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Analysis of An Orthogonal Grid Composed of Straight and Circular Members

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ANALYSIS OF AN ORTHOGONAL GRID COMPOSED OF STRAIGHT AND CIRCULAR MEMBERS

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ABSTRACT

This thesis has been written for the purpose of using the stiffness method to analyze an orthogonal grid composed of straight and circular members that is loaded perpendicular to the plane of grid. In the development of theory, each member is assumed to carry bending, torsion and shearing force, and the axial force is neglected.

It is much more complicated to derive the stiffness matrix for a circular member than that for a straight member. The stiffness matrix for a circular member is derived by using the strain energy and Castigliano's theorem while the stiffness matrix for a straight member is found by using the slope-deflection method. Having obtained the individual member stiffness matrix, it is then shown how the equilibrium equations of a joint are expressed in matrix form. Direct superposition of stiffness coefficients associated with each individual loading state can therefore be used in forming the equilibrium equations for the whole structure.

Based on the above, a general computer program is written for evaluating the joint displacements and end actions of each member. Then a numerical example illustrates the application of the method and permits an evaluation of its accuracy and efficiency. By checking the computed results of joint displacements and member end actions, it is indicated that the method is workable and surprisingly accurate.

NOTATIONS

E	Modulus of elasticity
G	Shearing modulus
I	Moment of inertia of a cross-sectional area with respect to the neutral axis
J	Torsion constant of a cross-sectional area
k	EI/GJ
M_{xAB}	Bending or twisting moment about x-axis at end A of member AB (Bending for straight member, twisting for circular member)
$M_{x\theta}$	Twisting moment at a general section of circular member
M_{yAB}	Bending or twisting moment about y-axis at end A of member AB (Twisting for straight member, bending for circular member)
$M_{y\theta}$	Bending moment at a general section of circular member
$M^F{}_{xAB}$	Fixed-end moment about x-axis at end A of member AB
$M^F{}_{yAB}$	Fixed-end moment about y-axis at end A of member AB
P	Concentrated load
R	Radius of curvature of a circular member
V_{AB}	Shearing force along z-axis at end A of member AB
$V^F{}_{AB}$	Fixed-end shearing force along z-axis at end A of member AB
V_θ	Shearing force at a general section of circular member
v	Poisson's ratio
w	Distributed load per unit length

δ_A	Deflection along z-axis at joint A
ϕ_{xA}	End rotation about x-axis at joint A
ϕ_{yA}	End rotation about y-axis at joint A
θ	An arbitrary angle subtending the arc of a circular member
β	Entire subtending angle of a circular member
γ	Constant for formula, $J=\gamma b^3h$, of rectangular section
$\{A\}$	Column matrix of joint actions
$\{D\}$	Column matrix of joint displacements
$\{F\}$	Column matrix of fixed-end actions
$[S]$	Matrix of stiffness influence coefficients
$[T]$	Transformation matrix
$[S]^T$	Transpose matrix
$[\ ]$	Square matrix
$\{\ \}$	Column matrix
x-axis	In-plane axis which is always tangent at a joint on the circular member
y-axis	In-plane radial axis
z-axis	The axis which is always perpendicular to the plane of grid

SIGN CONVENTIONS

The sign conventions for positive coordinate axes, actions and displacements are shown in Figure 1. Note that the double-headed arrow notation is used for the vector representation of moments and rotational displacements, with right-hand rule used to indicate their senses.

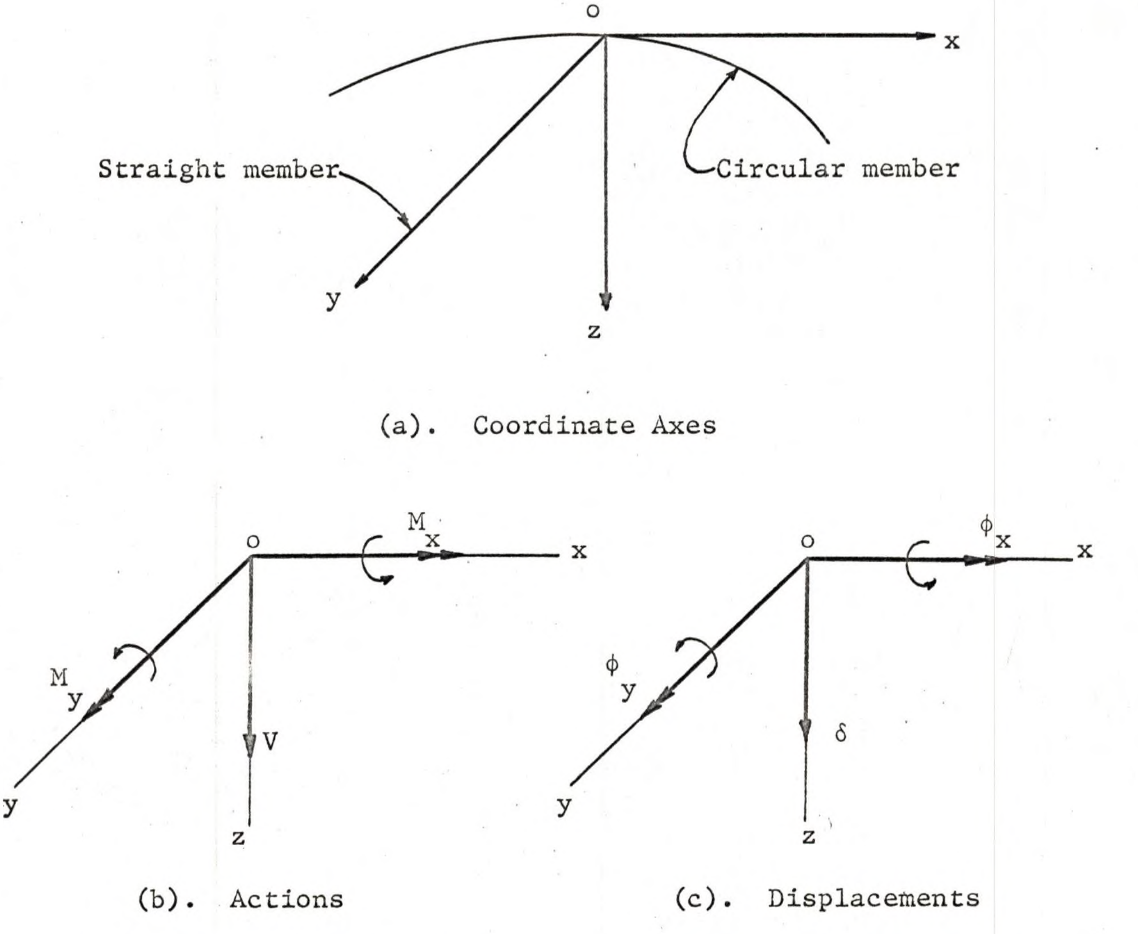

Fig. 1.--Positive Coordinate Axes, Actions and Displacements

INTRODUCTION

Floor systems of curved bridges, ramps, and buildings often consist of mutually perpendicular straight and circular members in a plane. The plane grid system may be bounded by points of intersection with other orthogonal members at both ends, or by an exterior support at one end and a point of intersection at the other.

There are generally two external force systems which apply to a plane frame. One acts in the plane of the frame while the other acts in a plane perpendicular to the plane. These two force systems can be analyzed separately. Then superposition of these two analyses will produce the total solution of the problem. This thesis will analyze a plane grid loaded perpendicular to its own plane only.

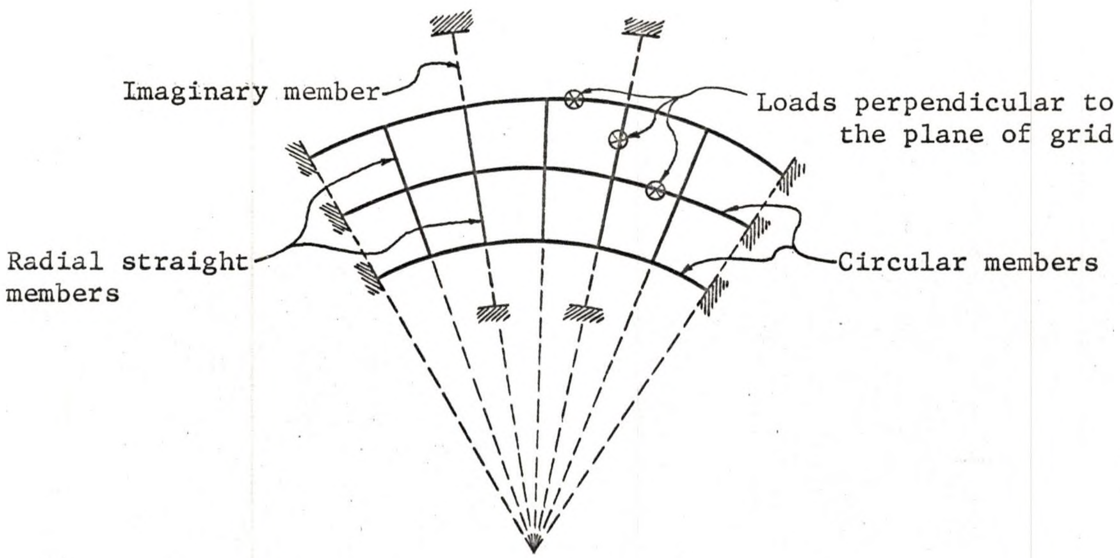

Fig. 2.--Typical Structure.

In the most general case, a joint, the intersection of two or more spatial members, should have six degrees of freedom. For the special case, when all the external loads act in a direction which is perpendicular to the plane of the grid, the in-plane axial deformations of the grid members are small in comparison with their deformations in the transverse direction. Therefore, we may assume the magnitude of the axial forces to be negligible, and that each intersection will be limited to three degrees of freedom. These are the two rotations about the orthogonal axes in the plane and the one deflection perpendicular to the plane (Wang, 1966).

Figure 3 shows the coordinate system for any joint A, with the x and y axes lying in the plane of the grid and the z-axis perpendicular to this plane. Neglecting the secondary actions V_x, V_y, and M_z, we consider the primary actions M_x, M_y, and V_z for this analysis.

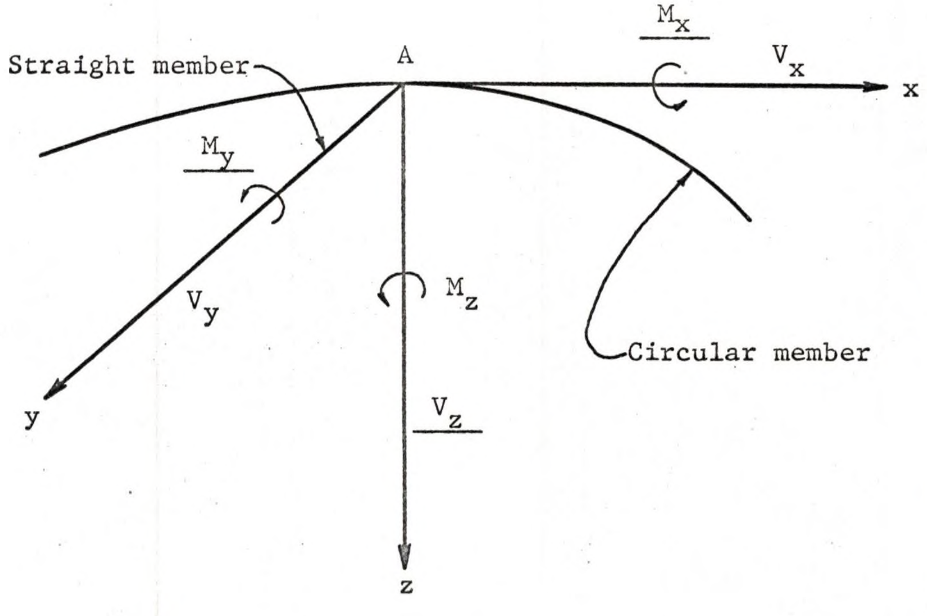

Fig. 3.--Coordinate System of Joint A.

