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Development of Procedures for Determination of Yield Lines

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DEVELOPMENT OF PROCEDURES FOR DETERMINATION OF YIELD LINES

by

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ABSTRACT

This thesis presents methods for determining yield line patterns of isotropic plates that are uniform in thickness and subjected to a uniform loading. The importance of these methods arises from the mathematical complications involved in finding yield line patterns by the present procedures.

Since yielding patterns are dependent on the shape of a plate as well as the support conditions, the methods recommended here will be based on assuming a yielding pattern and checking for its correctness. The use of the computer can provide many trial solutions in a very short time. The correct pattern will be that which gives the same value of the yielding moment along all yield lines.

The benefit of the recommended methods is that no mathematical complications are involved and a very elementary knowledge of computer programming is sufficient.

INTRODUCTION

Safety and economy are two criteria that a structural engineer strives to satisfy. These two criteria were, and are, the basis for most changes that occur in design codes. Not long ago, the basis for design was the Elastic Method or what is known as Working Strength Design (WSD). Today, this method is almost completely replaced by a new one, namely the Ultimate Strength Design (USD).

Working Strength Design

The Working Strength Design method uses the elastic behavior of the material as a basis for design. In this method the yield strength of the material is reduced to an allowable working value and the design is carried out on the basis of that value. This presents some problems to the engineers. The factor of safety against failure is not really defined. The method assumes an elastic stress condition but does not allow a solution for loading on a plate or a beam that produces a non-elastic stress distribution. A more serious limitation is in the analysis of plates that have irregular shapes. The present design codes list coefficients for the purpose of analyzing regularly shaped plates or slabs. The analysis is not quite that easy in the case of plates having irregular shapes and various support conditions.

Ultimate Strength Design

This is one of the accepted design procedures at the present time in the area of reinforced concrete. In this method design is carried out on the basis of an ultimate load which is equal to the working load multiplied by a load factor. The strength of the material used is that of the yielding strength.

For purposes of safety, the load factor is subdivided into two factors, namely the overload factor (U) and the undercapacity factor (ϕ). To be certain that the loading on a structure is not underestimated, the overload factor is applied. The American Concrete Institute Code [1] of 1963 specifies that when wind and earthquake loading are not critical, (U) can be computed by the following equation:

$$U = 1.5 \text{ dead load} + 1.8 \text{ live load}$$

To correct for errors in the quality of a material, quality of workmanship, accuracy of calculations and other approximations, the undercapacity factor (ϕ) is applied. This factor takes on different values depending on the function a member serves. Knowing U and ϕ, the load factor is then computed as U ϕ.

The Ultimate Strength Design applies for two dimensional members; however, in the case of slabs or plates, the Yield Line Theory is applied.

Yield Line Theory

The Yield Line Theory for slabs or plates is a relatively new concept of analysis. Even though test results show that the concept is an accurate one, the United States design codes have not yet adopted it. Up to this date most of the literature concerning the Yield Line Theory is still in foreign languages since most of the pioneers in this area, such as K. W. Johansen, are Europeans.

The basic concept of the Yield Line Theory states that failure does not occur until a mechanism is formed. Consider a fixed end beam subjected to a uniform loading that causes the end portions to reach their plastic moment. This does not signify failure. The beam will continue to carry an additional loading until plastic moment in the middle is reached, thus forming a mechanism. In the case of a plate, the same concept is involved. When a plate is subjected to some loading, various points will have different stresses. The addition of loading that causes some points to reach their yielding stress does not signify failure. The plate continues to take more loading until more adjacent points reach their yielding strength to form a mechanism. In joining the points that form the mechanism, the yield line pattern is obtained. The Yield Line Theory is only concerned with bending and completely ignores deflection and shear.

Purpose of Thesis

The purpose of this thesis is the development of methods by which the location of yield lines for plates of various shapes and support conditions can be determined. All plates will have a uniform thickness and will be acted upon by uniformly distributed loads. It is quite easy to determine the yield lines for symmetrical plates. However, this is not true in the case of non-symmetrical plates. It is hoped that with the information contained in this thesis, one could apply this theory to determine the yield lines of any plate.

SIGN CONVENTION

In this thesis, the sign of an angle of rotation will be determined by the right hand rule. This angle will be represented by a single headed arrow (——→) as shown in Figure 1.

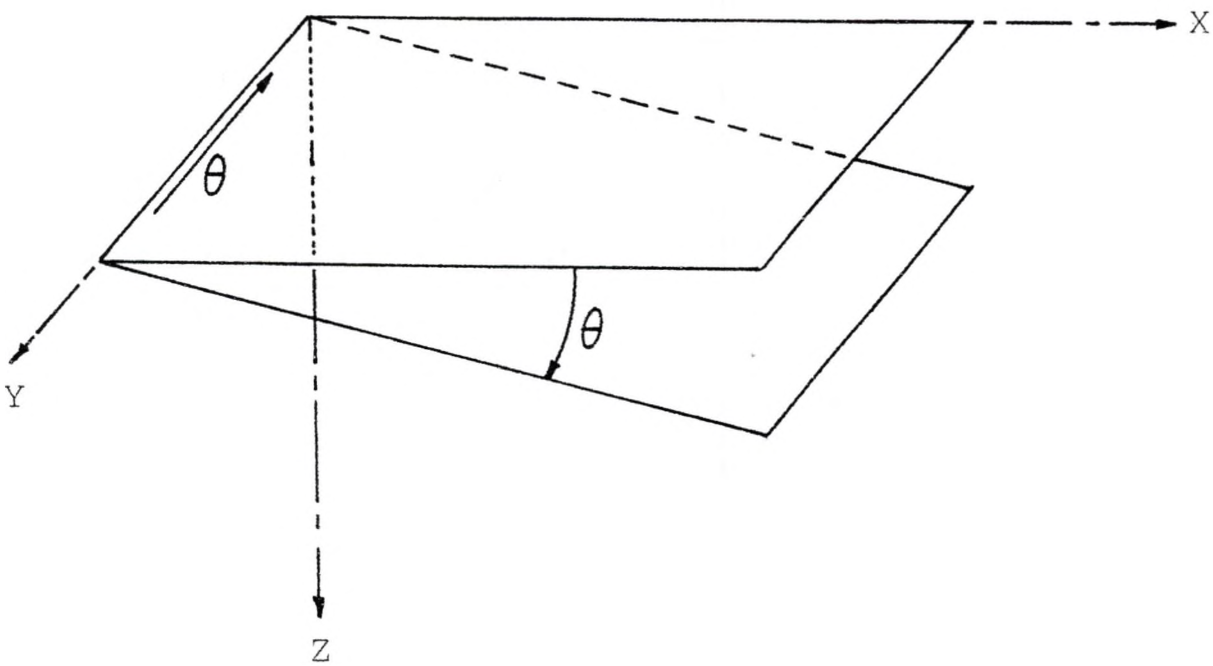


Fig. 1.--A Negative Angle of Rotation

A positive moment is a moment that tends to produce compressive stresses on the top fibers of a plate. Each moment will be represented by double headed arrows (——→→) placed along a yield line. An arrow lying in a region represents a moment acting on that region. The right hand rule will be used to establish the sense of that moment. Figure 2 illustrates a positive yielding moment acting along a yield line.

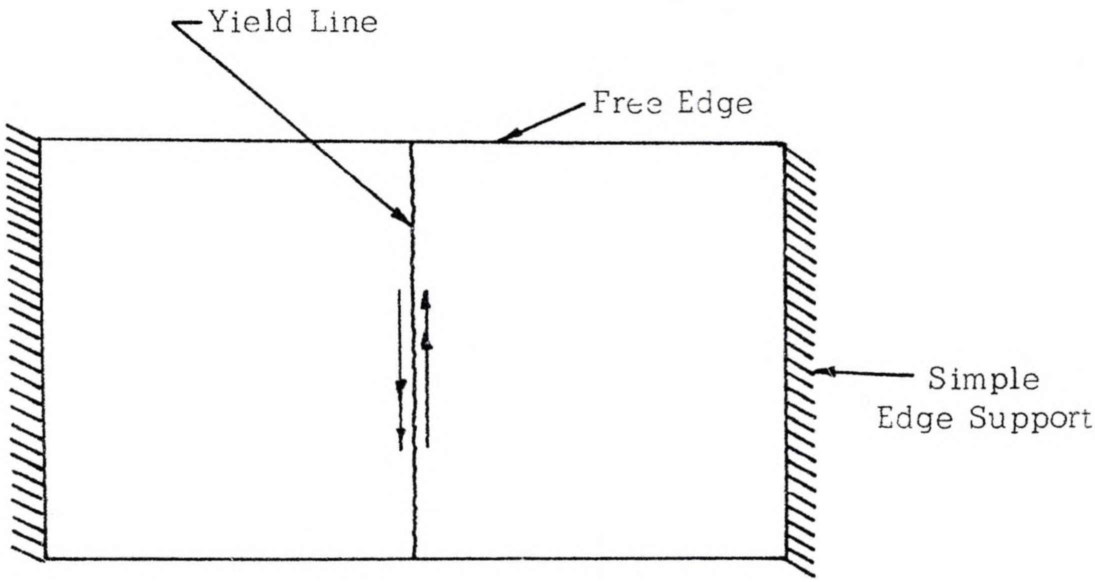


Fig. 2.--A Positive Yielding Moment

RULES FOR DETERMINING YIELD LINES

In discussing this section, it is essential to define the terms, an axis of rotation and a rigid region.

An Axis of Rotation

An axis of rotation is a line about which a portion of a plate rotates. For a plate that is simply supported along an edge, that edge will serve as an axis of rotation whose direction is well determined. For a plate that is supported on a column, the axis of rotation passes over the column but its direction is not known. Figure 3 illustrates the above mentioned cases of axes of rotation.

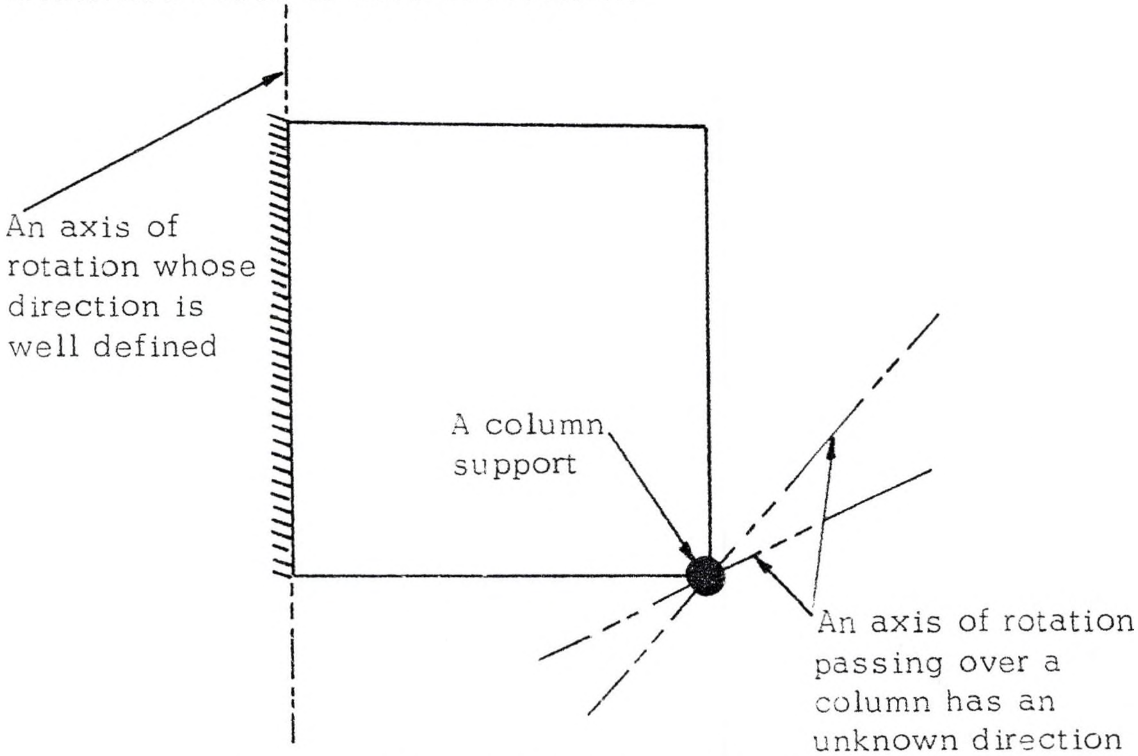


Fig. 3.--Cases of Axes of Rotation

A Rigid Region

Failure in a plate is characterized by the appearance of yield lines in some pattern. Figure 4 shows a square plate, simply supported along four edges, with the yield lines dividing it into regions A, B, C, and D.

