992	359
2	.03490	1.99970	358
3	.05236	1.99932	357
4	.06980	1.99878	356
5	.08724	1.99810	355
6	.10468	1.99726	354
7	.12210	1.99626	353
8	.13952	1.99412	352
9	.15692	1.99384	351
10	.17432	1.99238	350
11	.19170	1.99080	349
12	.20906	1.98904	348
13	.22640	1.98714	347
14	.24374	1.98510	346
15	.26106	1.98288	345
16	.27834	1.98054	344
17	.29562	1.97804	343
18	.31286	1.97438	342
19	.33010	1.97258	341
20	.34730	1.96962	340
21	.36448	1.96650	339

DEGREES	bos	sub	degrees
22	.38162	1.96326	338
23	.39874	1.95984	337
24	.41582	1.95630	336
25	.43888	1.95260	335
26	.44990	1.94874	334
27	.46690	1.94474	333
28	.48384	1.94060	332
29	.50076	1.93630	331
30	.51764	1.93186	330
31	.53448	1.92726	329
32	.55128	1.92252	328
33	.56804	1.91764	327
34	.58474	1.91260	326
35	.60142	1.90744	325
36	.61804	1.90212	324
37	.63460	1.89664	323
38	.65114	1.89104	322
39	.66762	1.88528	321
40	.68404	1.87938	320
41	.70042	1.87234	319
42	.71674	1.86716	318
43	.73300	1.86084	317
44	.74922	1.85436	316
45	.76536	1.84776	315
46	.78146	1.84100	314
47	.79750	1.83412	313

Degrees	bos	sub	degrees
48	.81348	1.82710	312
49	.82838	1.81992	311
50	.84524	1.81262	310
51	.86102	1.80518	309
52	.87674	1.79758	308
53	.89240	1.78986	307
54	.90798	1.78202	306
55	.92350	1.77402	305
56	.93894	1.76590	304
57	.95432	1.75764	303
58	.96962	1.74924	302
59	.98484	1.74072	301
60	1.00000	1.73206	300
61	1.01508	1.72326	299
62	1.03008	1.71434	298
63	1.04500	1.70376	297
64	1.05984	1.69610	296
65	1.07460	1.68678	295
66	108928	1.67734	294
67	1.10388	1.66778	293
68	1.11838	1.65808	292
69	1.13282	1.64826	291
70	1.14716	1.63830	290
71	1.16140	1.62824	289
72	1.17558	1.61804	288
73	1.18964	1.60772	287

degrees	bos	sub	degrees
74	1.20364	1.59728	286
75	1.21752	1.58670	285
76	1.23132	1.57602	284
77	1.24502	1.56522	283
78	1.25864	1.55430	282
79	1.27216	1.54324	281
80	1.28458	1.53208	280
81	1.29890	1.52082	279
82	1.31212	1.50942	278
83	1.32524	1.49792	277
84	1.33826	1.48628	276
85	1.35118	1.47456	275
86	1.36400	1.46270	274
87	1.37670	1.45074	273
88	1.38932	1.43868	272
89	1.40182	1.42650	271
90	1.41422	1.41422	270
91	1.42650	1.40182	269
92	1.43868	1.38932	268
93	1.45074	1.37670	267
94	1.46270	1.36400	266
95	1.47456	1.35118	265
6 96	1.48628	1.33826	264
97	1.49792	1.32524	263
98	1.50942	1.31212	262
99	1.52082	1.29890	261
100	1.53208	1.28458	260

DEGREES	bos	sub	degrees
101	1. 54324	1.27216	259
102	1.55430	1.25864	258
103	1.56522	1.24502	257
104	1.57602	1.23132	256
105	1.58670	1.21752	255
106	1.59788	1.20364	254
107	1.60772	1.18964	253
108	1.61804	1.17558	252
109	1.62824	1.16140	251
110	1.63830	1.14716	250
111	1.64826	1.13282	249
112	1.65808	1.11838	248
113	1.66778	1.10388	247
114	1.67734	1.08928	246
115	1.68678	1.07460	245
116	1.69610	1.05984	244
117	1.70376	1.04500	243
118	1.71434	1.03008	242
119	1.72326	1.01508	241
120	1.73206	1.00000	240
121	1.74072	.98484	239
122	1.74924	.96962	238
123	1.75764	.95432	237
124	1.76590	.93894	236
125	1.77462	.92350	235
126	1.78202	.90798	234
127	1.78986	.89240	233

Degree	bos	sub	Degrees
128	1.79758	.87674	232
129	1.80518	.86102	231
130	1.81262	.84524	230
131	1.81992	.82838	229
132	1.82710	.81348	228
133	1.83412	.79750	227
134	1.84100	.78146	226
135	1.84776	.76536	225
136	1.85436	.74922	224
137	1.86084	.73300	223
138	1.86716	.71674	222
139	1.87234	.70042	221
140	1.87938	.68404	220
141	1.88528	.66762	219
142	1.89104	.65114	218
143	1.89664	.63460	217
144	1.90212	.61804	216
145	1.90744	.60142	215
146	1.91260	.58474	214
147	1.91764	.56804	213
148	1.92252	.55128	212
149	1.92726	.53448	211
150	1.93186	.51764	210
151	1.93639	.50076	209
152	1.94060	.48384	208
153	1.94474	.46690	207
154	1.94874	.44990	206

DEGREES	bos	sub	degrees
155	1.95260	.43238	205
156	1.95630	.41582	204
157	1.95984	.39874	203
158	1.96326	.38162	202
159	1.96650	.36448	201
160	1.96962	.34730	200
161	1.97258	.33010	199
162	1.97438	.31286	198
163	1.97804	.29562	197
164	1.98054	.27834	196
165	1.98288	.26106	195
166	1.98510	.24374	194
167	1.98714	.22640	193
168	1.98904	.20906	192
169	1.99080	.19170	191
170	1.99238	.17432	190
171	1.99384	.15692	189
172	1.99412	.13952	188
173	1.99626	.12210	187
174	1.99726	.10468	186
175	1.99810	.08724	185
176	1.99878	.06980	184
177	1.99932	.05236	183
178	1.99970	.03490	182
179	1.99992	.01746	181
180	2.00000	.00000	180

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