In analyzing this grid, we can derive the stiffness matrix for each circular member by using the strain energy and Castigliano's theorem, and find the stiffness matrix for each straight member by using the slope-deflection method. Then we can establish the equations of equilibrium for each joint. Summation of all moments acting about each in-plane axis must be zero and the summation of forces acting at the joint and perpendicular to the plane must be zero. The Number "N" of equations of equilibrium is

$$N = 3j-s$$

where "j" is the number of intersections and "s" is the number of supported joints. A general computer program will be presented for analyzing the above mentioned structure and a numerical example will be provided (Martin, 1966).

The analysis of this thesis will be based upon the following assumptions and limitations:

1. All materials follow Hooke's law.
2. All members have constant sections and are rigidly connected at the joints.
3. The cross-sectional dimensions of members will be taken as small compared with the radii of curvature of circular members.
4. The displacements of the structure are small in comparison with its overall dimensions.
5. Effects of shearing deformations are very small and hence are neglected in this analysis.

GENERALIZED SLOPE-DEFLECTION EQUATIONS OF A CIRCULAR MEMBER

A member stiffness is defined as actions required at the end of a restrained member to produce a unit displacement of the end of the member. The stiffness of a member expressed in matrix form is called member stiffness matrix. It is convenient to find the member stiffness matrix by inverting the member flexibility matrix. The problem will be solved in three steps (Carpenter, 1960 and Martin, 1966).

Step (1): Development of Slope-deflection Equations

The relation of actions and displacements is shown in Figure 4, in which the arc between joint A and B is a portion of a plane circular curve and point C is any position with an angular distance of θ from joint B. Now we write the following equations for shear and moments.

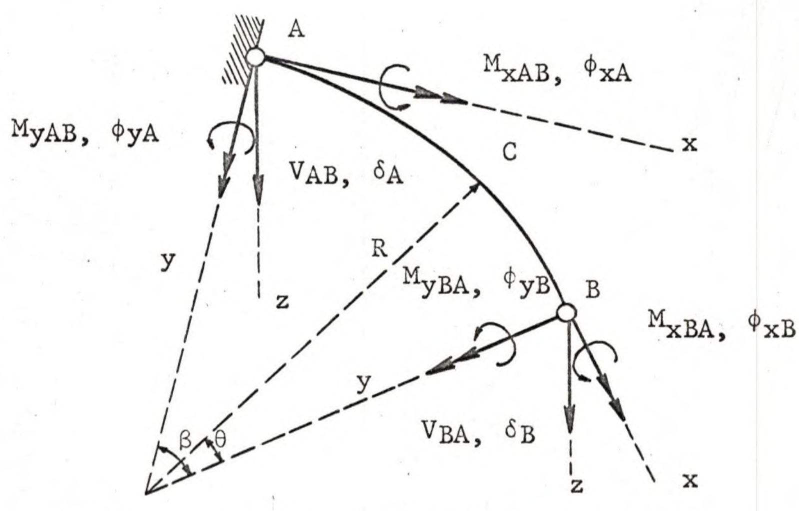

Fig. 4.--Actions and Displacements Applied to a Circular Member (End A is fixed).

The actions at a general section C in terms of the joint actions at B are

$$M_{x\theta} = M_{xBA}\cos\theta - M_{yBA}\sin\theta + V_{BA}R(1-\cos\theta) \qquad (1)$$

$$M_{y\theta} = M_{xBA}\sin\theta + M_{yBA}\cos\theta - V_{BA}R\sin\theta \qquad (2)$$

$$V_{\theta} = V_{BA} \qquad (3)$$

Expressed in matrix form

$$\begin{Bmatrix} M_{x\theta} \\ M_{y\theta} \\ V_{\theta} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & R(1-\cos\theta) \\ \sin\theta & \cos\theta & -R\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} \qquad (4)$$

Neglecting the effect of shearing deformations, the total strain energy of member AB is

$$U = \int_0^{\beta} \frac{(M_{y\theta})^2}{2EI} ds + \int_0^{\beta} \frac{(M_{x\theta})^2}{2GJ} ds \qquad (5)$$

and introducing $Rd\theta = ds$ into Eq. (5)

$$U = \int_0^{\beta} \frac{(M_{y\theta})^2}{2EI} Rd\theta + \int_0^{\beta} \frac{(M_{x\theta})^2}{2GJ} Rd\theta$$

Applying the Castigliano's theorem and with $\frac{EI}{GJ} = k$ gives the displacements at joint B as

$$\phi_{xB} = \frac{\partial U}{\partial M_{xBA}}$$

$$= \frac{1}{EI}\int_0^{\beta} M_{y\theta}\left(\frac{\partial M_{y\theta}}{\partial M_{xBA}}\right)Rd\theta + \frac{1}{GJ}\int_0^{\beta} M_{x\theta}\left(\frac{\partial M_{x\theta}}{\partial M_{xBA}}\right)Rd\theta$$

$$= \frac{RM_{xBA}}{EI}\left[\left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right) + k\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right)\right]$$

$$+ \frac{RM_{yBA}}{EI}\left[(1-k)\frac{1}{2}\sin^2\beta\right] + \frac{R^2V_{BA}}{EI}\left[\left(-\frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right)\right.$$

$$\left. - k\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta - \sin\beta\right)\right] \qquad (6)$$

$$\phi_{yB} = \frac{\partial U}{\partial M_{yBA}}$$

$$= \frac{1}{EI}\int_0^{\beta} M_{y\theta}\left(\frac{\partial M_{y\theta}}{\partial M_{yBA}}\right)Rd\theta + \frac{1}{GJ}\int_0^{\beta} M_{x\theta}\left(\frac{\partial M_{x\theta}}{\partial M_{yBA}}\right)Rd\theta$$

$$= \frac{RM_{xBA}}{EI}\left[(1-k)\frac{1}{2}\sin^2\beta\right] + \frac{RM_{yBA}}{EI}\left[\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right)\right.$$

$$\left. + k\left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right)\right]$$

$$+ \frac{R^2V_{BA}}{EI}\left[- \frac{1}{2}\sin^2\beta + k\left(\frac{1}{2}\sin^2\beta + \cos\beta - 1\right)\right] \tag{7}$$

$$\delta_B = \frac{\partial U}{\partial V_{BA}}$$

$$= \frac{1}{EI}\int_0^{\beta} M_{y\theta}\left(\frac{\partial M_{y\theta}}{\partial V_{BA}}\right)Rd\theta + \frac{1}{GJ}\int_0^{\beta} M_{x\theta}\left(\frac{\partial M_{x\theta}}{\partial V_{BA}}\right)Rd\theta$$

$$= \frac{R^2M_{xBA}}{EI}\left[\left(- \frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right) - k\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta - \sin\beta\right)\right]$$

$$+ \frac{R^2M_{yBA}}{EI}\left[- \frac{1}{2}\sin^2\beta + k\left(\frac{1}{2}\sin^2\beta + \cos\beta - 1\right)\right]$$

$$+ \frac{R^3V_{BA}}{EI}\left[\left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right)\right.$$

$$\left. + k\left(\frac{3\beta}{2} - 2\sin\beta + \frac{1}{4}\sin 2\beta\right)\right] \tag{8}$$

Eqs. (6), (7), and (8) can be simplified by letting

$$a' = \left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right) + k\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right)$$

$$b' = \frac{1}{2}(1-k)\sin^2\beta$$

$$c' = \left(- \frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right) - k\left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta - \sin\beta\right)$$

$$d' = \left(\frac{\beta}{2} + \frac{1}{4}\sin 2\beta\right) + k\left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right)$$

$$e' = \left(\frac{\beta}{2} - \frac{1}{4}\sin 2\beta\right) + k\left(\frac{3}{2}\beta - 2\sin\beta + \frac{1}{4}\sin 2\beta\right)$$

$$f' = - \frac{1}{2}\sin^2\beta + k\left(\frac{1}{2}\sin^2\beta + \cos\beta - 1\right)$$

Eqs. (6), (7), and (8) may then be simply expressed in matrix form

$$\begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} = \frac{1}{EI} \begin{bmatrix} Ra' & Rb' & R^2c' \\ Rb' & Rd' & R^2f' \\ R^2c' & R^2f' & R^3e' \end{bmatrix} \begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} \tag{9}$$

We solve Eq. (9) for the joint actions in terms of joint displacements to obtain

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} Ra' & Rb' & R^2c' \\ Rb' & Rd' & R^2f' \\ R^2c' & R^2f' & R^3e' \end{bmatrix}^{-1} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} \tag{10}$$

Let

$$\begin{bmatrix} a' & b' & c' \\ b' & d' & f' \\ c' & f' & e' \end{bmatrix}^{-1} = \begin{bmatrix} a & b & c \\ b & d & f \\ c & f & e \end{bmatrix}$$

Eq. (10) can then be written as

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} a/R & b/R & c/R^2 \\ b/R & d/R & f/R^2 \\ c/R^2 & f/R^2 & e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} \tag{11}$$

or $$\begin{Bmatrix} A_{BA} \end{Bmatrix} = EI \begin{bmatrix} S_{BB} \end{bmatrix} \begin{Bmatrix} D_B \end{Bmatrix} \tag{11-a}$$

In Eq. (11-a), $\{A_{BA}\}$ is the column matrix of joint actions at B, $[S_{BB}]$ is the square matrix of stiffness influence coefficients, $\{D_B\}$ is the column matrix of joint displacements at B.

The transformation matrix from B to A is

$$\left[T_{AB}\right] = \begin{bmatrix} \cos\beta & -\sin\beta & R(1-\cos\beta) \\ \sin\beta & \cos\beta & -R\sin\beta \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the carry-over actions from B to A are

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = - \begin{bmatrix} \cos\beta & -\sin\beta & R(1-\cos\beta) \\ \sin\beta & \cos\beta & -R\sin\beta \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix}$$

or

$$\left\{A_{AB}\right\} = - \left[T_{AB}\right] \left\{A_{BA}\right\} \tag{12}$$

Substituting Eq. (11-a) into Eq. (12), we obtain

$$\left\{A_{AB}\right\} = -EI \left[T_{AB}\right] \left[S_{BB}\right] \left\{D_B\right\} \tag{13}$$

where

$$- \left[T_{AB}\right] \left[S_{BB}\right]$$

$$= \begin{bmatrix} -\dfrac{(a-c)\cos\beta-b\sin\beta+c}{R} & -\dfrac{(b-f)\cos\beta-d\sin\beta+f}{R} & -\dfrac{(c-e)\cos\beta-f\sin\beta+e}{R^2} \\ \dfrac{(c-a)\sin\beta-b\cos\beta}{R} & \dfrac{(f-b)\sin\beta-d\cos\beta}{R} & \dfrac{(e-c)\sin\beta-f\cos\beta}{R^2} \\ -\dfrac{c}{R^2} & -\dfrac{f}{R^2} & -\dfrac{e}{R^3} \end{bmatrix} \tag{14}$$

Let

$$\left.\begin{aligned} r &= (a-c)\cos\beta-b\sin\beta+c \\ s &= (b-f)\cos\beta-d\sin\beta+f \\ t &= (c-e)\cos\beta-f\sin\beta+e \\ u &= (c-a)\sin\beta-b\cos\beta \\ v &= (f-b)\sin\beta-d\cos\beta \\ w &= (e-c)\sin\beta-f\cos\beta \end{aligned}\right\} \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13), we obtain

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = EI \begin{bmatrix} -r/R & -s/R & -t/R^2 \\ u/R & v/R & w/R^2 \\ -c/R^2 & -f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} \tag{16}$$

or

$$\{A_{AB}\} = EI \quad [S_{AB}] \{D_B\} \tag{16-a}$$

Eq. (16) expresses the joint actions at A due to the displacements at joint B.

Step (2): Development of Slope-deflection Equations

The relation of actions and displacements is shown in Figure 5, in which the arc between joint A and B is a portion of a plane circular curve and point C is any position with an angular distance of θ from joint A. Now we write the following equations for shear and moments.