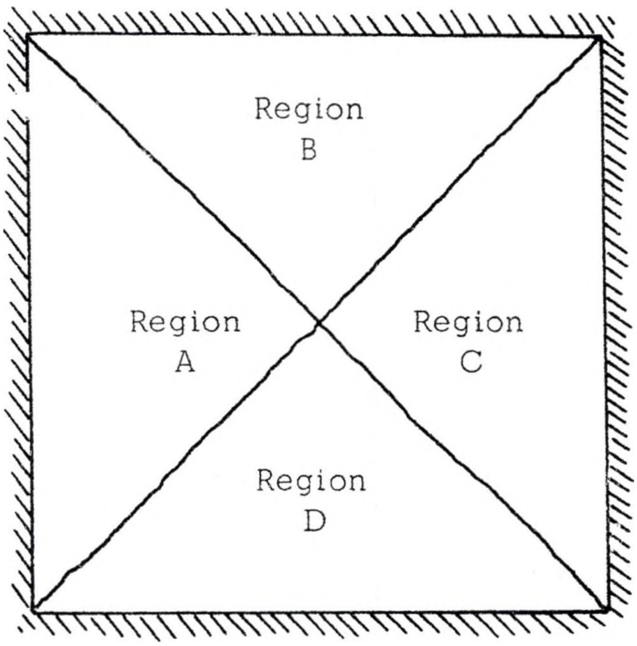


Fig. 4.--A Plate Divided into Four Regions

The deformation in each region of the plate shown in the above figure is elastic, and for all practical purposes each of these regions will be considered as a plane rigid region, thus ignoring all elastic deformations.

Johansen's Rules

The following two rules have been presented by K. W. Johansen [2] for determining yield lines:

(a) A yield line between two regions will pass through the intersection of their axes of rotation.

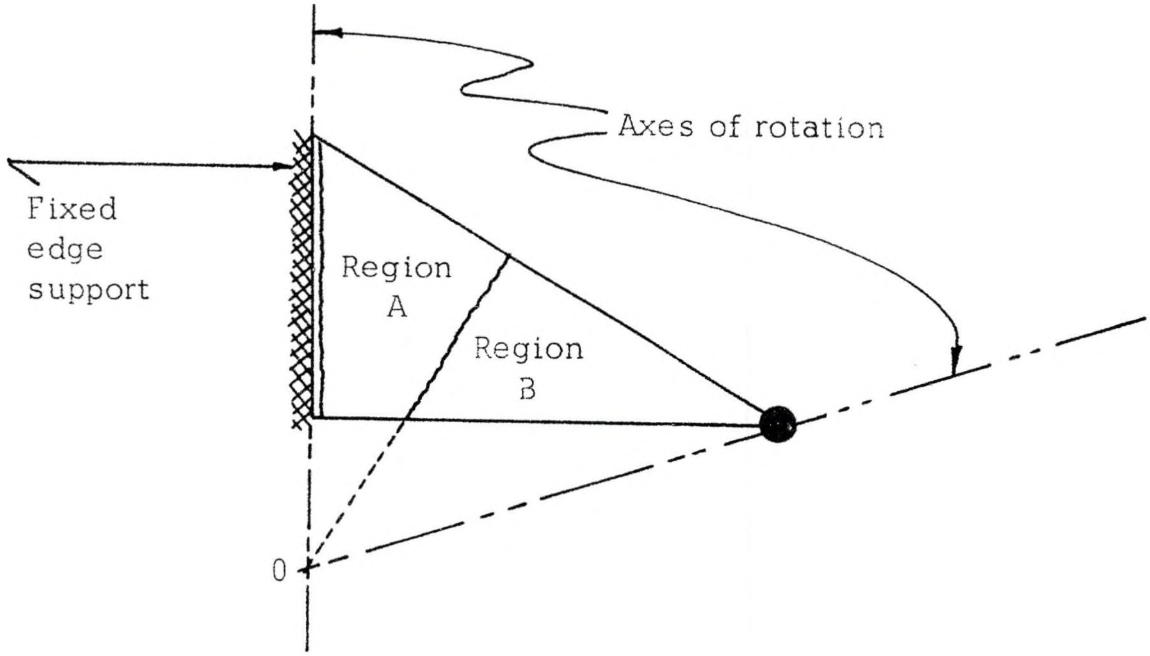


Fig. 5.--Johansen's Rule (a)

(b) The yield line pattern is determined by the axes of rotation of the various regions of a plate and the ratios between their rotations.

To prove this rule, consider a plate similar to that shown in Figure 6.

Let plane A'B'C'D' be passed at a distance of Δ from the undeflected surface of the plate. This plane, parallel to the original position of the plate, will cut regions I, II, III, and IV in lines A'B', B'C', C'D' and D'A'. These lines will be consecutively parallel to the axes of rotation of the regions. Let X_1 be the distance between AB and A'B', and let θ_1 be the angle that region I rotates about AB.

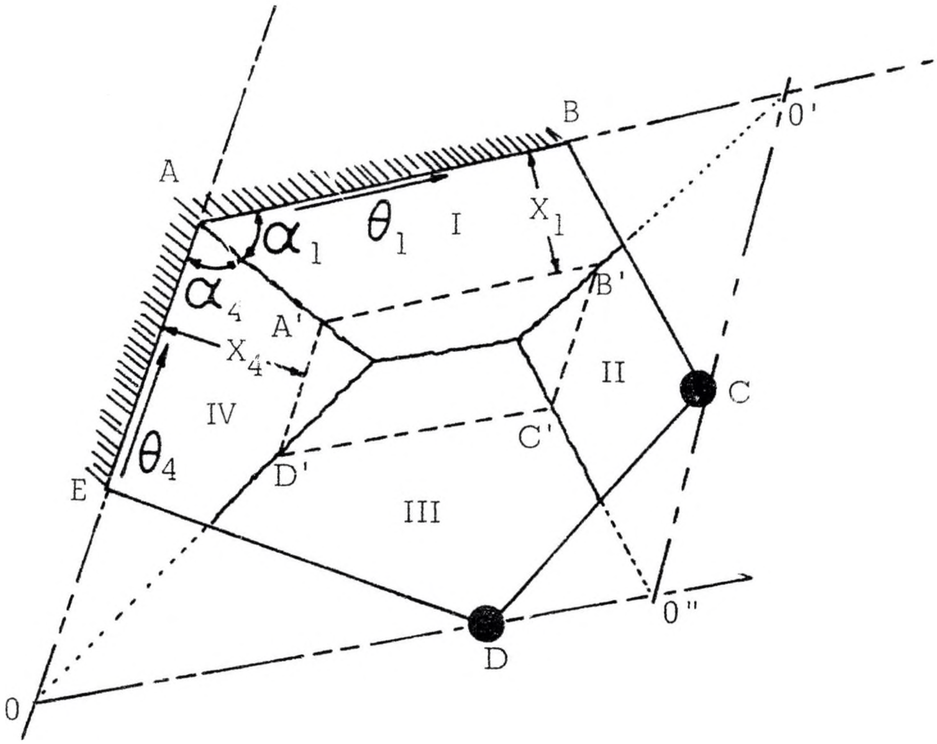


Fig. 6.--Johansen's Rule (b)

$$X_1 = \frac{\Delta}{\theta_1}$$

$$AA' = \frac{\Delta}{\theta_1 \sin\alpha_1} \qquad (1)$$

Let θ_4 be the angle that region IV rotates about AE, and let X_4 be the distance between AE and A'D'.

$$X_4 = \frac{\Delta}{\theta_4}$$

$$AA' = \frac{\Delta}{\theta_4 \sin\alpha_4} \qquad (2)$$

By equating equation 1 to equation 2 the following will be obtained.

$$\frac{\Delta}{\theta_4 \sin\alpha_4} = \frac{\Delta}{\theta_1 \sin\alpha_1}$$

$$\sin\alpha_4 = \frac{\theta_1}{\theta_4} \sin\alpha_1 \qquad (3)$$

Equation 3 shows that the direction of the yield line between regions I and IV is dependent on the rotations θ_1 and θ_4.

Similarly, it can be proven that an identical relation holds true for all regions.

With these two rules one can determine the apparent locations of the yield lines.

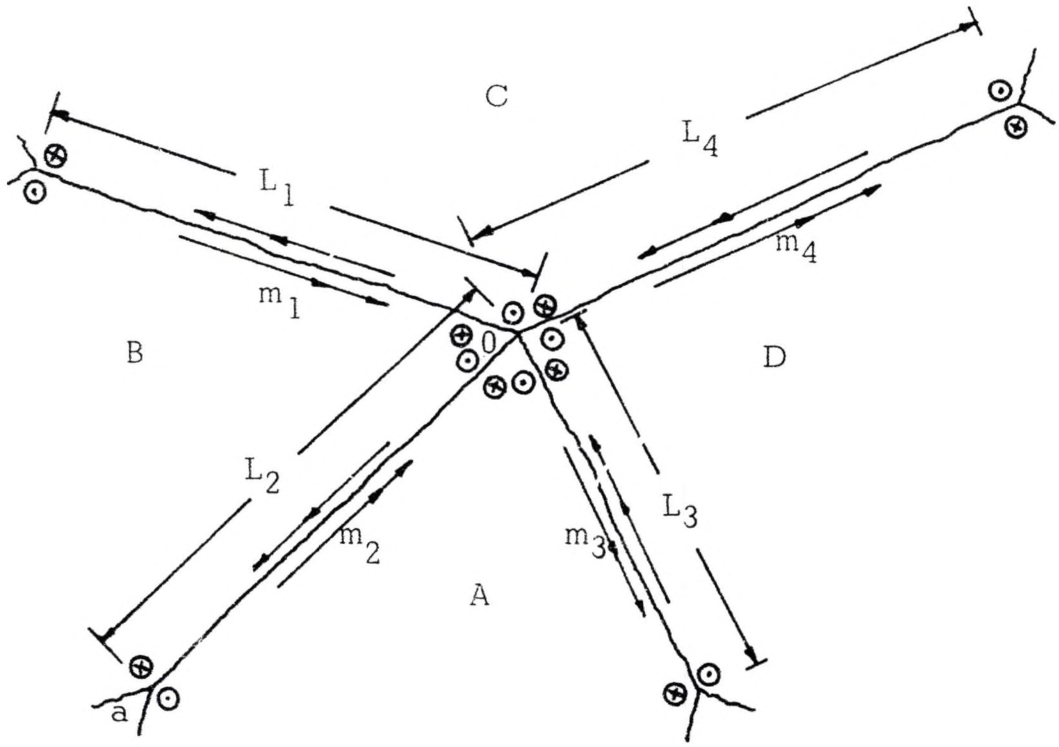
NODAL FORCES

In determining yield line patterns, equilibrium must exist in all parts of a plate that are bounded by yield lines. The distribution of shearing stresses along yield lines is of no concern in this thesis. For that reason, these stresses can be replaced by two forces that are equal in magnitude, opposite in direction, and acting at the ends of a yield line. It must be noted that these forces will cause a torsional moment that will induce torsional stresses along each yield line. However, the main concern here is not the state of stress at various points but the equilibrium condition of each region of a plate.

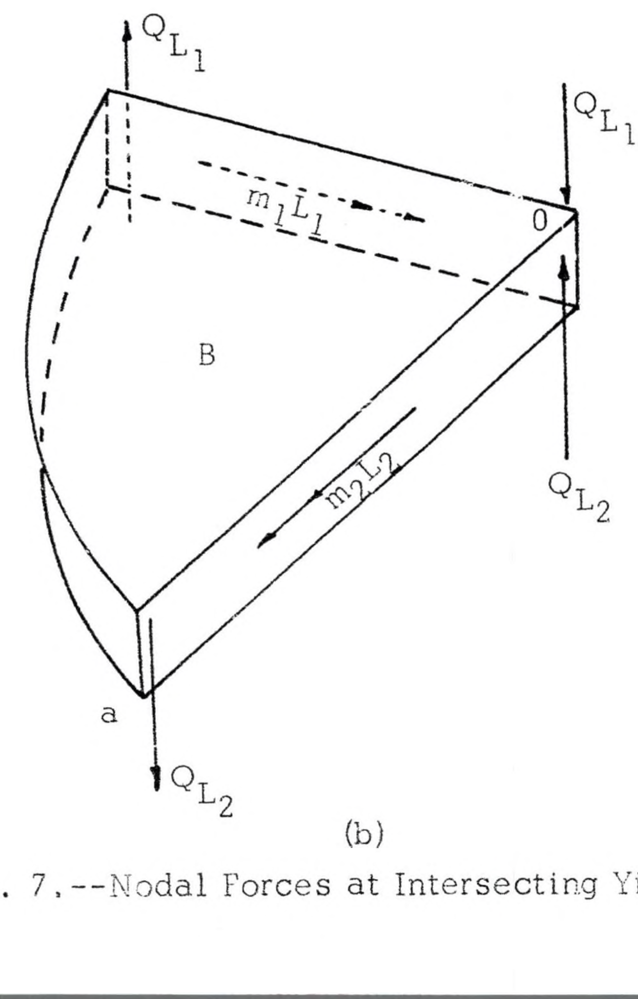
Nodal Forces at Intersecting Yield Lines

Figure 7 illustrates a case of four yield lines intersecting at point 0. Let m_1, m_2, m_3 and m_4 represent moments per unit length along L_1, L_2, L_3 and L_4, respectively. The symbol, $\odot$, represents a force acting upward while the symbol, $\otimes$, is used to represent a force acting downward.

Examination of Figure 7 (b) indicates that at point 0 there exists two forces, Q_{L_1} and Q_{L_2}, that are parallel, unequal and directionally opposite. Let Q_B represent the resultant of these two forces (i.e.,



(a)



(b)

Fig. 7.--Nodal Forces at Intersecting Yield Lines

$Q_B = Q_{L_2} - Q_{L_1}$). The same argument holds true for the other regions.

Thus:

$$Q_A = Q_{L_3} - Q_{L_2}$$

$$Q_B = Q_{L_2} - Q_{L_1}$$

$$Q_C = Q_{L_1} - Q_{L_4}$$

$$Q_D = Q_{L_4} - Q_{L_3}$$

But

$$Q_A + Q_B + Q_C + Q_D = 0$$

This proves that there can be an infinite number of yield lines intersecting at one point.

Magnitude of Nodal Forces

Figure 8 shows a small element of region B that is adjacent to region A. A free body diagram of this element is shown in Figure 9.

An assumption is made here that the element is small enough that dm_2 may be considered negligibly small. The summation of moments about a0' yields the following:

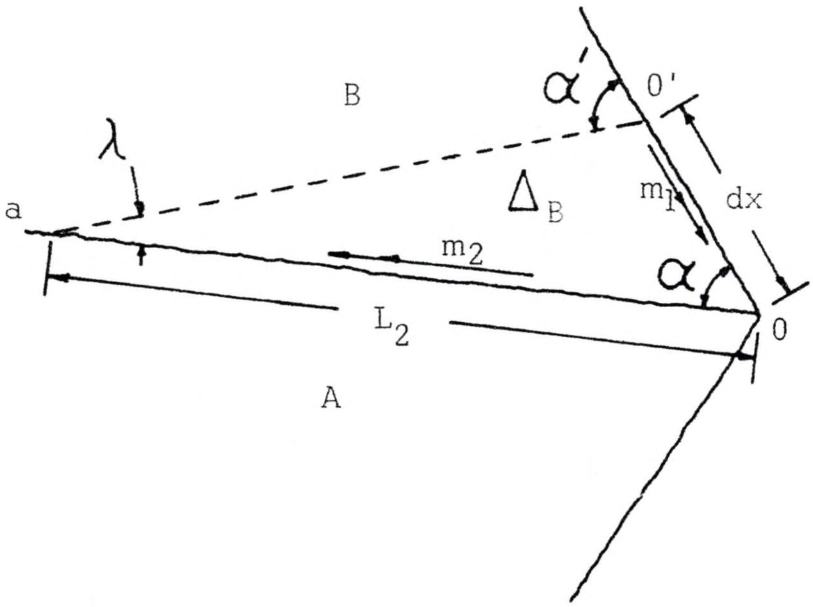


Fig. 8.--A Small Element Taken from Region B

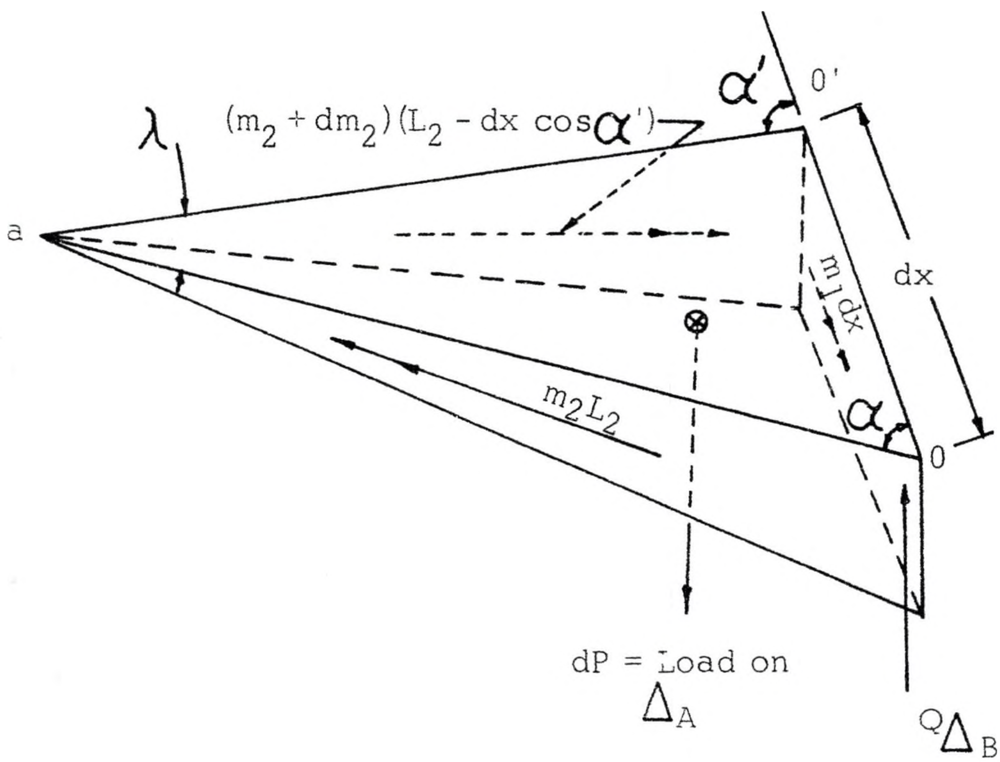


Fig. 9.--Free Body Diagram of the Element Shown in Figure 8

$$m_2L_2 \cos\lambda + Q_{\Delta_B} \, dx \sin\alpha' - m_1 dx \cos\alpha' -$$

$$(m_2L_2 - m_2 dx \cos\alpha') - dP \frac{dx \sin\alpha'}{3} = 0 \qquad (4)$$

The term dP is small enough that the value of $dP \frac{dx \sin\alpha'}{3}$ can be neglected. As 0' approaches 0, α' approaches α, λ approaches zero and equation 4 becomes:

$$m_2L_2 + Q_{\Delta_B} \, dx \sin\alpha - m_1 \, dx \cos\alpha - m_2L_2 +$$
$$m_2 \, dx \cos\alpha = 0$$

$$Q_{\Delta_B} = (m_1 - m_2) \frac{\cos\alpha}{\sin\alpha}$$

$$Q_{\Delta_B} = (m_1 - m_2) \cot\alpha \qquad (5)$$

Equation 5 gives direct computation for the nodal force that acts on a region at the intersection of two yield lines. This force is equal to the difference of moments along these yield lines multiplied by the cotangent of the angle between them.

Cases of Nodal Forces at Intersecting Yield Lines

Having obtained an expression for the nodal forces, various cases of intersecting yield lines will be examined.

(a) Three Yield Lines of the Same Sign Intersecting at a Point

Figure 10 shows three yield lines of the same sign intersecting at point 0.

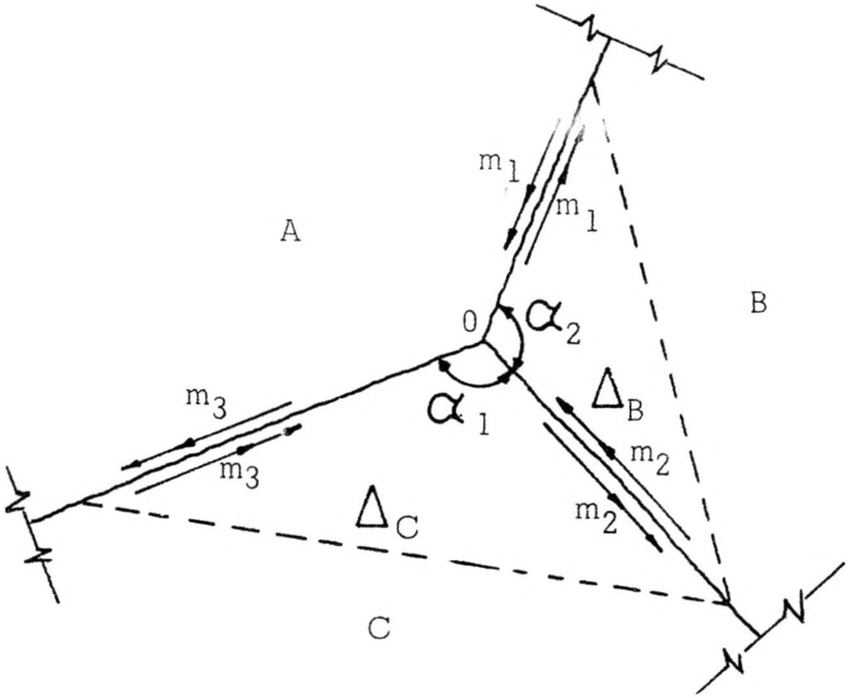


Fig. 10--Case (a) of Intersecting Yield Lines

Two segments, Δ_B and Δ_C, are shown in the above figure. In an isotropic plate that has uniform thickness, the moments per foot of length along yield lines of the same sign are equal in magnitude. Hence:

$$m_1 = m_2 = m_3$$

Equation 5 yields:

$$Q\Delta_C = 0 \cot \alpha_1 = 0$$

$$Q\Delta_B = 0 \cot \alpha_2 = 0$$

But $Q_A + Q\Delta_C + Q\Delta_B$ must be equal to zero to satisfy the condition of equilibrium. This would make $Q_A = 0$. Using the same analysis, it can be shown that $Q_B = Q_C = Q_A = 0$. Thus, the following can be concluded: In an isotropic plate the nodal forces at the intersection of yield lines of the same sign are all equal to zero.