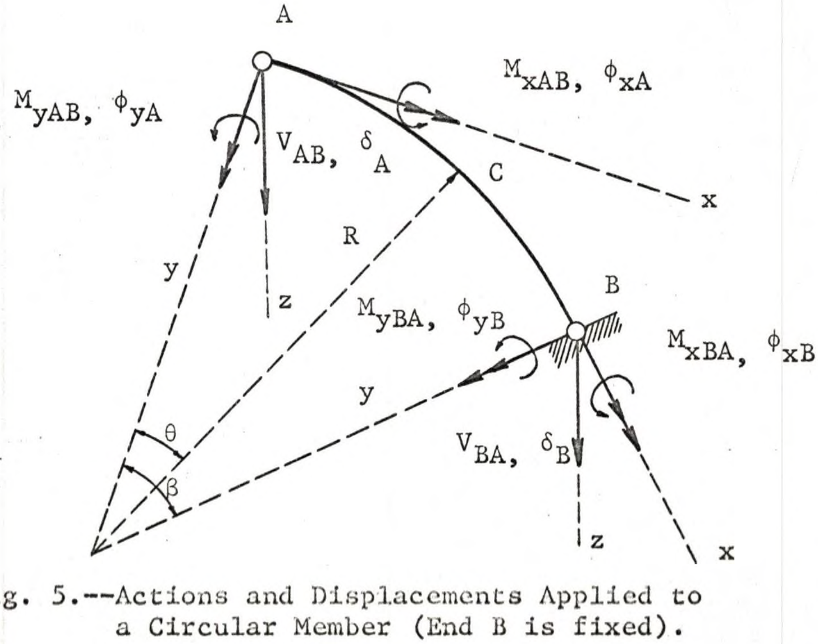

Fig. 5.--Actions and Displacements Applied to a Circular Member (End B is fixed).

The actions at a general section C in terms of the joint actions at A are

$$\left.\begin{aligned} M_{x\theta} &= M_{xAB}\cos\theta + M_{yAB}\sin\theta + V_{AB}R(1-\cos\theta) \\ M_{y\theta} &= -M_{xAB}\sin\theta + M_{yAB}\cos\theta + V_{AB}R\sin\theta \\ V_{\theta} &= V_{AB} \end{aligned}\right\} \quad (17)$$

By using the same procedure as Step (1), we obtain

$$\begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} = \frac{1}{EI}\begin{bmatrix} Ra' & -Rb' & R^2c' \\ -Rb' & Rd' & -R^2f' \\ R^2c' & -R^2f' & R^3e' \end{bmatrix}\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} \quad (18)$$

Solving for the joint actions in terms of the joint displacements to obtain

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = EI\begin{bmatrix} a/R & -b/R & c/R^2 \\ -b/R & d/R & -f/R^2 \\ c/R^2 & -f/R^2 & e/R^3 \end{bmatrix}\begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} \quad (19)$$

or

$$\{A_{AB}\} = EI\,[S_{AA}]\,\{D_A\} \quad (19\text{-a})$$

The definitions of a', b', c', d', e' and f' in Eq. (18) are the same as those of Eq. (9), and the definitions of a, b, c, d, e and f are also the same as those of Eq. (11).

The transformation matrix from A to B is

$$[T_{BA}] = \begin{bmatrix} \cos\beta & \sin\beta & R(1-\cos\beta) \\ -\sin\beta & \cos\beta & R\sin\beta \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the carry-over forces from A to B are

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = - \begin{bmatrix} \cos\beta & \sin\beta & R(1-\cos\beta) \\ -\sin\beta & \cos\beta & R\sin\beta \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix}$$

Substituting Eq. (19) into above equation and using the relation given in Eq. (15), we obtain

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} -r/R & s/R & -t/R^2 \\ -u/R & v/R & -w/R^2 \\ -c/R^2 & f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} \tag{20}$$

or

$$\{A_{BA}\} = EI \left[S_{BA}\right] \{D_A\} \tag{20-a}$$

By employing the reciprocal theorem, we find the matrices of stiffness influence coefficients $\left[S_{AB}\right]$ and $\left[S_{BA}\right]$ are equal to one another, and the matrix of stiffness influence coefficients is equal to its transpose matrix (Gere and Weaver, 1965).

$$\left[S_{AB}\right] = \left[S_{BA}\right] = \left[S_{BA}\right]^T$$

or

$$\begin{bmatrix} -r/R & -s/R & -t/R^2 \\ u/R & v/R & w/R^2 \\ -c/R^2 & -f/R^2 & -e/R^3 \end{bmatrix} = \begin{bmatrix} -r/R & -u/R & -c/R^2 \\ s/R & v/R & f/R^2 \\ -t/R^2 & -w/R^2 & -e/R^3 \end{bmatrix}$$

By employing the theory of matrix, we find

$$\left.\begin{aligned} u &= s \\ t &= c \\ w &= f \end{aligned}\right\} \tag{21}$$

Substituting Eq. (21) into Eqs. (16) and (20), we obtain

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = EI \begin{bmatrix} -r/R & -s/R & -c/R^2 \\ s/R & v/R & f/R^2 \\ -c/R^2 & -f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} \tag{22}$$

or

$$\{A_{AB}\} = EI \left[S_{AB}\right] \{D_B\} \tag{22-a}$$

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} -r/R & s/R & -c/R^2 \\ -s/R & v/R & -f/R^2 \\ -c/R^2 & f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} \tag{23}$$

or

$$\{A_{BA}\} = EI \left[S_{BA}\right] \{D_A\} \tag{23-a}$$

Step (3): Generalized Slope-deflection Equations of a Circular Member

For a circular member subjected to external forces causing displacements at both ends of the member, we find the generalized slope-deflection equations are as follows:

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = EI \begin{bmatrix} a/R & -b/R & c/R^2 \\ -b/R & d/R & -f/R^2 \\ c/R^2 & -f/R^2 & e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} + EI \begin{bmatrix} -r/R & -s/R & -c/R^2 \\ s/R & v/R & f/R^2 \\ -c/R^2 & -f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} + \begin{Bmatrix} M^F{}_{xAB} \\ M^F{}_{yAB} \\ V^F{}_{AB} \end{Bmatrix} \tag{24}$$

or

$$\{A_{AB}\} = EI \left[S_{AA}\right] \{D_A\} + EI \left[S_{AB}\right] \{D_B\} + \{F_{AB}\} \tag{24-a}$$

and

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} a/R & b/R & c/R^2 \\ b/R & d/R & f/R^2 \\ c/R^2 & f/R^2 & e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix}$$

$$+ EI \begin{bmatrix} -r/R & s/R & -c/R^2 \\ -s/R & v/R & -f/R^2 \\ -c/R^2 & f/R^2 & -e/R^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} + \begin{Bmatrix} M^F{}_{xBA} \\ M^F{}_{yBA} \\ V^F{}_{BA} \end{Bmatrix} \quad (25)$$

or

$$\{A_{BA}\} = EI\,[S_{BB}]\,\{D_B\} + EI\,[S_{BA}]\,\{D_A\} + \{F_{BA}\} \quad (25\text{-a})$$

In Eqs. (24-a) and (25-a), the matrices $\{A_{ij}\}$ and $\{F_{ij}\}$ represent the end actions and fixed-end actions, respectively, at i-end of member ij. The matrix $[S_{ii}]$ represents the direct stiffness influence coefficients at i, and $[S_{ij}]$ represents the carry-over stiffness influence coefficients transferred from j to i. The matrix $\{D_i\}$ represents the displacements at i.

GENERALIZED SLOPE-DEFLECTION EQUATIONS OF A STRAIGHT MEMBER

The two-joint member shown in Figure 6 is subjected to end-actions and corresponding end-displacements. There are three degrees of freedom to be considered at each joint. M_x, M_y, and V represent the bending moment, torsional moment and shearing force of the member, respectively. Positive directions for bending moment and torsional moment are indicated by the double-headed arrows. These are to be used in conjunction with the right-hand rule. The shearing force is positive in the z-axis direction. Since the member has three degrees of freedom at each joint, the order of the stiffness matrix will be 3 x 3. In this case, the generalized slope-deflection equations can be expressed as follows (Jenkins, 1969):

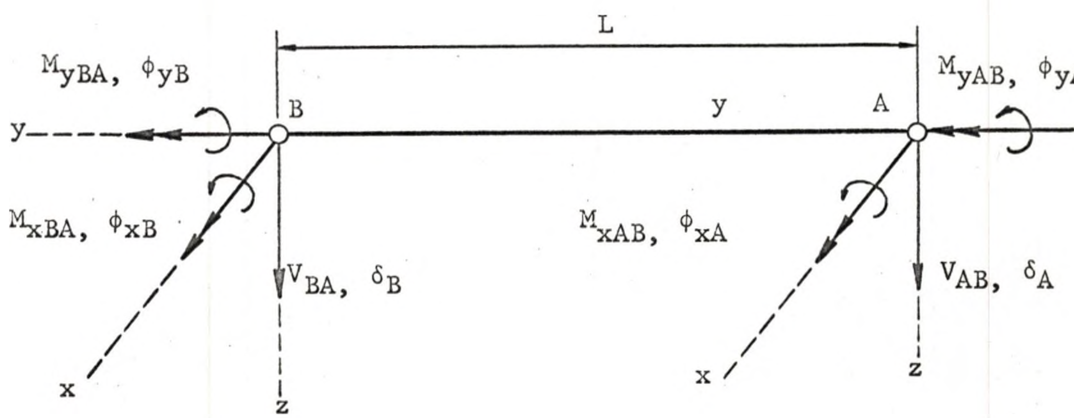

Fig. 6.--Actions and Displacements Applied to a Straight Member.

$$\begin{Bmatrix} M_{xAB} \\ M_{yAB} \\ V_{AB} \end{Bmatrix} = EI \begin{bmatrix} 4/L & 0 & 6/L^2 \\ 0 & 1/kL & 0 \\ 6/L^2 & 0 & 12/L^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix}$$

$$+ EI \begin{bmatrix} 2/L & 0 & -6/L^2 \\ 0 & 1/kL & 0 \\ 6/L^2 & 0 & -12/L^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix} + \begin{Bmatrix} M^F{}_{xAB} \\ M^F{}_{yAB} \\ V^F{}_{AB} \end{Bmatrix} \qquad (26)$$

or

$$\{A_{AB}\} = EI\ [S_{AA}]\{D_A\} + EI\ [S_{AB}]\{D_B\} + \{F_{AB}\} \qquad (26\text{-a})$$

and

$$\begin{Bmatrix} M_{xBA} \\ M_{yBA} \\ V_{BA} \end{Bmatrix} = EI \begin{bmatrix} 4/L & 0 & -6/L^2 \\ 0 & 1/kL & 0 \\ -6/L^2 & 0 & 12/L^3 \end{bmatrix} \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ \delta_B \end{Bmatrix}$$

$$+ EI \begin{bmatrix} 2/L & 0 & 6/L^2 \\ 0 & -1/kL & 0 \\ -6/L^2 & 0 & -12/L^3 \end{bmatrix} \begin{Bmatrix} \phi_{xA} \\ \phi_{yA} \\ \delta_A \end{Bmatrix} + \begin{Bmatrix} M^F{}_{xBA} \\ M^F{}_{yBA} \\ V^F{}_{BA} \end{Bmatrix} \qquad (27)$$

or

$$\{A_{BA}\} = EI\ [S_{BB}]\{D_B\} + EI\ [S_{BA}]\{D_A\} + \{F_{BA}\} \qquad (27\text{-a})$$

In Eqs. (26-a) and (27-a), $\{A_{ij}\}$ is the column matrix of joint actions at i, $[S_{ii}]$ is the square matrix of direct stiffness influence coefficients at i, $[S_{ij}]$ is the square matrix of carry-over stiffness influence coefficients transferred from j to i, $\{D_i\}$ is the column matrix of joint displacements at i, and $\{F_{ij}\}$ is the column matrix of fixed-end actions at i.

EQUILIBRIUM EQUATIONS OF A JOINT

Each joint is in equilibrium under the action of the forces acting on it by the members meeting there, together with any external loads which are applied directly to the joint. Each member is also in equilibrium under the joint reactions applied at its ends together with any external loads applied directly to the member.

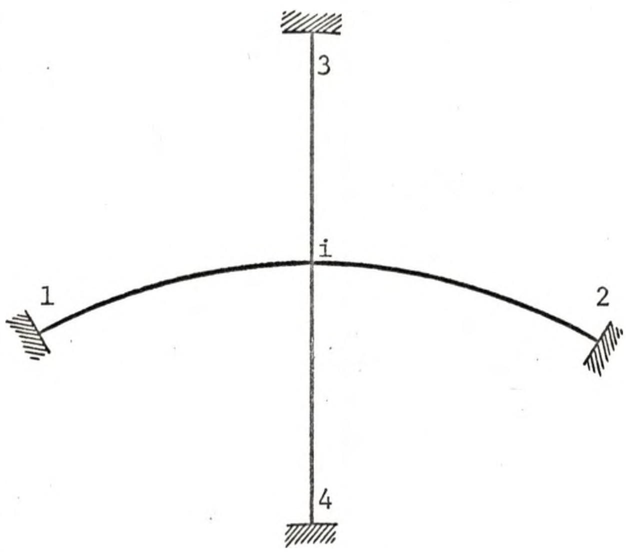

Fig. 7.--A Joint Connected to Two Circular Members and Two Straight Members.

In Figure 7, a typical joint i is orthogonally connected to two circular members i1 and i2 and two straight members i3 and i4. By employing the Eqs. (24-a), (25-a), (26-a) and (27-a), the end actions of each member attached to i can be expressed as (Hall and Woodhead, 1967):

$$\{A_{i1}\} = (EI)_1 [S_{ii}]_1 \{D_i\} + (EI)_1 [S_{i1}] \{D_1\} + \{F_{i1}\} \quad (28)$$

$$\{A_{i2}\} = (EI)_2 [S_{ii}]_2 \{D_i\} + (EI)_2 [S_{i2}] \{D_2\} + \{F_{i2}\} \quad (29)$$

$$\{A_{i3}\} = (EI)_3 [S_{ii}]_3 \{D_i\} + (EI)_3 [S_{i3}] \{D_3\} + \{F_{i3}\} \quad (30)$$

$$\{A_{i4}\} = (EI)_4 [S_{ii}]_4 \{D_i\} + (EI)_4 [S_{i4}] \{D_4\} + \{F_{i4}\} \quad (31)$$

Let

$$[B_{ii}]_j = (EI)_1 [S_{ii}]_1 + (EI)_2 [S_{ii}]_2 + (EI)_3 [S_{ii}]_3 + (EI)_4 [S_{ii}]_4 \quad (32)$$

$$[C_{ij}] = (EI)_j [S_{ij}] \quad (33)$$

$$\{F_{ij}\} = \{F_{i1}\} + \{F_{i2}\} + \{F_{i3}\} + \{F_{i4}\} \quad (34)$$

By using the equilibirum condition, the sum of all external forces must be equal to the sum of all member actions. Therefore, the sum of right-hand side of Eqs. (28), (29), (30) and (31) is equal to the sum of external forces $\{P_i\}$ at joint i. By employing the relations of Eqs. (32), (33) and (34), the equilibrium equations can be simplified as

$$[B_{ii}]_j \{D_i\} + [C_{i1}] \{D_1\} + [C_{i2}] \{D_2\} + [C_{i3}] \{D_3\} + [C_{i4}] \{D_4\} = \{P_i\} - \{F_{ij}\} \quad (35)$$

In Eq. (35), $[B_{ii}]_j$ is the direct stiffness matrix, $[C_{ij}]$ is the carry-over stiffness matrix, $\{P_i\}$ is the matrix of external forces acting directly at joint i, and $\{F_{ij}\}$ is the matrix of fixed-end actions.

If a joint is connected to less than four members, we can add the imaginary members by letting the moduli of rigidity, EI, equal to zero for these members.

ASSEMBLING THE EQUILIBRIUM EQUATIONS OF OVER-ALL JOINTS

The general form of the equilibrium equations of a typical joint is expressed in Eq. (35). In preparation for the analysis of a grid by computer, the equilibrium equations of over-all joints of the grid should be presented. For this purpose, a plane grid is illustrated.

A numbering system for members and joints is given in Figure 8. The sequences of equilibrium equations are presented by the order of $M_x = 0$, $M_y = 0$, $V = 0$. The displacements of a joint are presented by the order of ϕ_x, ϕ_y, δ. The equations of equilibrium for over-all joints are obtained by simply summing the equilibrium equations of individual joints. These are given by Eq. (36).

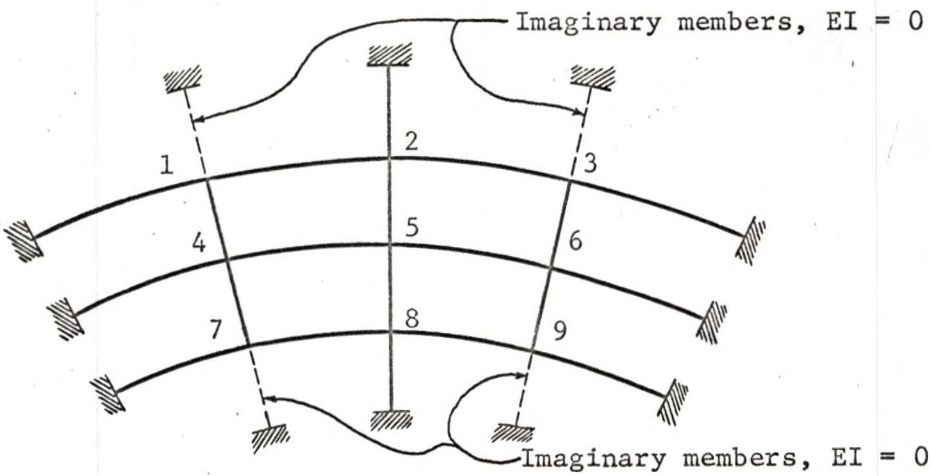

Fig. 8.--Numbering System of an Illustrated Structure.

$$\begin{bmatrix}
[B_{11}] & [C_{12}] & & [C_{14}] & & & & & \\
[C_{21}] & [B_{22}] & [C_{23}] & & [C_{25}] & & & & \\
& [C_{32}] & [B_{33}] & & & [C_{36}] & & & \\
[C_{41}] & & & [B_{44}] & [C_{45}] & & [C_{47}] & & \\
& [C_{52}] & & [C_{54}] & [B_{55}] & [C_{56}] & & [C_{58}] & \\
& & [C_{63}] & & [C_{65}] & [B_{66}] & & & [C_{69}] \\
& & & [C_{74}] & & & [B_{77}] & [C_{78}] & \\
& & & & [C_{85}] & & [C_{87}] & [B_{88}] & [C_{89}] \\
& & & & & [C_{96}] & & [C_{98}] & [B_{99}]
\end{bmatrix}
\begin{Bmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \\ \{D_4\} \\ \{D_5\} \\ \{D_6\} \\ \{D_7\} \\ \{D_8\} \\ \{D_9\} \end{Bmatrix}
=
\begin{Bmatrix} \{P_1\} \\ \{P_2\} \\ \{P_3\} \\ \{P_4\} \\ \{P_5\} \\ \{P_6\} \\ \{P_7\} \\ \{P_8\} \\ \{P_9\} \end{Bmatrix}
-
\begin{Bmatrix} \{F_1\} \\ \{F_2\} \\ \{F_3\} \\ \{F_4\} \\ \{F_5\} \\ \{F_6\} \\ \{F_7\} \\ \{F_8\} \\ \{F_9\} \end{Bmatrix} \tag{36}$$

or $[S]\{D\} = \{P\} - \{F\}$ (36-a)

Let NJ = Number of joints (end supports are excluded)

In Eq. (36-a), [S] is the (3NJ x 3NJ) stiffness matrix, {D} is the column matrix of displacements, {P} is the column matrix of external forces acting at joints, and {F} is the column matrix of fixed-end actions.

THE GENERAL COMPUTER PROGRAM

In previous formulation the stiffness method of analysis was presented in a form which is suitable for computer programming. We will now present a general computer program for this analysis. The general program consists of a main program and two subroutines. The main program is used for generating the joint displacements of each joint and end actions of each member. Two IBM subroutines, MINV and GMPRD, are used in the main program for matrix inversion and multiplication. The relevant expressions for the joint displacements are given in Eq. (36) and for end actions in Eqs. (24) to (27). The program is written in FORTRAN IV for the IBM System 360, Model 40, computer. As an illustration of the use of the general computer program, a typical grid problem is analyzed later in this part.

Arrangement of Data for Program

The following indentifiers are used for representing the input and output data:

I. Input Data

1. NJ, NJH, NJV

   NJ = Number of intersections (End supports excluded)

   NJH = Number of intersections counted circumferentially

   NJV = Number of intersections counted radially

2. Elements of Member Stiffness Matrices

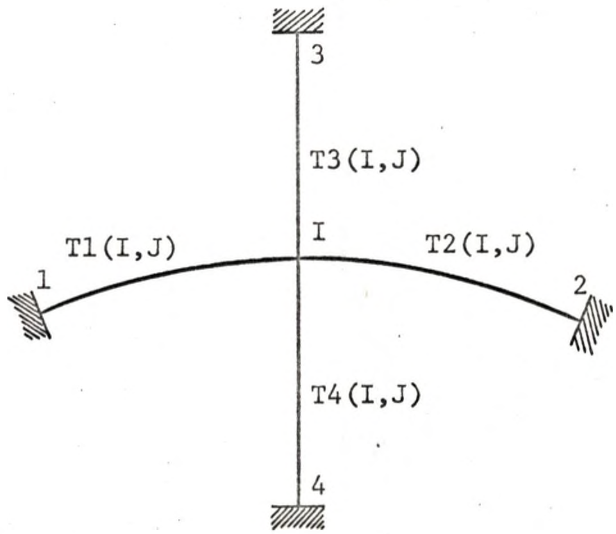

Fig. 9.--Elements of Member Stiffness Matrices

a. For circular members

(Refer to Eqs. (24), (25) and Figure 9)

I = 1,2,.....,NJ

J = 1,2,.....,9

T1(I,1), T2(I,1) = a/R

T1(I,2), T2(I,2) = b/R

T1(I,3), T2(I,3) = c/R^2

T1(I,4), T2(I,4) = d/R

T1(I,5), T2(I,5) = e/R^3

T1(I,6), T2(I,6) = f/R^2

T1(I,7), T2(I,7) = r/R

T1(I,8), T2(I,8) = s/R

T1(I,9), T2(I,9) = v/R

b. <u>For straight members</u>

(Refer to Eqs. (26), (27) and Figure 9).

T3(I,1), T4(I,1) = 4/L

T3(I,2), T4(I,2) = 1/kL

T3(I,3), T4(I,3) = $12/L^3$

T3(I,4), T4(I,4) = $6/L^2$

3. <u>Flexural Stiffness of Members</u>

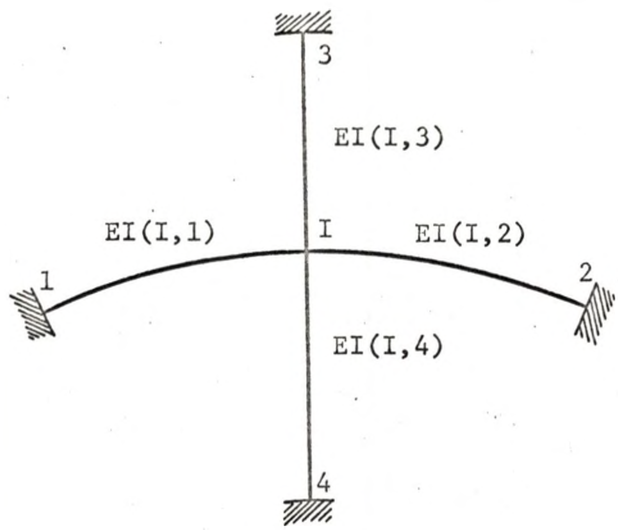

Fig. 10.--Flexural Stiffness of Members

EI(I,J)

I = 1,2,.....,NJ

J = 1,2,3,4

4. <u>Fixed-end Actions</u>

FX(I,J), FY(I,J), FZ(I,J)

I = 1,2,.....,NJ

J = 1,2,3,4

FX = M_X^F, fixed-end moment about x-axis

FY = M_Y^F, fixed-end moment about y-axis

FZ = V_z^F, fixed-end shearing force along z-axis

5. External Forces Applying at Joints

P(I)

I = 1,2,.....,NJ

II. Output Results

1. Displacements

DS(I)

I = 1,2,.....,NJ

The order is ϕ_x, ϕ_y, δ for each joint.