(b) Two Yield Lines of the Same Sign Intersecting a Third One of a Different Sign

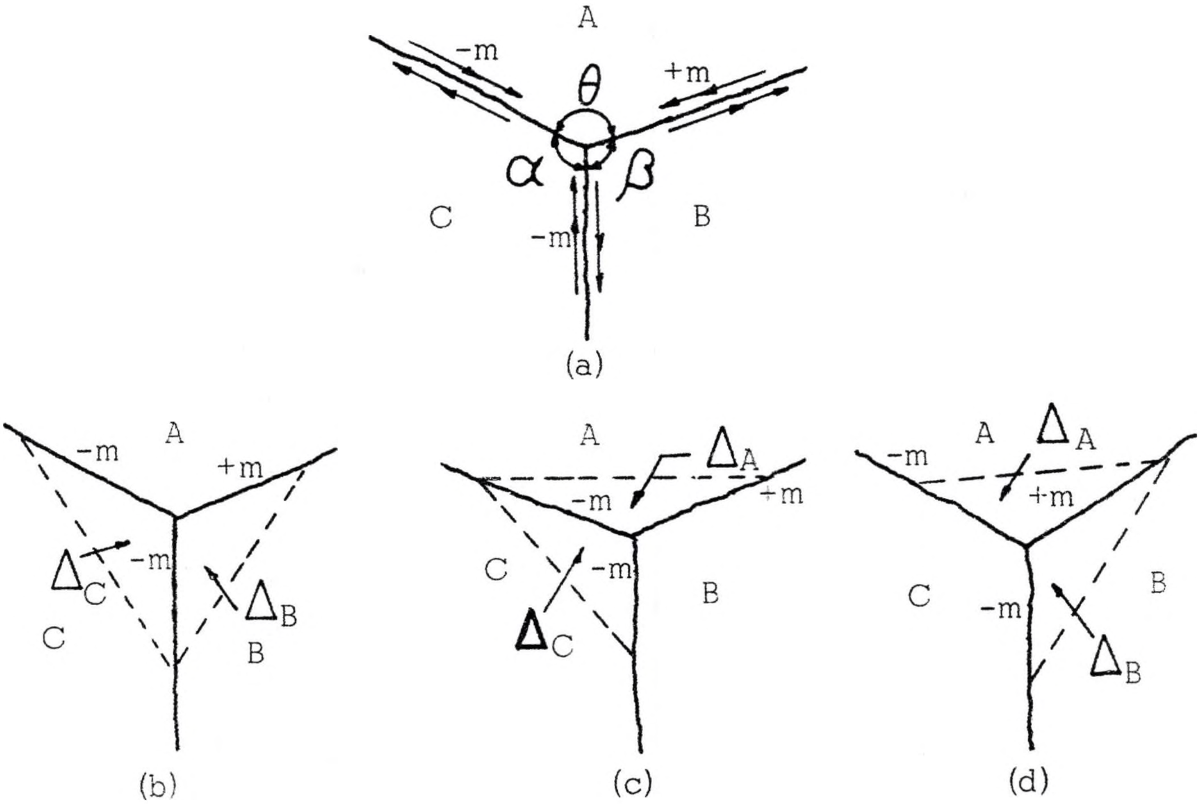


Fig. 11.--Case (b) of Intersecting Yield Lines

To evaluate Q_A use Figure 11 (b).

$$Q_{\Delta_C} = (-m + m)\cot\alpha = 0$$

$$Q\Delta_B = (m + m) \cot\beta = 2m \cot\beta$$

But

$$Q_A + Q\Delta_C + Q\Delta_B = 0$$

Therefore:

$$Q_A = -Q\Delta_C - Q\Delta_B = -2m \cot\beta$$

To evaluate Q_B refer to Figure 11 (c).

$$Q\Delta_C = 0$$

$$Q\Delta_A = 2m \cot\theta$$

But

$$Q_B = -Q\Delta_C - Q\Delta_A$$

Therefore:

$$Q_B = -2m \cot\theta$$

For evaluating Q_C refer to Figure 11 (d).

$$Q\Delta_A = -2m \cot\theta$$

$$Q\Delta_B = -2m \cot\beta$$

Therefore:

$$Q_C = 2m\,(\cot\theta + \cot\beta)$$

If the summation of Q_A, Q_B and Q_C is equal to zero, then the case of two yield lines of the same sign intersecting a third one of a different sign is possible.

$$Q_A + Q_B + Q_C = 2m\,(\cot\theta + \cot\beta) - 2m\cot\beta - 2m\cot\theta = 0$$

Therefore, the mentioned case is possible. To generalize, the following can be stated: It is possible to have two yield lines of the same sign intersecting a third one of a different sign.

(c) <u>Intersection of Three Yield Lines of the Same Sign with a Fourth of a Different Sign</u>

To evaluate Q_B refer to Figure 12 (b).

$$Q_{\Delta C} = 0\tan\theta = 0$$

$$Q_{\Delta A} = 0\tan(\alpha + \gamma) = 0$$

But

$$Q_{\Delta C} + Q_{\Delta A} + Q_B = 0$$

Therefore:

$$Q_B = 0$$

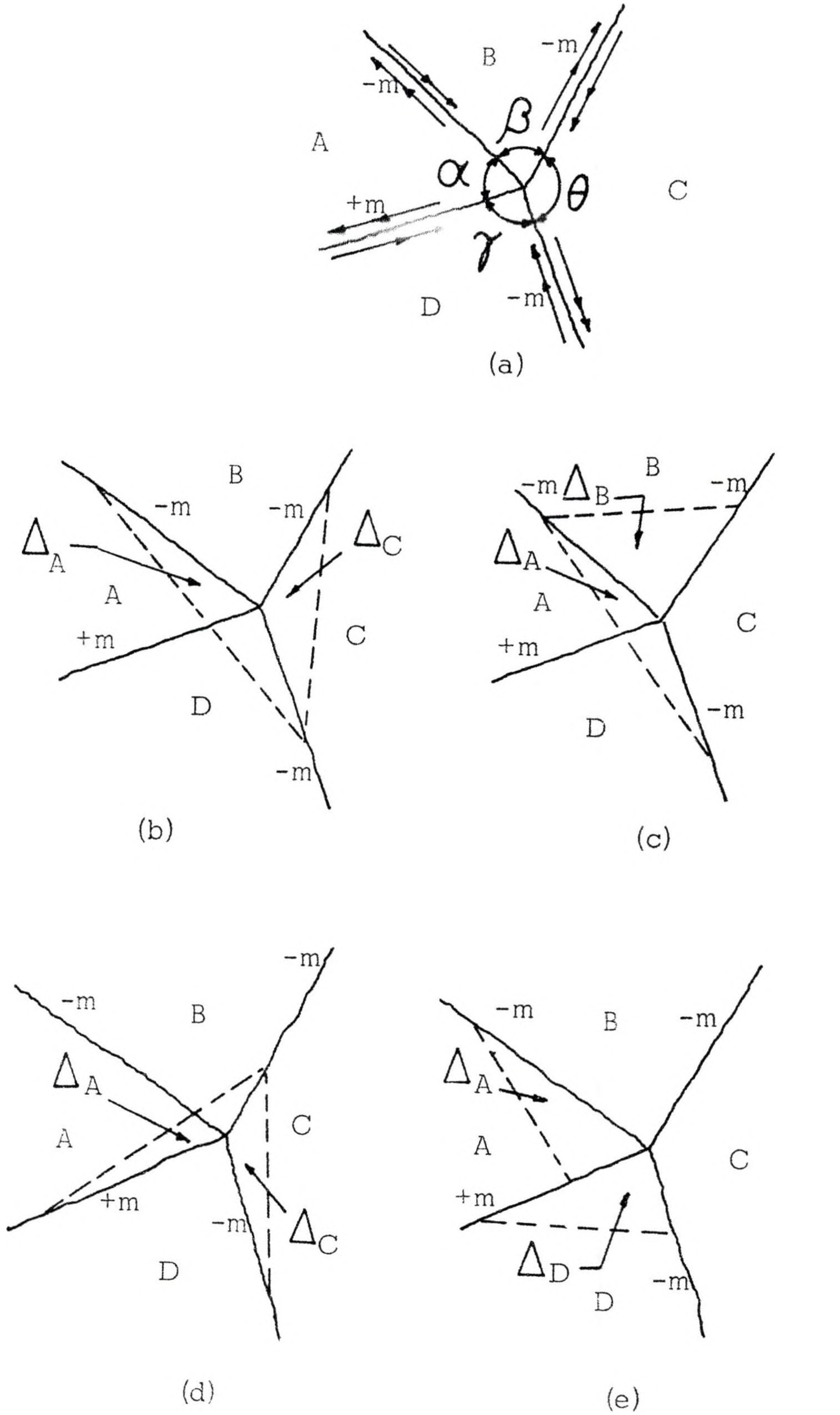


Fig. 12.--Case (c) of Intersecting Yield Lines

To evaluate Q_C refer to Figure 12 (c).

$$Q_{\Delta_B} = 0 \tan\beta = 0$$

$$Q_{\Delta_D} = 0 \tan(\alpha+\gamma) = 0$$

Therefore:

$$Q_C = -Q_{\Delta_B} - Q_{\Delta_D} = 0$$

$$Q_C = 0$$

To evaluate Q_D refer to Figure 12 (d)

$$Q_{\Delta_A} = -2m \cot(\alpha+\beta)$$

$$Q_{\Delta_C} = (-m + m) \cot\theta = 0$$

Therefore:

$$Q_D = 2m \cot(\alpha+\beta)$$

To evaluate Q_A refer to Figure 12 (e)

$$Q_{\Delta_A} = (-m - m) \cot(\alpha) = -2m \cot\alpha$$

$$Q_{\Delta_D} = (0) \cot(\gamma) = 0$$

Therefore:

$$Q_A = Q_{\Delta_A} - Q_{\Delta_D} = -2m \cot \alpha$$

To satisfy the equilibrium condition:

$$Q_A + Q_B + Q_C + Q_D = 0 \text{ or } Q_A + Q_D = 0$$

Therefore:

$$(2m)\,[\cot(\alpha + \beta) - \cot \alpha] = 0$$

If $\beta \neq 0$, then m = 0. This shows that the yield line between A and D cannot exist. If there was a yield line between regions A and D, it would have to be of a different sign. This leads to the conclusion that when there are yield lines of one sign in at least three directions, no yield line of opposite sign can intersect them. Conversely, it can be stated that when yield lines of different signs intersect, they cannot radiate in more than three directions.

Nodal Forces at a Free Edge

Equation 5 gives direct computation for the nodal force at a free edge. Referring to Figure 13:

$$Q_{\Delta_A} = (0 - m) \cot \alpha = -m \cot \alpha$$

But

$$Q_{\Delta_A} + Q_B = 0$$

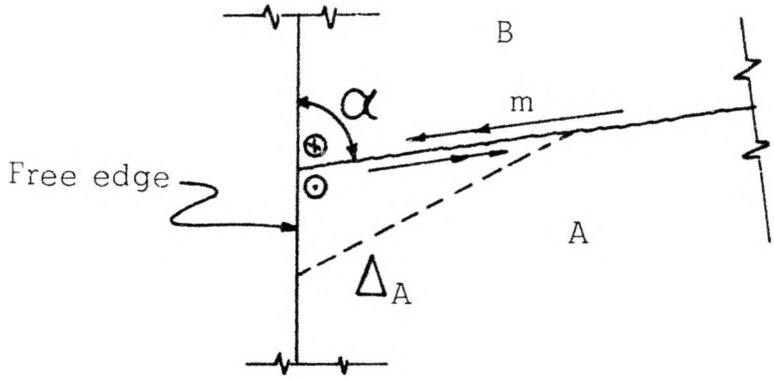


Fig. 13.--A Yield Line Intersecting a Free Edge

Therefore:

$$Q_B = m \cot \alpha \tag{5a}$$

If α is an acute angle, the force, Q_A will be acting upward while Q_B, which is equal in magnitude to Q_A, will act downward.

The study of the nodal forces presented here is adequate and sufficient for the use of the yield line theory, and with this information procedures for determination of yield line patterns will be established.

THE EQUILIBRIUM CONDITIONS

Having studied the nodal forces and the geometric layout of yielding patterns, the equilibrium conditions of a yielded plate can now be introduced.

As mentioned by K. W. Johansen [2], statical equilibrium must exist in each region of a yielded plate. This implies that three equilibrium conditions must be satisfied. Two of these conditions state that the sum of moments about any two non-parallel axes in the plane of a region must equal zero. The third condition specifies that the sum of forces in a direction perpendicular to the region must also equal zero. To apply these conditions, prepare a free body diagram of each region as shown in Figure 14.

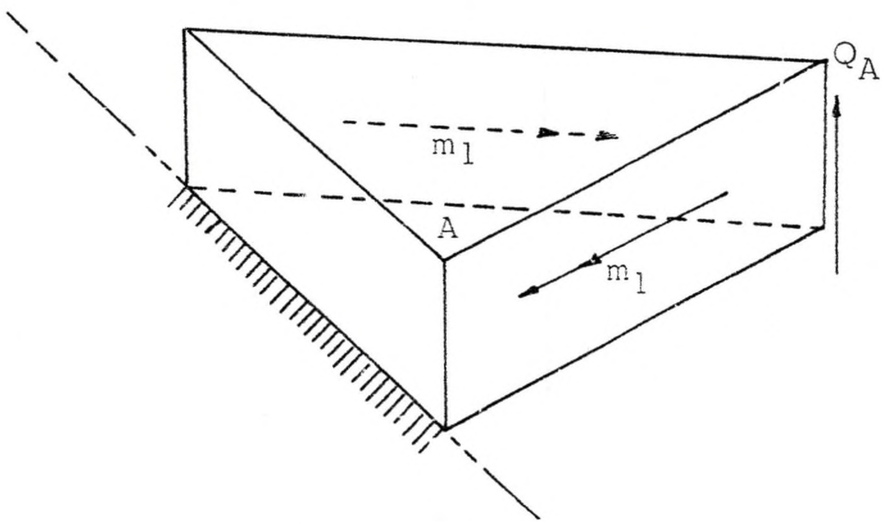


Fig. 14.--A Free Body Diagram of a Region

For a region that is supported along an edge, the magnitude and the distribution of the reaction are not known. Summing moments about that edge will give a direct computation of the yielding moment in that region while the application of the remaining two conditions will determine the magnitude of the resultant of the reaction and its point of action.