2. End Actions

AX(I,J), AY(I,J), AZ(I,J)

I = 1,2,.....,NJ

J = 1,2,3,4

AX = M_x, end moment about x-axis

AY = M_y, end moment about y-axis

AZ = V, shearing force along z-axis

Computer Program

```
C                                      MAIN PROGRAM
C***********************************************************************
      DOUBLE PRECISION T1(9,9),T2(9,9),T3(9,4),T4(9,4),FX(9,4),
     1FY(9,4),FZ(9,4),EI(9,4),P(9),S(27,27),A(27),LLL(27,27),
     2MMM(27,27),DS(27),AX(27),AY(27),AZ(27)
C    TO READ THE NUMBER OF JOINTS
      READ(1,1) NJ,NJH,NJV
    1 FORMAT(3I2)
C    TO READ THE ELEMENTS OF MEMBER STIFFNESS MATRICES
      READ(1,2) ((T1(I,J),J=1,9),I=1,NJ)
    2 FORMAT(3D16.8)
      READ(1,2) ((T2(I,J),J=1,9),I=1,NJ)
      READ(1,3) ((T3(I,J),J=1,4),I=1,NJ)
    3 FORMAT(4D16.8)
      READ(1,3) ((T4(I,J),J=1,4),I=1,NJ)
C    TO READ FLEXURAL STIFFNESS OF MEMBERS
      READ(1,3) ((EI(I,J),J=1,4),I=1,NJ)
C    TO READ FIXED-END ACTIONS
      DO 20 I=1,NJ
      DO 20 J=1,4
   20 READ(1,2) FX(I,J),FY(I,J),FZ(I,J)
C    TO READ EXTERNAL FORCES APPLYING AT EACH JOINT
      READ(1,2) (P(I),I=1,9)
C    TO GENERATE THE STIFFNESS MATRIX OF OVER-ALL JOINTS
      N=3*NJ
      DO 21 I=1,N
      DO 21 J=1,N
   21 S(I,J)=0.0
      M=1
      DO 22 I=1,NJ
      J=3*I-2
      J1=J+1
      J2=J+2
      J3=J+3
      J4=J+4
      J5=J+5
      J6=3*(I+NJH)-2
      J7=J6+1
      J8=J7+1
      S(J,J)=EI(I,1)*T1(I,1)+EI(I,2)*T2(I,1)+EI(I,3)*T3(I,1)+EI
     1(I,4)*T4(I,1)
      S(J,J1)=EI(I,1)*T1(I,2)-EI(I,2)*T2(I,2)
      S(J,J2)=EI(I,1)*T1(I,3)+EI(I,2)*T2(I,3)-EI(I,3)*T3(I,4)+E
     1I(I,4)*T4(I,4)
      S(J1,J1)=EI(I,1)*T1(I,4)+EI(I,2)*T2(I,4)+EI(I,3)*T3(I,2)+
     1EI(I,4)*T4(I,2)
      S(J1,J2)=EI(I,1)*T1(I,6)-EI(I,2)*T2(I,6)
      S(J2,J2)=EI(I,1)*T1(I,5)+EI(I,2)*T2(I,5)+EI(I,3)*T3(I,3)+
     1EI(I,4)*T4(I,3)
```

```
      DO 23 K=M,NJV
      IF (I-K*NJH) 23,80,23
   23 CONTINUE
      S(J,J3)=-EI(I,2)*T2(I,7)
      S(J,J4)=-EI(I,2)*T2(I,8)
      S(J,J5)=-EI(I,2)*T2(I,3)
      S(J1,J3)=-S(J,J4)
      S(J1,J4)=EI(I,2)*T2(I,9)
      S(J1,J5)=EI(I,2)*T2(I,6)
      S(J2,J3)=S(J,J5)
      S(J2,J4)=-S(J1,J5)
      S(J2,J5)=-EI(I,2)*T2(I,5)
      GO TO 32
   80 M=M+1
   32 IF (I-NJ+NJH) 34,34,36
   34 S(J,J6)=EI(I,4)*T4(I,1)/2.0
      S(J,J8)=-EI(I,4)*T4(I,4)
      S(J1,J7)=-EI(I,4)*T4(I,2)
      S(J2,J6)=EI(I,4)*T4(I,4)
      S(J2,J8)=-EI(I,4)*T4(I,3)
   36 A(J)=-(FX(I,1)+FX(I,2)+FX(I,3)+FX(I,4))
      A(J1)=-(FY(I,1)+FY(I,2)+FY(I,3)+FY(I,4))
      A(J2)=P(I)-(FZ(I,1)+FZ(I,2)+FZ(I,3)+FZ(I,4))
   22 CONTINUE
      DO 24 I=1,N
      DO 24 J=1,N
   24 S(J,I)=S(I,J)
C    TO FIND THE FLEXIBILITY MATRIX BY INVERTING THE STIFFNESS MATRIX
      CALL MINV (S,N,D,LLL,MMM)
C    TO MULTIPLY FLEXIBILITY MATRIX BY FORCE MATRIX TO FIND THE
     DISPLACEMENT MATRIX
      CALL GMPRD (S,A,DS,N,N,1)
      WRITE(3,4)
    4 FORMAT(2X,'THE DISPLACEMENTS OF EACH JOINT ARE',//)
      WRITE(3,5) (DS(I),I=1,N)
    5 FORMAT(4X,3D16.8,//)
C    TO FIND THE END ACTIONS OF EACH MEMBER
      M1=1
      M2=1
      DO 50 I=1,NJ
      J=3*I-2
      JM3=J-3
      JM2=J-2
      JM1=J-1
      J1=J+1
      J2=J+2
      J3=J+3
      J4=J+4
      J5=J+5
      JD1=3*(I+NJH)-2
      JD2=JD1+1
```

```
      JD3=JD2+1
      JU1=3*(I-NJH)-2
      JU2=JU1+1
      JU3=JU2+1
      DO 52 K=M1,NJV
      IF (I+NJH-K*NJH-1) 52,54,52
   52 CONTINUE
C   TO FIND THE END ACTIONS OF LEFT MEMBERS OF EACH JOINT
      ALX1=EI(I,1)*(-T1(I,7)*DS(JM3)+T1(I,8)*DS(JM2)-T1(I,3)*DS
     1(JM1))
      ALY1=EI(I,1)*(-T1(I,8)*DS(JM3)+T1(I,9)*DS(JM2)-T1(I,6)*DS
     1(JM1))
      ALZ1=EI(I,1)*(-T1(I,3)*DS(JM3)+T1(I,6)*DS(JM2)-T1(I,5)*DS
     1(JM1))
      GO TO 53
   54 ALX1=0.0
      ALY1=0.0
      ALZ1=0.0
      M1=M1+1
   53 ALX=EI(I,1)*(T1(I,1)*DS(J)+T1(I,2)*DS(J1)+T1(I,3)*DS(J2))
      ALY=EI(I,1)*(T1(I,2)*DS(J)+T1(I,4)*DS(J1)+T1(I,6)*DS(J2))
      ALZ=EI(I,1)*(T1(I,3)*DS(J)+T1(I,6)*DS(J1)+T1(I,5)*DS(J2))
      AX(I,1)=ALX+ALX1+FX(I,1)
      AY(I,1)=ALY+ALY1+FY(I,1)
      AZ(I,1)=ALZ+ALZ1+FZ(I,1)
C   TO FIND THE END ACTIONS OF RIGHT MEMBERS OF EACH JOINT
      DO 58 K=M2,NJV
      IF (I-K*NJH) 58,60,58
   58 CONTINUE
      ARX1=EI(I,2)*(-T2(I,7)*DS(J3)-T2(I,8)*DS(J4)-T2(I,3)*DS(J
     15))
      ARY1=EI(I,2)*(+T2(I,8)*DS(J3)+T2(I,9)*DS(J4)+T2(I,6)*DS(J
     15))
      ARZ1=EI(I,2)*(-T2(I,3)*DS(J3)-T2(I,6)*DS(J4)-T2(I,5)*DS(J
     15))
      GO TO 59
   60 ARX1=0.0
      ARY1=0.0
      ARZ1=0.0
      M2=M2+1
   59 ARX=EI(I,2)*(T2(I,1)*DS(J)-T2(I,2)*DS(J1)+T2(I,3)*DS(J2))
      ARY=EI(I,2)*(-T2(I,2)*DS(J)+T2(I,4)*DS(J1)-T2(I,6)*DS(J2))
      ARZ=EI(I,2)*(T2(I,3)*DS(J)-T2(I,6)*DS(J1)+T2(I,5)*DS(J2))
      AX(I,2)=ARX+ARX1+FX(I,2)
      AY(I,2)=ARY+ARY1+FY(I,2)
      AZ(I,2)=ARZ+ARZ1+FZ(I,2)
C   TO FIND THE END ACTIONS OF UPPER MEMBERS OF EACH JOINT
      IF (I-NJH) 66,66,63
   63 AUX1=EI(I,3)*(T3(I,1)*DS(JU1)/2.0+T3(I,4)*DS(JU3))
      AUY1=-EI(I,3)*T3(I,2)*DS(JU2)
      AUZ1=-EI(I,3)*(T3(I,4)*DS(JU1)+T3(I,3)*DS(JU3))
```

```
      GO TO 64
   66 AUX1=0.0
      AUY1=0.0
      AUZ1=0.0
   64 AUX=EI(I,3)*(T3(I,1)*DS(J)-T3(I,4)*DS(J2))
      AUY=EI(I,3)*T3(I,2)*DS(J1)
      AUZ=EI(I,3)*(-T3(I,4)*DS(J)+T3(I,3)*DS(J2))
      AX(I,3)=AUX+AUX1+FX(I,3)
      AY(I,3)=AUY+AUY1+FY(I,3)
      AZ(I,3)=AUZ+AUZ1+FZ(I,3)
C     TO FIND THE END ACTIONS OF LOWER MEMBERS OF EACH JOINT
      IF (I+NJH-NJ) 70,70,72
   70 ADX1=EI(I,4)*(T4(I,1)*DS(JD1)/2.0-T4(I,4)*DS(JD3))
      ADY1=-EI(I,4)*T4(I,2)*DS(JD2)
      ADZ1=EI(I,4)*(T4(I,4)*DS(JD1)-T4(I,3)*DS(JD3))
      GO TO 74
   72 ADX1=0.0
      ADY1=0.0
      ADZ1=0.0
   74 ADX=EI(I,4)*(T4(I,1)*DS(J)+T4(I,4)*DS(J2))
      ADY=EI(I,4)*T4(I,2)*DS(J1)
      ADZ=EI(I,4)*(T4(I,4)*DS(J)+T4(I,3)*DS(J2))
      AX(I,4)=ADX+ADX1+FX(I,4)
      AY(I,4)=ADY+ADY1+FY(I,4)
      AZ(I,4)=ADZ+ADZ1+FZ(I,4)
      WRITE(3,10) I
   10 FORMAT(T35,'JOINT NO.',I2,/)
      WRITE(3,11)
   11 FORMAT(T15,'LEFT', 11X,'RIGHT',11X,'UPPER',11X,'LOWER',/)
      WRITE(3,12) AX(I,1),AX(I,2),AX(I,3),AX(I,4)
   12 FORMAT(2X,'MX=',2X,4D16.8,/)
      WRITE(3,13) AY(I,1),AY(I,2),AY(I,3),AY(I,4)
   13 FORMAT(2X,'MY=',2X,4D16.8,/)
      WRITE(3,14) AZ(I,1),AZ(I,2),AZ(I,3),AZ(I,4)
   14 FORMAT(2X,'V=',3X,4D16.8,//)
   50 CONTINUE
      STOP
      END
```

Example

The plane grid system of Figure 11 is constructed from twelve circular members and six straight members. Each member has a rectangular cross-section 1.5' x 3.0'. The 1.5' dimension in each member lies in a horizontal plane. All joints are orthogonally and rigidly connected by the circular and straight members. The supports A, B, C, D, E and F are assumed to be completely fixed. The elastic moduli are $E = 10^5$ ksf and $G = 0.4 \times 10^5$ ksf. The grid is to be analyzed for three systems of loading condition:

(a). Distributed loads of 0.2 k/ft. on the straight member 1-4, and 0.4 k/ft. on the circular member 4-5.