If a plate has "n" regions, "n" values of the yielding moment along the yield lines can be determined. The critical location of the yield lines is that which causes all "n" values of the yielding moments to be equal. This is due to the fact that the yielding moment at any point of an isotropic plate of uniform thickness has the same value. With this criteria in mind, the validity of a yielding pattern can be determined. Methods are established for determining yield lines in a later section of this thesis.

A commonly occurring region in yielded plates is a trapezoidal region which is bounded by yield lines and simply supported along one edge as shown in Figure 15.

Equating the summation of moments about AA' to zero yields:

$$m = 1/6\ ph^2\ (1 + 2\frac{b2}{b1}) \qquad (6)$$

where p is the intensity of loading per unit area.

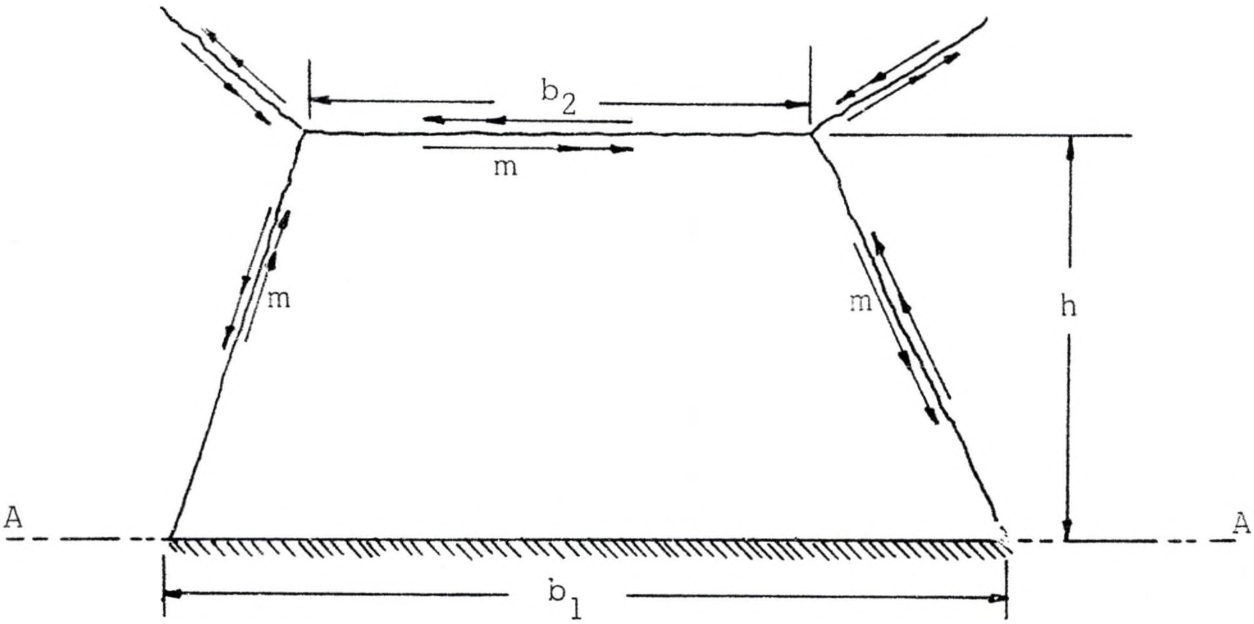


Fig. 15.--A Trapezoidal Region

In a following section of this thesis methods for determining yield line patterns are developed. These methods are based on the equilibrium conditions mentioned in this section.

PROCEDURES FOR DETERMINATION OF YIELD LINE PATTERNS

All shapes of plates can be classified as symmetrical or non-symmetrical. Symmetrical plates are defined as those that possess a geometric symmetry and whose supports give symmetric yielding lines about some axis. Figure 16 shows a symmetrical rectangular plate, simply supported along all four edges and subjected to a uniformly distributed load p. Non-symmetric plates are those that do not possess an axis of symmetry.

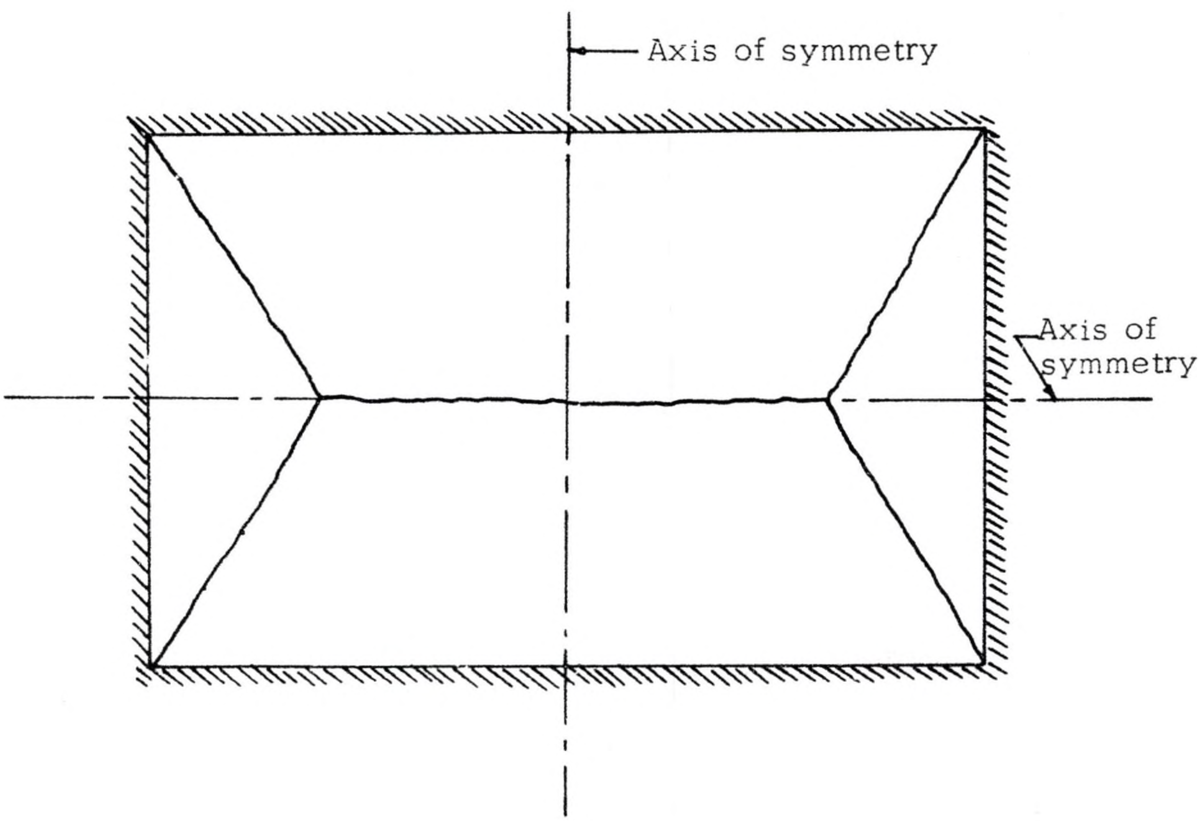


Fig. 16.--A Case of Symmetrical Plates

In this thesis three methods are recommended for the determination of yield line patterns. These are mathematical method, semi-graphical solution and thin membrane analogy solution.

Mathematical Method

The location of a yielding pattern in an isotropic plate, having uniform thickness, is critical when the values of the yielding moment along all yield lines are equal. With this in mind, the following steps describe the mathematical method as well as the procedure.

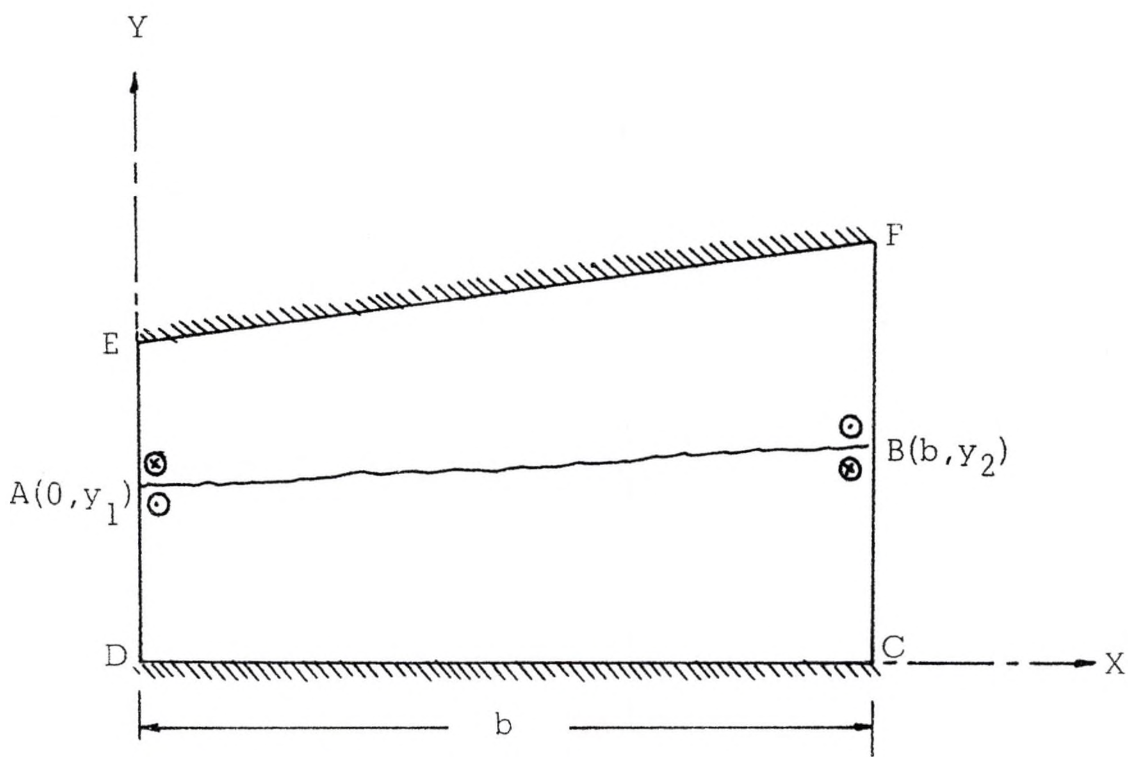


Fig. 17.--Application of the Mathematical Method

(a) Assume a failure pattern using Johansen's rules as mentioned before.

(b) Select a convenient set of axes.

(c) Let the coordinates of the points that determine the location of the yield lines be defined in terms of some unknowns x and y.

(d) Place nodal forces where needed with their values computed in terms of the chosen unknowns.

(e) Define all dependent unknowns in terms of those that can vary independently. If point A of Figure 17 is to be an independent variable point, point B will have to be related to point A since the yield line AB has to pass through the intersection of the axes of rotation of regions I and II.

(f) Considering each region separately and using the equilibrium conditions, obtain expressions for the moment along a yield line. If there are n independent variables that determine a yielding pattern, (n + 1) equations must be obtained since the value of the yielding moment is an unknown.

(g) Let the chosen variables, such as y_1 or y_2 of Figure 17, take all possible values. This could be performed best by the use of a computer. The change in the variables will cause the yield line pattern to take all possible locations. When the values of the yielding moments are equal to each other, within some accuracy, print the values of the variables. This will determine the exact location of the yield lines.

Attempts were made to find the solution for yielding patterns of non-symmetrical plates. However, the calculations involved made it a tedious task due to their complexity. In cases of symmetrical plates, the variables are reduced in number and the solution is an easy one.

Semi-Graphical Solution

A quick process to perform the operations of the mathematical method is the semi-graphical solution. The following steps describe the semi-graphical solution as well as the procedure:

(a) Draw the plate to scale.

(b) Assume a yielding pattern according to Johansen's rules.

(c) Scale the values needed to use the equation of equilibrium for each region.

(d) Obtain the values of the yielding moment in each region by simply applying the equation of statics.

(e) Repeat the process until values of moments for all regions compare.

The method is fairly simple and does not involve any mathematical complications.

For certain shapes the mathematical method consumes a substantial amount of computer time due to the number of trials needed. One can easily cut down this time by the application of the semi-graphical solution by which the yielding pattern is confined within some limits.

The semi-graphical method is applicable to all plates regardless of shape.

Thin Membrane Analogy

In this thesis deformation in a plate, subjected to loadings, is considered to be concentrated at the location of yield lines. All regions will rotate, referenced to the original position of the plate, about their axes of rotation. If planes parallel to the original position of the plate are passed through the plate they will intersect the deflected surface of the plate in lines. These lines may be thought of as contour lines showing certain deflections from the original surface of the plate. The change in direction of a contour line simply represents the existence of a yield line at that location as shown in Figure 18.

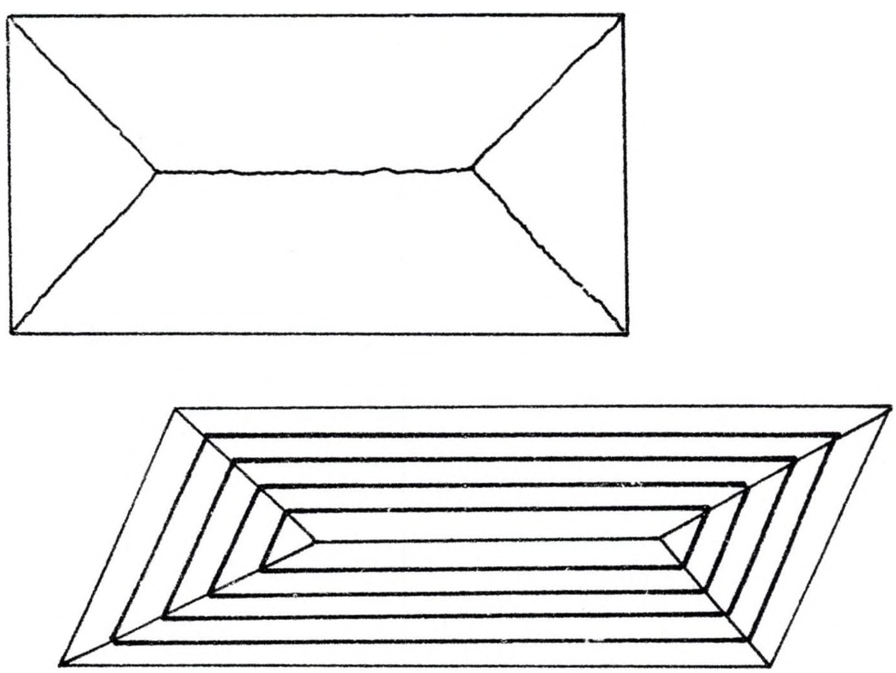


Fig. 18.--Deflected Surface of a Yielded Plate

In a thesis submitted to the faculty of the University of North Dakota, Mr. G. Bihnam [3] presented an adequate procedure for determining the contour lines of any shape of a plate subjected to a uniform loading. This was done by taking a metallic sheet and cutting out the shape of the plate. A thin membrane was used to cover the hole. Pressure was then applied to the thin membrane, and an optical comparator was used for measuring defections, from which contour lines were plotted. Having obtained the contour lines of the deflected surface, the points where these lines change direction could be joined, and the results obtained would be, as previously discussed, the yield line pattern. It must be noted here that contour lines will take a relatively straight direction at some points of the plate. If there is a considerable change in direction, it means that the contour line has passed through a point on a yield line. To determine the locations of the points where contour lines change directions, interpolation must be applied as done in the following paragraph.

The square section used by Mr. G. Bihnam is a good illustration for the purposes of this thesis. Referring to Figure 19, the change in direction of contour lines of the deflected surface of a square shaped plate can be easily noted. Where contour lines are reasonably straight, tangents are drawn. The intersections of these tangents define the path of a yield line. This is a rather crude and lengthy method for determining the location of yield lines. However, for certain plates that have odd shapes, the method can be used well.

One of the many difficulties that exists in using this method is in providing fixed end supports on a membrane since the membrane cannot resist moment.

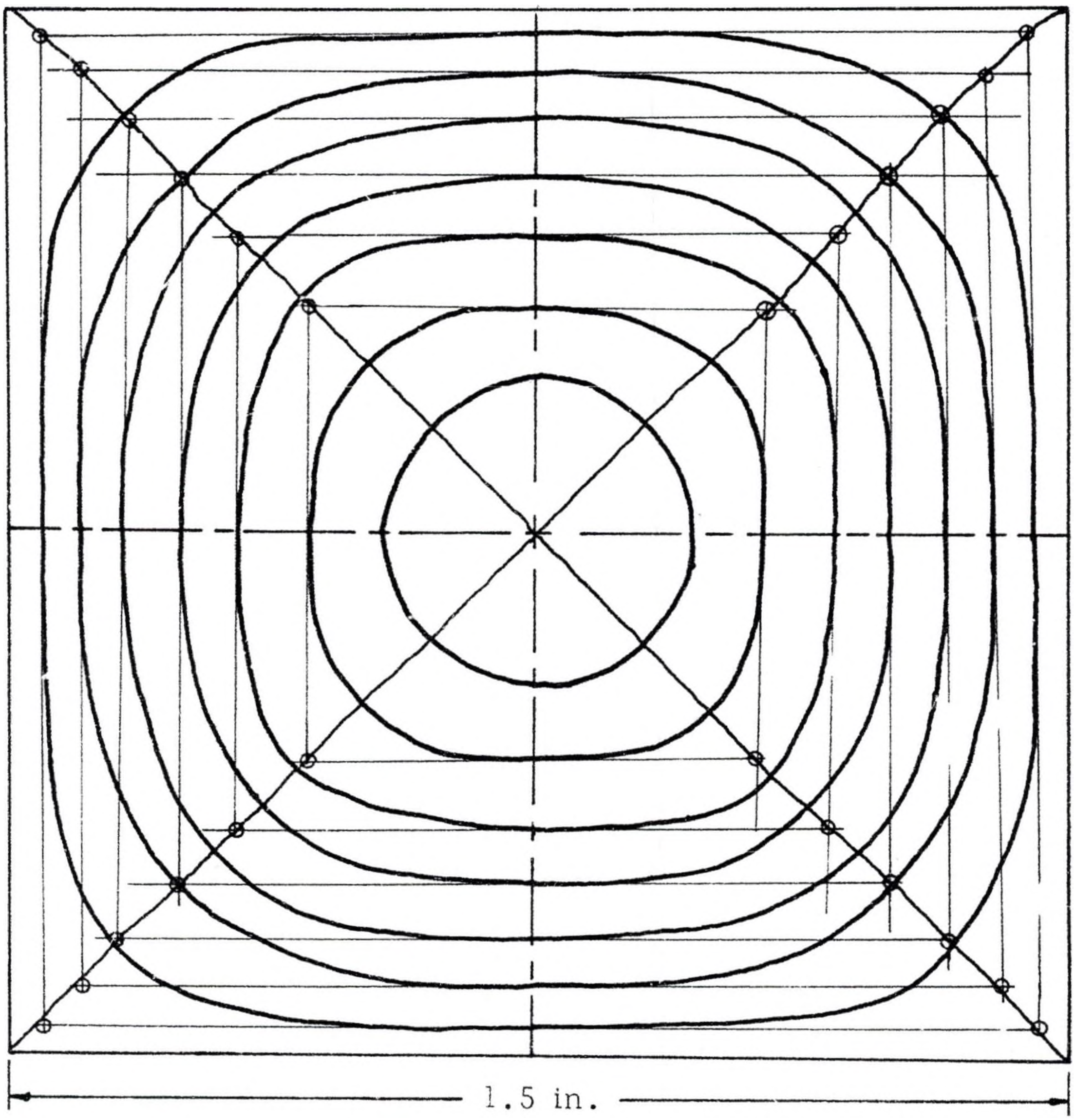


Fig. 19.--Contour Map of a Square Plate

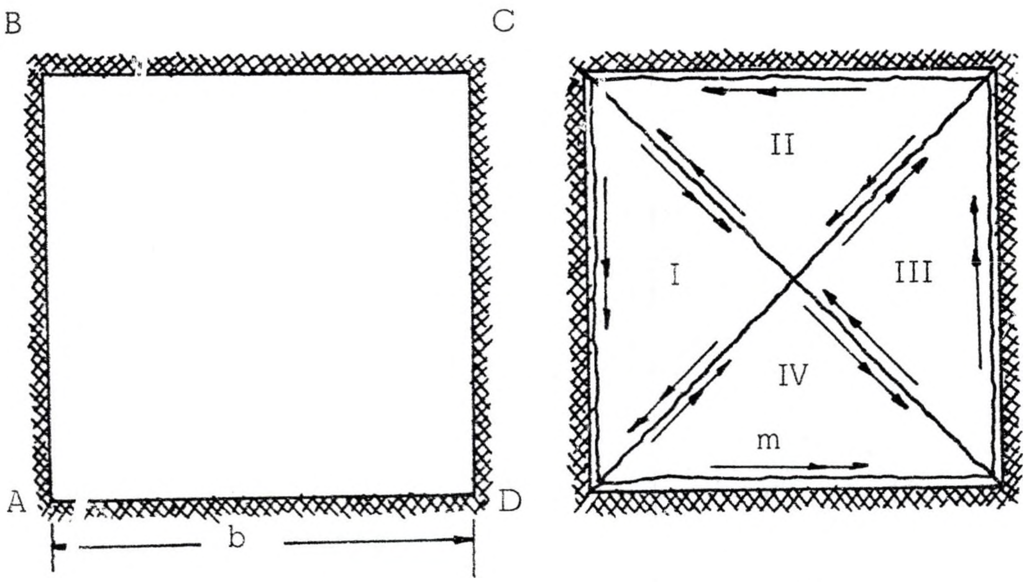
PROBLEM ILLUSTRATIONS

The following problems were chosen to demonstrate the validity of the proposed methods as well as the use of the Yield Line Theory. These problems consist of two groups: (a) symmetrical plates and (b) non-symmetrical plates.

Symmetrical Plates

Problem 1.S.P.

Determine the location of yield lines of the plate shown in Figure 20 (a) The plate is subjected to a uniform load of (p) pounds per square foot.



(a) (b)

Fig. 20.--Problem 1.S.P.

Solution

From symmetry the pattern is shown in Figure 20 (b).

$$\sum M_{AB} = 0$$

$$(2)(m)(b) = (P)(\frac{b}{2})(b)(\frac{1}{2})(\frac{b}{2}\cdot\frac{1}{3})$$

$$m = \frac{Pb^2}{48}$$

From symmetry the same value of m is obtained for region II. Therefore, the pattern assumed is correct.

Problem 2.S.P.

Determine the location of yield lines of the plate shown in Figure 21 (a). The plate is subjected to a uniform load of one kip per square foot and is simply supported along all edges.

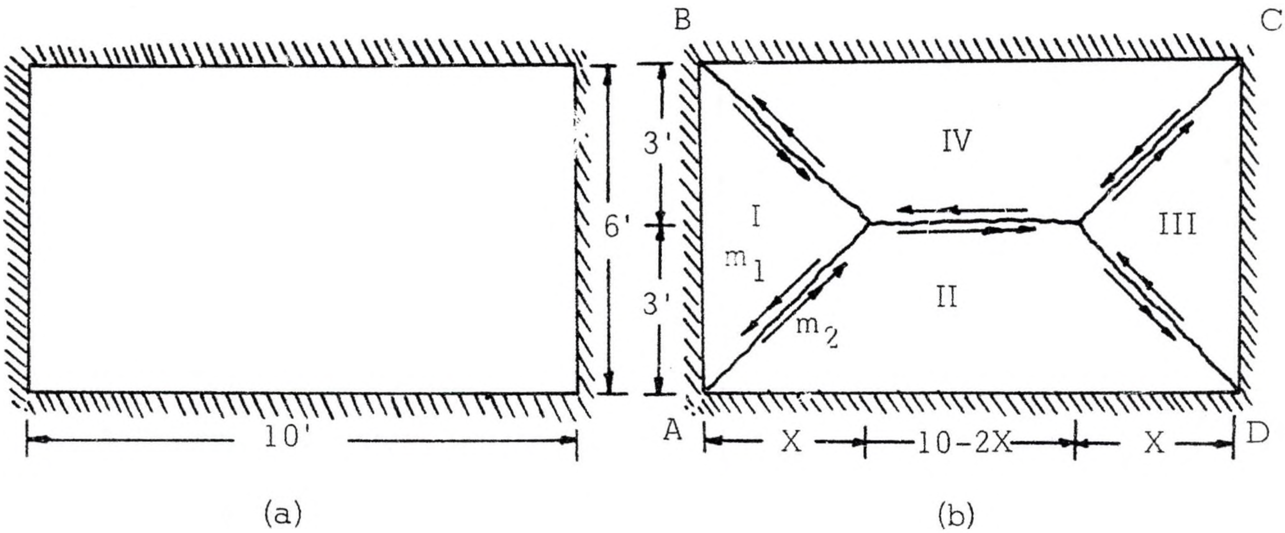


Fig. 21.--Problem 2.S.P.