(b). Concentrated loads of 6 kips at the center of circular member 2-3, 8 kips at the point 10 ft. from the end 7 of straight member 4-7, and 8 kips at a point 5° from the end 8 of circular member 8-9.

(c). Concentrated loads 3 kips, 4 kips, 2 kips, 5 kips, 1 kip, 2 kips and 3 kips are acting at joints 2, 3, 5, 6, 7, 8 and 9, respectively.

For each load, calculate the displacements of each joint and the end actions of each member. Use radian, feet and kip units.

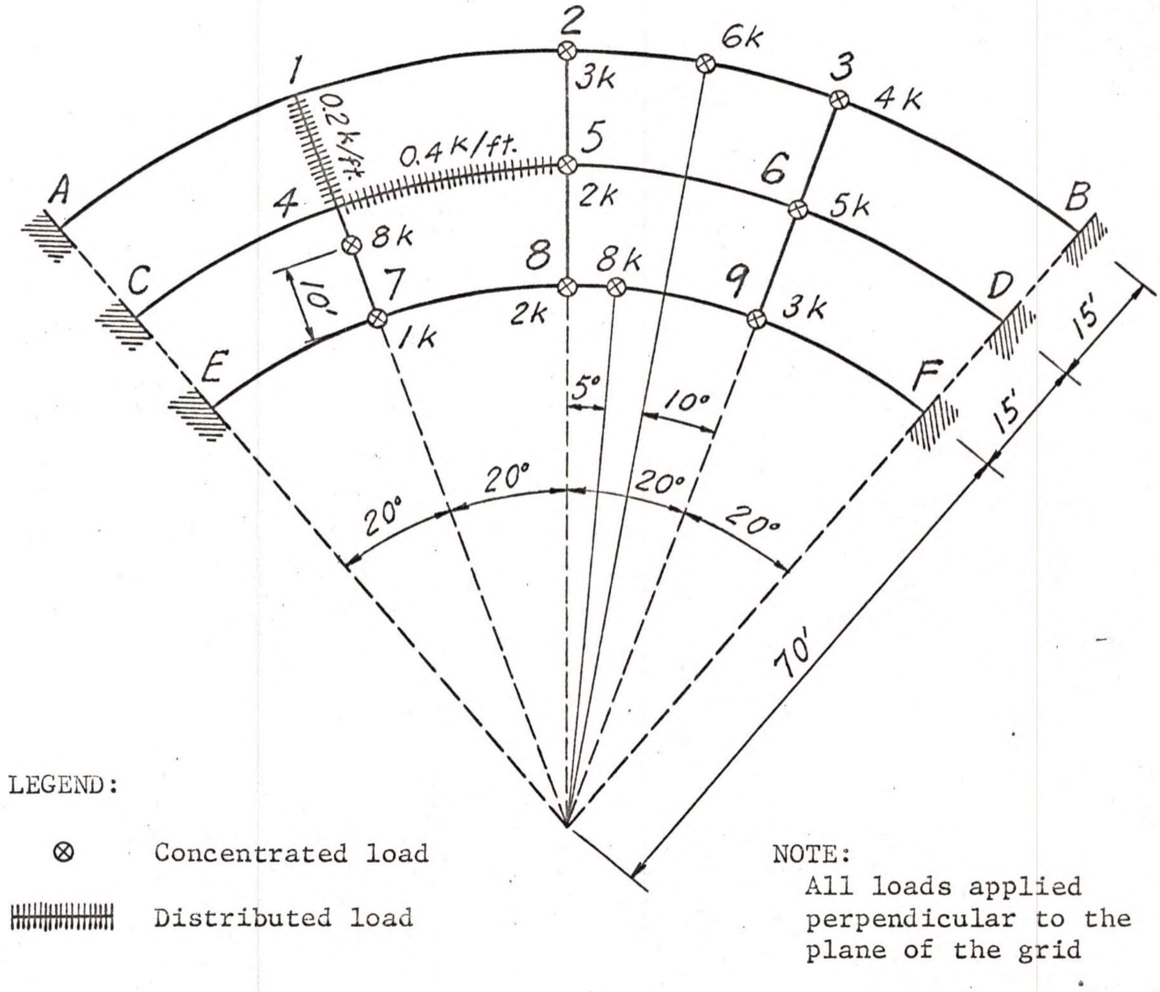

Fig. 11.--The Grid System for Example

Solution:

The problem will be solved in the following three steps:

I. Preliminary Calculation for Input Data

1. The Properties of the Cross-section of Members

$E = 10^5$ ksf, $G = 0.4 \times 10^5$ ksf, $E = 2.5\ G$

$I_x = I_y = bh^3/12 = 1.5 \times 3^3/12 = 3.375$ ft.^{4}

$J = \gamma b^3 h = 0.229 \times 1.5^3 \times 3.0 = 2.318625$ ft.^{4}

$EI = EI_x = 337500$ k-ft.^{2}

$GJ = 92745$ k-ft.^{2}

$k = EI/GJ = 3.6390102$

2. Elements of Member Stiffness Matrices

a. For circular members

Based on Eqs. (9), (10), (11), (15) and (16), a computer program is written for computing the elements of matrices for circular members in APPENDIX A.

(1) For $\beta = 20°$, R = 100 ft.,

$k = EI/GJ = 3.6390102$

a/R = 0.90026514D-02   b/R = 0.97255074D-02

c/R^2 = 0.36253967D-03   d/R = 0.11203393D+00

e/R^3 = 0.27972848D-03 f/R^2 = 0.48684516D-02

r/R = 0.73197880D-02   s/R = 0.18149103D-03

v/R = 0.57907054D-01

(2) For $\beta = 20°$, R = 85 ft.,

k = 3.6390102

a/R = 0.10591355D-01   b/R = 0.11441773D-01

c/R^2 = 0.50178501D-03   d/R = 0.13180462D+00

$e/R^3 = 0.45549111D-03$ $f/R^2 = 0.67383412D-02$

$r/R = 0.86115153D-02$ $s/R = 0.21351886D-03$

$v/R = 0.68125946D-01$

(3) For $\beta = 20°$, $R = 70$ ft.,

$k = 3.6390102$

$a/R = 0.12860931D-01$ $b/R = 0.13893582D-01$

$c/R^2 = 0.73987687D-03$ $d/R = 0.16004847D+00$

$e/R^3 = 0.81553492D-03$ $f/R^2 = 0.99356154D-02$

$r/R = 0.10456840D-01$ $s/R = 0.25927290D-03$

$v/R = 0.827243363D-01$

b. For straight members

$4/L = 4/15 = 0.26666667$

$1/kL = 1/(3.6390102 \times 15) = 0.01832$

$12/L^3 = 12/15^3 = 0.0035555556$

$6/L^2 = 6/15^2 = 0.026666667$

3. Fixed-end Actions

a. For circular members

The formulae for calculating the fixed-end actions of circular members due to concentrated load and distributed load are provided in APPENDIX B.

(1) For member 2-3

$\beta = 20°$, $R = 100'$, $EI/GJ = 3.6390102$, $P = 6k$

$M_{x23} = M_{x32} = -0.52179337D-01$

$M_{y23} = -M_{y32} = 0.26542496D+02$

$V_{23} = V_{32} = -0.30000000D+01$

(2) For member 4-5

$\beta = 20°$, $R = 85'$, $EI/GJ = 3.6390102$,

$w = 0.4$ k/ft.

$M_{x45} = M_{x54} = -0.45992725D-01$

$M_{y45} = -M_{y54} = 0.29664963D+02$

$V_{45} = V_{54} = -0.59341154D+01$

(3) For member 8-9

$\beta = 20°$, $R = 70'$, $EI/GJ = 3.6390102$, $P = 8k$

$M_{x89} = -0.18467140D+00$, $M_{x98} = -0.23947906D+00$

$M_{y89} = 0.27782471D+02$, $M_{y98} = -0.92500963D+01$

$V_{89} = -0.67553415D+01$, $V_{98} = -0.12446585D+01$

b. For straight members

(1) For member 1-4

$w = 0.2$ k/ft., $L = 15'$

$M_{x14} = -M_{x41} = -0.37500000D+01$

$M_{y14} = M_{y41} = 0.00000000D+00$

$V_{14} = V_{41} = -0.15000000D+01$

(2) For member 4-7

$M_{x47} = -0.17777778D+02$, $M_{x74} = 0.88888889D+01$

$M_{y47} = 0.00000000D+00$, $M_{y74} = 0.00000000D+00$

$V_{47} = -0.53333333D+01$, $V_{74} = -0.26666667D+01$

II. Input Data

All data cards are placed in the following sequence.

(1). NJ, NJH, NJV card

9 3 3

(2). T(1) cards

0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.10591355D-01	0.11441773D-01	0.50178501D-03
0.13180462D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.10591355D-01	0.11441773D-01	0.50178501D-03
0.13180462D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.10591355D-01	0.11441773D-01	0.50178501D-03
0.13180462D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03
0.16004847D+00	0.81553492D-03	0.99356154D-02
0.10456840D-01	0.25927290D-03	0.82724363D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03
0.16004847D+00	0.81553492D-03	0.99356154D-02
0.10456840D-01	0.25927290D-03	0.82724363D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03

0.16004847D+00	0.81553492D-03	0.99356154D-02
0.10456840D-01	0.25927290D-03	0.82724363D-01

(3). T(2) cards

0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.90026514D-02	0.97255074D-02	0.36253967D-03
0.11203393D+00	0.27972848D-03	0.48684516D-02
0.73197880D-02	0.18149103D-03	0.57907054D-01
0.10591335D-01	0.11441773D-01	0.50178501D-03
0.13180462D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.10591355D-01	0.11441773D-01	0.50178501D-03
0.13180462D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.10591355D-01	0.11441773D-01	0.50178501D-03
0.13180463D+00	0.45549111D-03	0.67383412D-02
0.86115153D-02	0.21351886D-03	0.68125946D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03
0.16004847D+00	0.81553492D-03	0.99356154D-02
0.10456840D-01	0.25927290D-03	0.82724363D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03
0.16004847D+00	0.81553492D-03	0.99356154D-02

0.10456840D-01	0.25927290D-03	0.82724363D-01
0.12860931D-01	0.13893582D-01	0.73987687D-03
0.16004847D+00	0.81553492D-03	0.99356154D-02
0.10456840D-01	0.25927290D-03	0.82724363D-01

(4). <u>T(3) cards</u>

0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01

(5). <u>T(4) cards</u>

0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.26666667D+00	0.18320000D-01	0.35555556D-02	0.26666667D-01
0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00

(6). EI cards

0.33750000D+06	0.33750000D+06	0.00000000D+00	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.00000000D+00	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.00000000D+00	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.33750000D+06
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.00000000D+00
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.00000000D+00
0.33750000D+06	0.33750000D+06	0.33750000D+06	0.00000000D+00

(7). FX, FY, FZ cards

0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.37500000D+01	0.00000000D+00	-0.15000000D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.52179337D-01	0.26542496D+02	-0.30000000D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.52179337D-01	-0.26542496D+02	-0.30000000D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.45992725D-01	0.29664963D+02	-0.59341154D+01

0.37500000D+01	0.00000000D+00	-0.15000000D+01
-0.17777778D+02	0.00000000D+00	-0.53333333D+01
-0.45992725D-01	-0.29664963D+02	-0.59341154D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.88888889D+01	0.00000000D+00	-0.26666667D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.18467140D+00	0.27782471D+02	-0.67553415D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
-0.23947906D+00	-0.92500963D+01	-0.12446585D+01
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00
0.00000000D+00	0.00000000D+00	0.00000000D+00