Solution

The nodal forces are all equal to zero. From symmetry the pattern of yield lines is shown in Figure 21 (b). $\sum M_{AB}$ for region I yields:

$$m_1 = \frac{(1)(x^2)}{6} \qquad (2.S.P.a)$$

Applying equation 6 for region II, the following can be obtained:

$$m_2 = \frac{1}{6}(1)(9)\left[1 + \frac{2\,(10 - 2x)}{10}\right] \qquad (2.S.P.b)$$

With the above equations one can use the proposed method to solve for x. However, the problem is quite simple, and there is no need to do so. Equating equation 2.S.P.a to 2.S.P.b, the following is obtained:

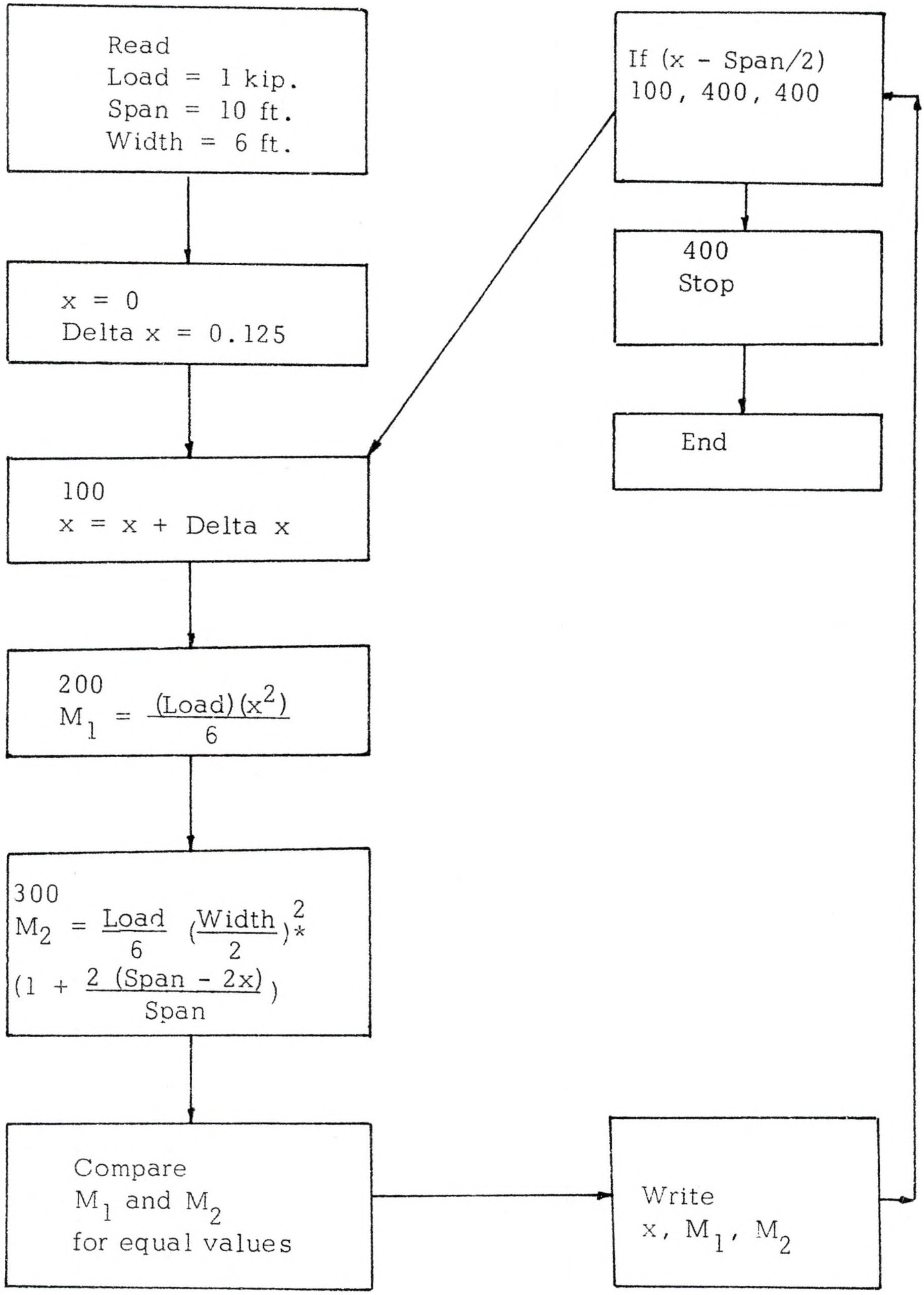
$$\frac{1}{6}x^2 = \frac{9}{6}\left[1 + \frac{2\,(10 - 2x)}{10}\right]$$

Which yields: x = 3.69 ft. and m = 2.275 ft. - kip/ft.

The same problem was solved on the computer with the use of the proposed method, and the results obtained were:

x = 3.687 and m = 2.277 ft. = kip/ft.

The programming involved is very simple and an elementary knowledge of it is sufficient. The following flow chart was used to solve the above problem.



Read
Load = 1 kip.
Span = 10 ft.
Width = 6 ft.
x = 0
Delta x = 0.125
100
x = x + Delta x
200
$M_1 = \frac{(Load)(x^2)}{6}$
300
$M_2 = \frac{Load}{6} (\frac{Width}{2})^2 *$
$(1 + \frac{2 (Span - 2x)}{Span})$
Compare
M_1 and M_2
for equal values
Write
x, M_1, M_2
If (x - Span/2)
100, 400, 400
400
Stop
End

Problem 3.S.P.

Repeat problem 2.S.P. using the semi-graphical solution.

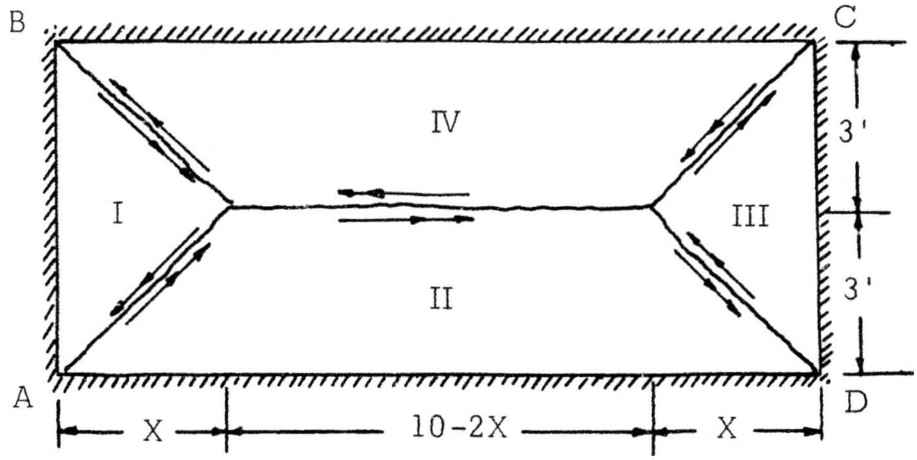


Fig. 22.--Problem 3.S.P.

The yielding pattern is shown in Figure 22.

Try x = 3.0

$\sum M_{AB}$ for region I yields:

$$m_1 = \frac{9}{6} = 1.5 \text{ ft. - kip/ft.}$$

Applying equation 6 for region II yields:

$$m_2 = \frac{9}{6} \left(1 + 2\frac{4}{10}\right) = 2.7 \text{ ft. - kip/ft.}$$

Try x = 3.5

$\sum M_{AB}$ for region I yields:

$$m_1 = \frac{(3.5)^2}{6} = 2.04 \text{ ft. - kip/ft.}$$

Applying equation 6 for region II yields:

$$m_2 = \frac{9}{6}\left(1 + 2\frac{3}{10}\right) = 2.40 \text{ ft. - kip/ft.}$$

Try x = 3.75

$\sum M_{AB}$ for region I yields:

$$m_1 = \frac{(3.75)^2}{6} = 2.35 \text{ ft. - kip/ft.}$$

$$m_2 = \frac{9}{6}\left(1 + \frac{2.5}{10}\right) = 2.25 \text{ ft. - kip/ft.}$$

The values of m_1 and m_2 compare well and there is no need for more trials.

Problem 4.S.P.

Determine the yield line pattern for the plate shown in Figure 23 (a). The plate is subjected to a uniform loading of (P) pounds per square foot.

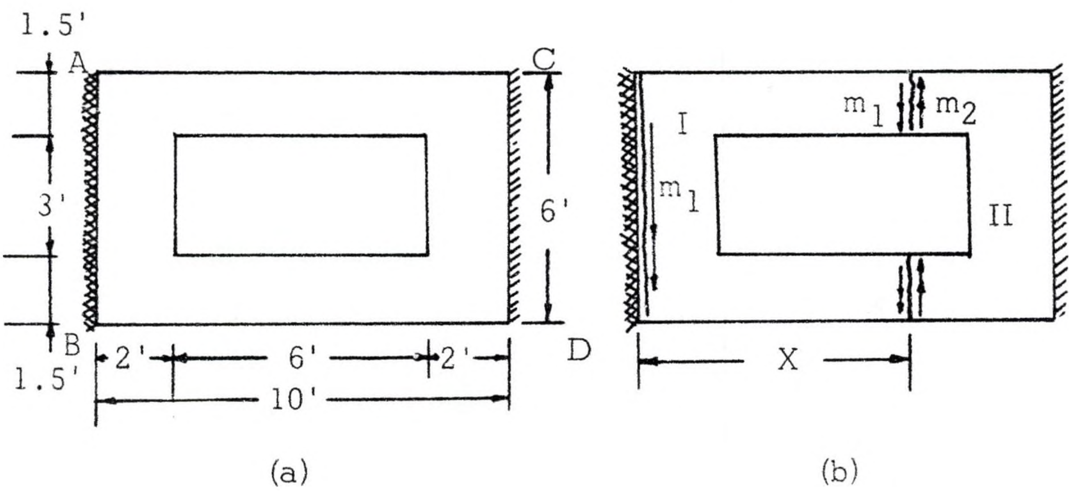


Fig. 23.--Problem 4.S.P.

Solution

All nodal forces are equal to zero. For region I

$$\sum M_{AB} = 0$$

$$m_1(6) + 3(m_1) = p[(6)(2)(1) + (x - 2)(1.5)(\frac{x + 2}{2})(2)]$$

$$m_1 = \frac{p}{9}[12 + 1.5(x^2 - 4)] \qquad (4.S.P.a)$$

$\sum M_{DC}$ for region II gives:

$$3m_2 = p[\frac{(10 - x)^2}{2} (1.5)(2) + (3)(2)(1)]$$

$$m_2 = \frac{p}{3}[(1.5)(10 - x)^2 + 6] \qquad (4.S.P.b)$$

Equating equation 3.S.P.a to equation 3.S.P.b, the following is obtained:

$$x^2 - 30x + 154 = 0$$

$$x = 6.43 \text{ ft.}$$

Problem 5.S.P.

Determine the yielding pattern for a circular isotropic plate which is subjected to a uniform loading of (p) pounds per square foot. The plate is simply supported along its edge.