(8). P cards

0.00000000+00	0.30000000D+01	0.40000000D+01

0.00000000D+00	0.20000000D+01	0.50000000D+01
0.10000000D+01	0.20000000+01	0.30000000D+01

III. Output Results

1. Displacements: with the units in radian or ft.

Joint	ϕx	ϕy	δ
1	-0.47828926D-02	-0.12888218D-01	0.31712200D+00
2	-0.12092830D-01	-0.10256663D-02	0.61136140D+00
3	-0.71891558D-02	0.13671305D-01	0.34428008D+00
4	-0.56712648D-02	-0.11416603D-01	0.24092814D+00
5	-0.11686115D-01	0.39097761D-03	0.43456842D+00
6	-0.68294385D-02	0.11000649D-01	0.23774336D+00
7	-0.67831387D-02	-0.82645940D-02	0.14445832D+00
8	-0.11508398D-01	-0.96066036D-04	0.26154970D+00
9	-0.63182620D-02	0.83700302D-02	0.13927615D+00

2. End Actions: with the units in ft.-k or k

JOINT NO. 1

	LEFT	RIGHT	UPPER	LOWER
M_x =	-0.18033920D+02	0.21706451D+02	0.0	-0.36726074D+01
M_y =	0.18042526D+02	-0.89433594D+01	0.0	-0.90989838D+01
V =	0.81770849D+01	-0.40223083D+01	0.0	-0.41547852D+01

JOINT NO. 2

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.69192352D+01	0.16173819D+02	0.0	-0.23093750D+02
M_y =	0.15340112D+03	-0.14464183D+03	0.0	-0.87591076D+01
V =	0.40223083D+01	0.83662415D+00	0.0	-0.18588867D+01

JOINT NO. 3

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.20110785D+02	-0.24592484D+02	0.0	0.44816895D+01
M_y =	0.86454923D+01	-0.25158157D+02	0.0	0.16512650D+02
V =	-0.68366241D+01	0.91599188D+01	0.0	0.16767578D+01

JOINT NO. 4

	LEFT	RIGHT	UPPER	LOWER
M_x =	-0.23556961D+02	0.24910672D+02	-0.36149414D+02	0.34795464D+02
M_y =	0.18159378D+02	-0.77696073D+01	0.90989838D+01	-0.19488876D+02
V =	0.10113491D+02	-0.96090879D+01	0.11547852D+01	-0.16591847D+01

JOINT NO. 5

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.81444522D+01	0.91062775D+01	-0.47890625D+01	-0.12459961D+02
M_y =	0.12088753D+03	-0.13265796D+03	0.87591076D+01	0.30113897D+01
V =	-0.22591429D+01	0.35283661D+01	0.18588867D+01	-0.11281738D+01

JOINT NO. 6

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.18727432D+02	-0.26630051D+02	0.20668945D+02	-0.12766357D+02
M_y =	0.25195068D+02	-0.24947510D+02	-0.16512651D+02	0.16265121D+02
V =	-0.35283661D+01	0.10373719D+02	-0.16767578D+01	-0.16865540D+00

JOINT NO. 7

	LEFT	RIGHT	UPPER	LOWER
M_x =	-0.32123550D+02	0.20695679D+02	0.11427951D+02	0.0
M_y =	0.61777849D+01	-0.25666504D+02	0.19488876D+02	0.0
V =	0.10353921D+02	-0.30131073D+01	-0.63408153D+01	0.0

JOINT NO. 8

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.20510559D+01	0.24114780D+01	-0.44626465D+01	0.0
M_y =	0.10333667D+03	-0.10032520D+03	-0.30113897D+01	0.0
V =	0.30131073D+01	-0.21412668D+01	0.11281738D+01	0.0

JOINT NO. 9

	LEFT	RIGHT	UPPER	LOWER
M_x =	0.21657585D+02	-0.31894165D+02	0.10236572D+02	0.0
M_y =	0.15504408D+01	0.14714861D+02	-0.16265121D+02	0.0
V =	-0.58587332D+01	0.86900930D+01	0.16865540D+00	0.0

DISCUSSIONS AND CONCLUSIONS

It is tedious to find the stiffness matrices of circular members by human labor and it is almost impossible to get an exact result. For this reason, a computer program for calculating the elements of stiffness matrices of circular members is presented in APPENDIX A.

When a straight member is subjected to transverse loading the resultant actions at any section are a bending moment and a shearing force. If, however, the member is curved in plan form, in which case it must be firmly held at the supports, there is a twisting moment in addition and this twisting action considerably complicates the arithmetical work of stress calculation. The formale for calculating the fixed-end actions of circular members due to concentrated load and distributed load are provided in APPENDIX B. The formulae for finding the fixed-end actions of straight members can be easily found in most text books of structural theory. Therefore, we omit the presentation of those formulae in this analysis (Pippard, 1952).

The torsion constant J for a member with circular cross-section is equal to the polar moment of inertia of the cross-section, $J = I_x + I_y$. For a member with rectangular cross-section, b x h ($b \leq h$), the value of J can be calculated by the formula $J = \gamma b^3 h$, where γ is a constant which is found by Saint-Venant. The values of γ are shown in APPENDIX C (Seely and Smith, 1952).

The general computer program provided in this thesis has been arranged for the solution of the illustrated example problem. For large problems the dimension statements should be modified for compatibility with the computer capacity being used.

For an ideal elastic material, the modulus of elasticity E and shearing modulus G are related with the Poisson's Ratio v of the material. The relation between E and G is $G = E/2(1+v)$.

In this analysis we assume that the displacements of the structure are small in comparison with its overall dimensions; in other words, the members are assumed not to change in length, even though they may be subjected to axial forces. The errors resulting from this assumption are very small, owing to the fact that member deformations in the transverse direction are many times larger than their axial deformations.

It is important that the final results be examined for their correctness before they are accepted for use. The end actions of members can be examined by making the usual statics check, which may be made by seeing that all joints and members are in equilibrium. The joint displacements may be roughly checked by visual inspection to see whether they are identical with the configuration and external loads of the structure.

Based on the results of this thesis, we draw the following conclusions:

1. This computer program will give a rapid and accurate result for a grid composed of straight and circular members due to loadings

perpendicular to the plane of the grid. For the computation of the above illustrated problem, it took only two or three minutes.

2. A high degree of accuracy is obtainable with this program. Numbers with eight effective digits were used in the calculation and resulted in answers that were accurate to five or six digits. See the statics check of Joint 2 and the elastic surface of the grid.

a. For illustration, the statics check of Joint 2 will be made. The same checks may be done for the remaining joints.

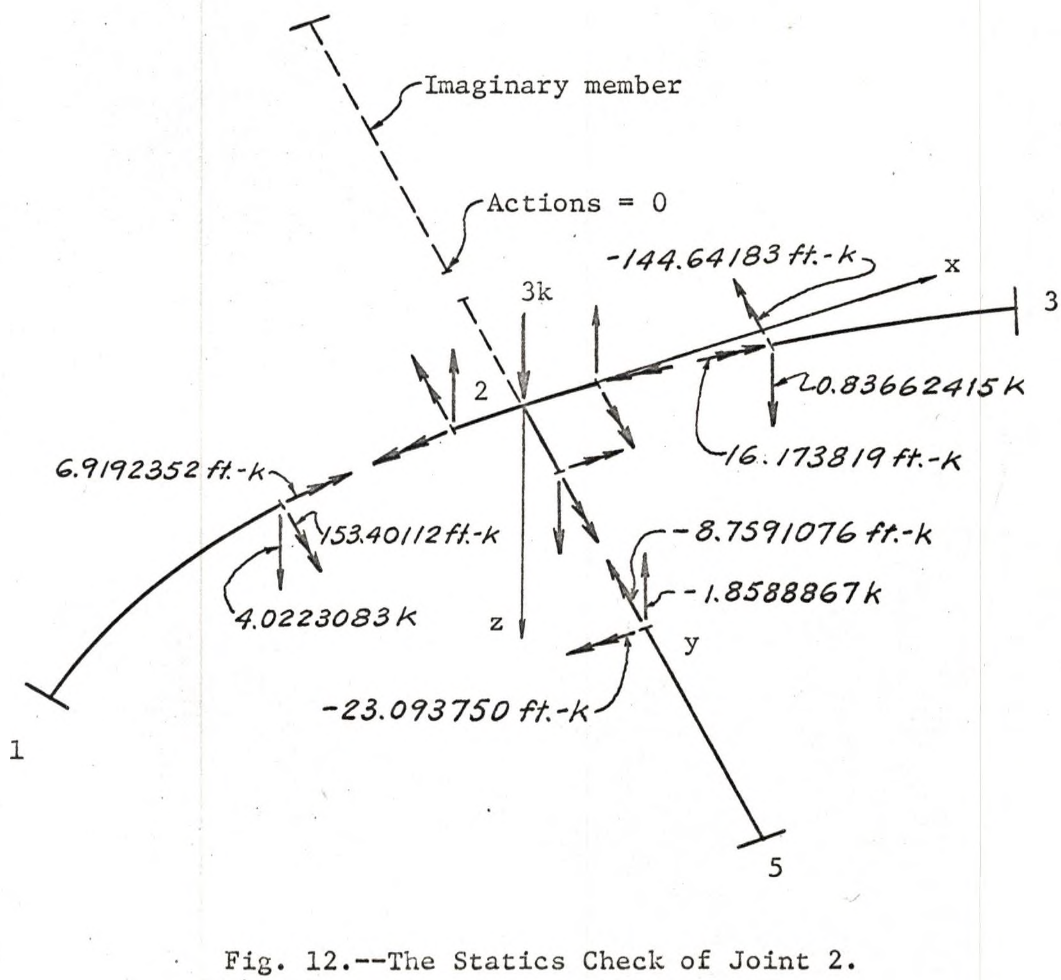

Fig. 12.--The Statics Check of Joint 2.

$\Sigma M_x = 0 - (6.9192352 + 16.173819 + 0 - 23.093750)$

$= 0.0006958 \doteqdot 0$ (OK)

$\Sigma M_y = 0 - (153.40112 - 144.64183 + 0 - 8.7591076)$

$= - 0.0001824 \doteqdot 0$ (OK)

$\Sigma V_z = 3 - (4.0223083 + 0.83662415 - 1.8588867)$

$= - 0.00004575 \doteqdot 0$ (OK)

b. Inspecting the elastic surface of the grid shown in Figure 13, it is indicated that the joint displacements are reasonably identical with the configuration and the external loads of the structure.

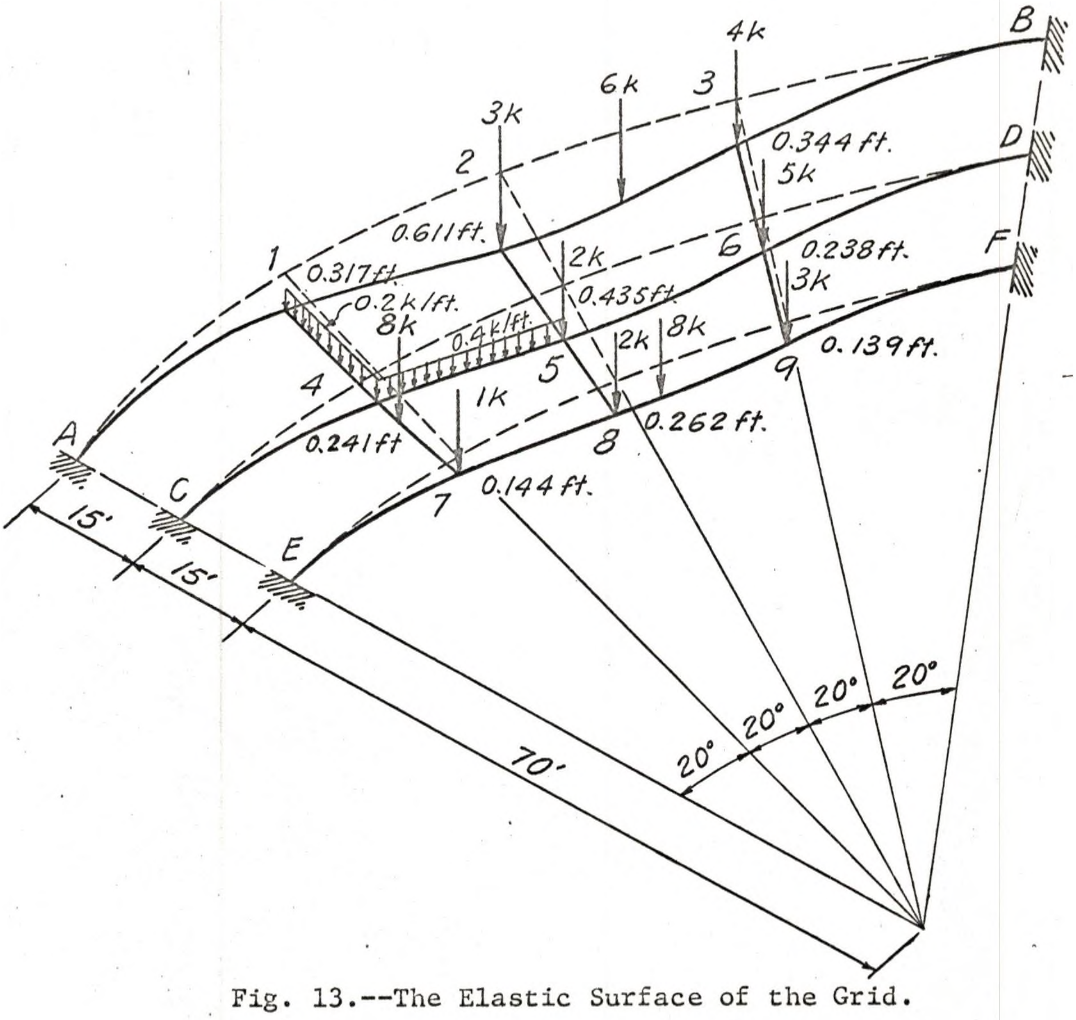

Fig. 13.--The Elastic Surface of the Grid.

APPENDICES

APPENDIX A

COMPUTER PROGRAM FOR CALCULATING THE ELEMENTS OF STIFFNESS MATRICES OF CIRCULAR MEMBERS

Based on Eqs. (6) to (16), a program is written in FORTRAN IV for IBM System, Model 40 Computer. This program calculates the elements of stiffness matrices of circular members with various subtending angles, radii and EI/GJ. The program consists of a main program and a subroutine. The main program is listed below and the IBM subroutine MINV is used in the main program for matrix inversion.

```
C                               MAIN PROGRAM
C***********************************************************************
C   THIS PROGRAM CALCULATES THE ELEMENTS OF STIFFNESS MATRICES
C   OF CIRCULAR MEMBERS
      DOUBLE PRECISION T(3,3),LLL(3,3),MMM(3,3),R(3),E(9,3)
C   B = SUBTENDING ANGLE OF CIRCULAR MEMBER
C   S1B = SIN B, S2B = SIN 2B, C1B = COS B, Q = EI/GJ
C   R(J) = RADII OF CIRCULAR MEMBERS
      READ(1,1) C1B,Q
    1 FORMAT(2D16.8)
      READ(1,2) B,S1B,S2B
    2 FORMAT(3D16.8)
      READ(1,2) (R(J),J=1,3)
      T(1,1)=(B/2.0-S2B/4.0)+Q*(B/2.0+S2B/4.0)
      T(1,2)=(1.0-Q)*(S1B**2)/2.0
      T(1,3)=(-B/2.0+S2B/4.0)-Q*(B/2.0+S2B/4.0-S1B)
      T(2.1)=T(1,2)
      T(2,2)=(B/2.0+S2B/4.0)+Q*(B/2.0-S2B/4.0)
      T(2,3)=-S1B**2/2.0+Q*(S1B**2/2.0+C1B-1.0)
      T(3,1)=T(1,3)
      T(3,2)=T(2,3)
      T(3,3)=(B/2.0-S2B/4.0)+Q*(3.0*B/2.0-2.0*S1B+S2B/4.0)
      CALL MINV (T,3,D,LLL,MMM)
```

```
      DO 10 J=1,3
      E(1,J)=T(1,1)/R(J)
      E(2,J)=T(1,2)/R(J)
      E(3,J)=T(1,3)/R(J)**2
      E(4,J)=T(2,2)/R(J)
      E(5,J)=T(3,3)/R(J)**3
      E(6,J)=T(2,3)/R(J)**2
      E(7,J)=((T(1,1)-T(1,3))*C1B-T(1,2)*S1B+T(1,3))/R(J)
      E(8,J)=((T(1,2)-T(2,3))*C1B-T(2,2)*S1B+T(2,3))/R(J)
      E(9,J)=((T(2,3)-T(1,2))*S1B-T(2,2)*C1B)/R(J)
   10 CONTINUE
      WRITE(3,5)
    5 FORMAT(2X,'THE ELEMENTS OF STIFFNESS MATRICES OF CIRCULAR
     MEMBERS ARE',//)
      WRITE(3,6) (R(J),J=1,3)
    6 FORMAT(8X,'R=',I3,11X,'R=',I3,11X,'R=',I3,/)
      WRITE(3,7) ((E(I,J),J=1,3),I=1,9)
    7 FORMAT(2X,3D16.8,/)
      STOP
      END
```

APPENDIX B

FIXED END ACTIONS FOR CIRCULAR MEMBERS

The Circular Member with A Concentrated Load

Figure 14, (a), shows a circular member lying in a horizontal plane and built in at both ends. Such a member subtending an angle β and carrying a single concentrated load P at an angular distance θ from the mid-point C. The problem can be solved by using the method of superposition and dividing the loading into a symmetrical and a skew-symmetrical system as shown at (b) and (c) in Figure 14 (Pippard, 1952).

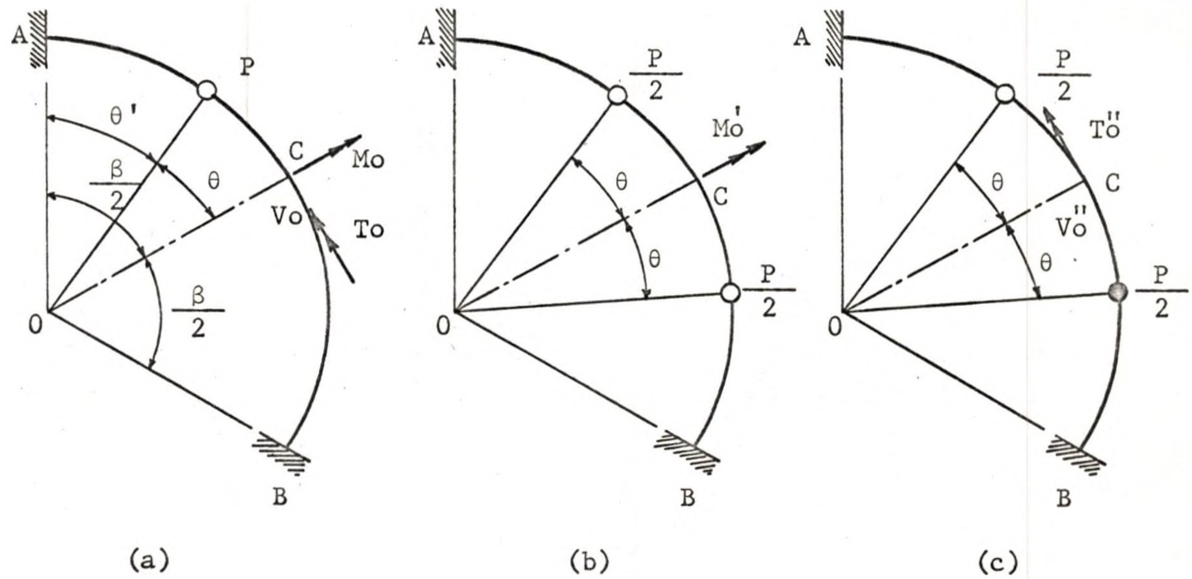

Fig. 14.--The Circular Member with a Concentrated Load

In the symmetrical system (b), equal loads $P/2$ act downward at the same distance θ from the center line OC and the resultant actions at the center reduce simply to a bending moment M_o'; both the twisting moment T_o' and the shearing force V_o' are zero. In the skew symmetrical system (c), a load $P/2$ acts downward at θ from OC in the segment CA and a load acts upward at θ from OC in the segment CB. The resultant actions at C are a twisting moment T_o'' and a shearing force V_o''; there is no bending moment.

For sign conventions of member end actions, bending moment M will be taken as positive when it produces convexity of the beam upwards, twisting moment T will be taken as positive when it produces clockwise rotation of a section viewed from the free end and shearing force V will be taken as positive when it acts downward.

Based on the assumptions mentioned above, the following relations are developed:

$$\frac{M_o'}{PR} = \frac{(k+1)\theta'\sin\theta+(k-1)\sin(\beta/2)\sin\theta'-2k[\cos\theta-\cos(\beta/2)]}{(k+1)\beta-(k-1)\sin\beta}$$

$$\frac{T_o''}{PR/2} = \{\theta'-\sin\theta'-\frac{V_o''}{P/2}[\beta/2-\sin(\beta/2)]\}\operatorname{cosec}(\beta/2)$$

$$\frac{V_o''}{P/2} = 1+$$

$$\frac{2(k+1)[\theta'\cos\theta\sin(\beta/2)-\beta\sin\theta'/2-\theta\beta/2]+4k\sin\theta\sin(\beta/2)-(k-1)\theta\sin\beta}{(k+1)\beta^2/2+(k-1)\beta\sin\beta/2-2k(1-\cos\beta)}$$

The total resultant actions are

$$M_A=M_o'\cos(\beta/2)+T_o''\sin(\beta/2)-V_o''R\sin(\beta/2)+PR\sin\theta'$$

$$T_A=-M_o'\sin(\beta/2)+T_o''\cos(\beta/2)+V_o''R[1-\cos(\beta/2)]-PR(1-\cos\theta')$$

$$V_A=V_o''-P$$

$M_B = M_o' \cos(\beta/2) - T_o'' \sin(\beta/2) + V_o'' R \sin(\beta/2)$

$T_B = M_o' \sin(\beta/2) + T_o'' \cos\ (\beta/2) + V_o'' R[1-\cos(\beta/2)]$

$V_B = -V_o''$

<u>The Circular Member with Distributed Load</u>

The circular member shown in Figure 15 carries a uniformly distributed load of intensity w over the whole length of the member.

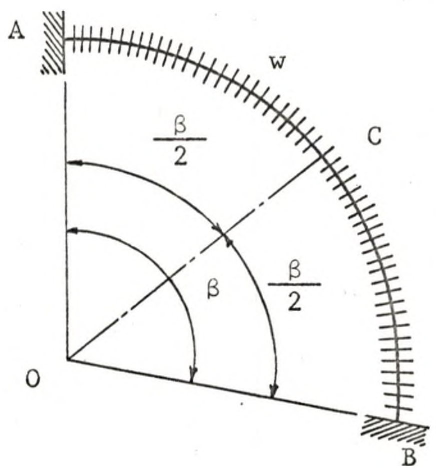

Fig. 15.--The Circular Member with Distributed Load

By using the same procedure and sign conventions as the previous analysis for the circular member with a concentrated load, we obtain the resultant actions as follows (Pippard, 1952).

$$M_A = M_B = -wR^2 \left\{ \frac{4\cos(\beta/2)[(k+1)\sin(\beta/2) - k\beta\cos(\beta/2)/2]}{(k+1)\beta - (k-1)\sin\beta} - 1 \right\}$$

$$T_A = -T_B = wR^2 \left\{ \frac{4\sin(\beta/2)[(k+1)\sin(\beta/2) - k\beta\cos(\beta/2)/2]}{(k+1)\beta - (k-1)\sin\beta} - \beta/2 \right\}$$

$V_A = V_B = -wR\beta/2$

APPENDIX C

TABLE 1

DATA FOR THE TORSION OF A MEMBER WITH RECTANGULAR CROSS-SECTION*

h/b	γ
1.0	0.141
1.5	0.196
2.0	0.229
2.5	0.249
3.0	0.263
4.0	0.281
6.0	0.299
10.0	0.312
∞	0.333

*Torsion constant J for a member with rectangular cross-section, b x h ($b \leq h$), is calculated by the formula $J = \gamma b^3 h$ (Seely and Smith, 1952).

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