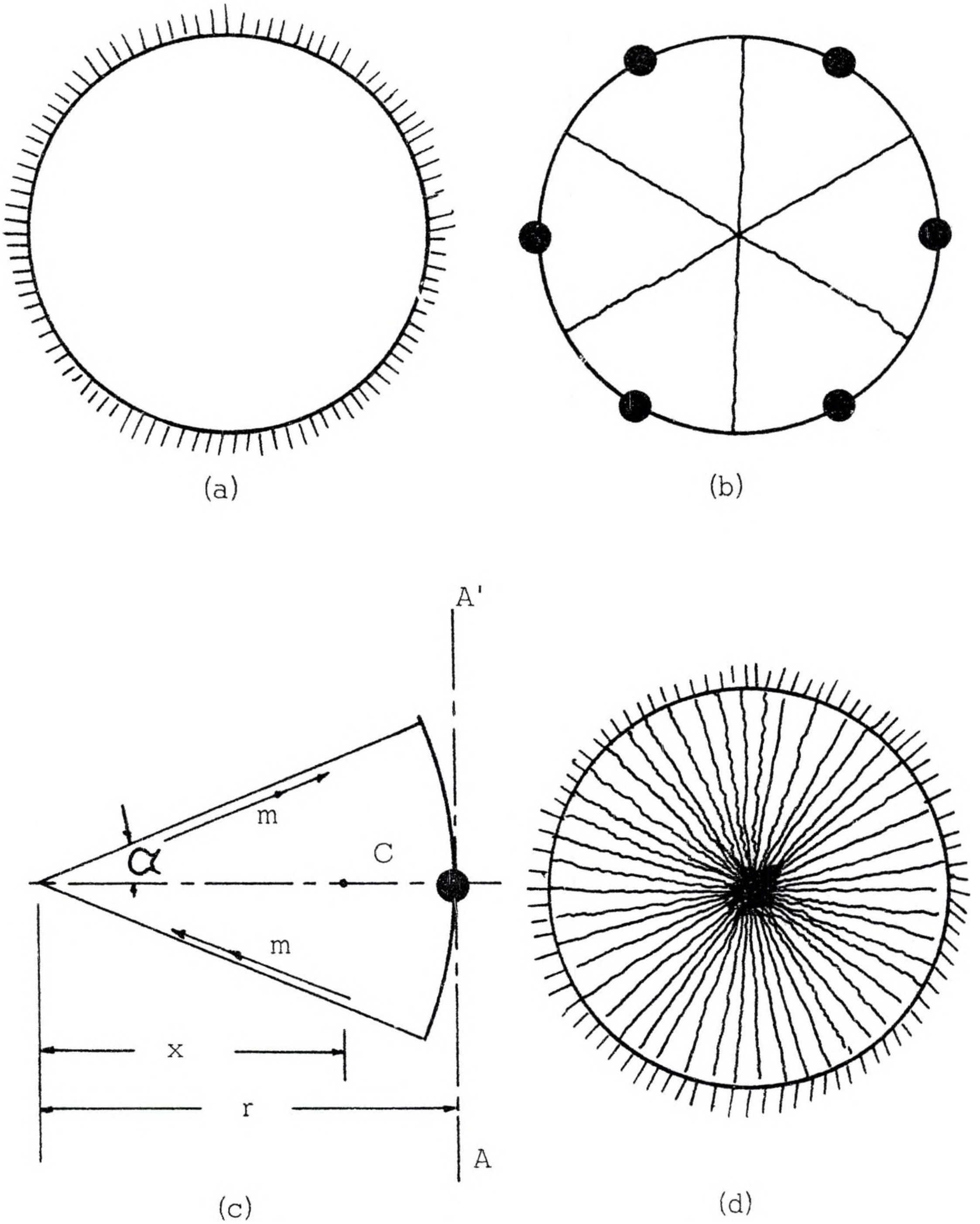


Fig. 24.--Problem 5.S.P.

Solution

The solution provided here is the same as that presented by K. W. Johansen [2]. Consider a plate, which is symmetrically supported on (n) columns. From symmetry the yielding pattern is shown in Figure 24 (b).

$$P = p\pi r^2$$

$$\alpha = \frac{\pi}{n}$$

$$x = \frac{2}{3} r \frac{\sin\alpha}{\alpha}$$

Referring to Figure 24 (c) and equating the sum of moments about AA' to zero, the following is obtained:

$$m = \frac{P}{2n \sin\left(\frac{\pi}{n}\right)} \left(1 - \frac{2}{3} \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}\right.$$

When n approaches ∞, m approaches $\frac{P}{6\pi}$. The problem is then reduced to that of a circular plate which is simply supported along all its edges, and the yielding pattern is identical to that shown in Figure 24 (d).

Non-Symmetrical Plates

Problem 1.N.S.P.

Knowing that the imposed loading is equal to one kip per square foot, find the exact yielding pattern of the plate shown in Figure 25 (a).

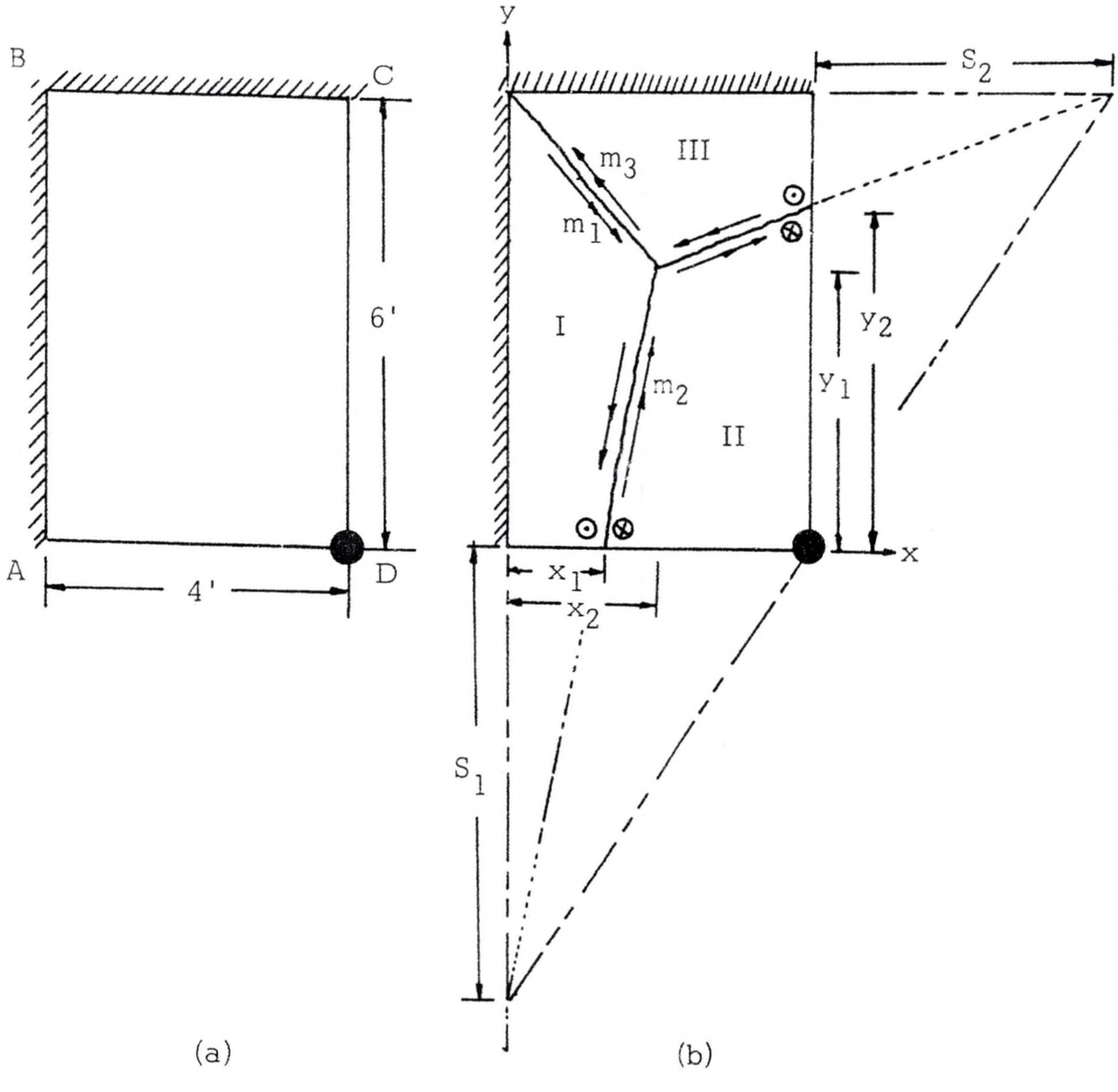


Fig. 25.--Problem 1.N.S.P.

Solution

The yielding pattern will be similar to that shown in Figure 25 (b). The nodal force, acting at the intersection of the yield line with edge AD, is equal to:

$$\frac{m(x_2 - x_1)}{y_1}$$

The nodal force, acting at the intersection of the yield line with edge DC, is equal to:

$$\frac{m(y_2 - y_1)}{(b - x_2)}$$

Summation of moments about line AB for region I yields the following:

$$(m_1)(L) + m_1 \frac{(x_2 - x_1)}{y_1}(x_1) = P\,[(L - y_1)\left(\frac{x_2}{2}\right)\left(\frac{x_2}{3}\right) + (x_1)(y_1)\left(\frac{x_2}{2}\right) + (x_2 - x_1)(y_1)\left(\frac{1}{2}\right)\left(\frac{x_2 - x_1}{3} + x_1\right)]$$

From which is obtained:

$$m_1 = \frac{py_1}{6} \quad \frac{x_2^2\,(L - y_1) + 3x_1^2\, y_1 + y_1(x_2 - x_1)(x_2 + 2x_1)}{(y_1)(L) + (x_2 - x_1)(x_1)}$$

(1.N.S.P.a)

Summation of moments about line DC for region II yields:

$$m_2 = p \quad \frac{\left\{ \begin{array}{l} \frac{(x_2 - x_1)}{2}(y_1)\,[b - x_2 + \frac{(x_2 - x_1)}{3}] + \frac{(b - x_2)^2}{6}(y_2 - y_1) \\ + y_1 \frac{(b - x_2)^2}{2} \end{array} \right\}}{[y_2 - \frac{x_2 - x_1}{y_1}(b - x_1)]}$$

(1.N.S.P.b)

Summation of moments about line AD for region II yields the following results:

$$m_2(b - x_1) - m_2 \frac{(y_2 - y_1)}{b - x_2} y_2 = p \quad (b - x_2)(y_1)(\frac{y_1}{2}) + (\frac{b - x_2}{2})(y_2 - y_1)(y_1 + \frac{y_2 - y_1}{3}) + y_1 (\frac{x_2 - x_1}{2}) (\frac{y_1}{3})$$

From which is obtained:

$$m_2 = \frac{p}{6} \frac{3y_1^2 \ (b - x_2) + (b - x_2)(y_2 - y_1)(2y_1 + y_2) + y_1^2 (x_2 - x_1)}{(b - x_1) - \frac{(y_2 - y_1)(y_2)}{b - x_2}}$$

(1.N.S.P.c)

Summation of moments about line BC for region III yields:

$$m_3(b) + m_3 (\frac{y_2 - y_1}{b - x_2})(L - y_2) = p \left\{ \frac{(x_2)(L - y_1)}{2} \cdot \frac{(L - y_1)}{3} + (b - x_2)(L - y_2) (\frac{L - y_2}{2}) + (y_2 - y_1)(b - x_2)(\frac{1}{2}) \left[(L - y_2) + \frac{y_2 - y_1}{3} \right] \right\}$$

which gives:

$$m_3 = \frac{p}{6}\,\frac{x_2(L - y_1)^2 + 3(L - y_2)^2(b - x_2) + (y_2 - y_1)\cdot(b - x_2)(3L - 2y_2 - y_1)}{b + \frac{(y_2 - y_1)}{(b - x_2)}\cdot(L - y_2)}$$

(1.N.S.P.d)

Next step is to establish the value of y_2 in terms of x_1, x_2, and y_1.

$$S_1 = \frac{-x_1\, y_1}{x_1 - x_2}$$

$$S_2 = \frac{b}{S_1}\; L$$

$$y_2 = L - \left(\frac{L - y_1}{b + S_2 - x_2}\right)\cdot S_2$$

The values of x_1, x_2, y_1, and y_2 that give relatively equal values for m_1, m_2, and m_3 are:

x_1 = 2.02 ft.

x_2 = 3.54 ft.

y_1 = 2.07 ft.

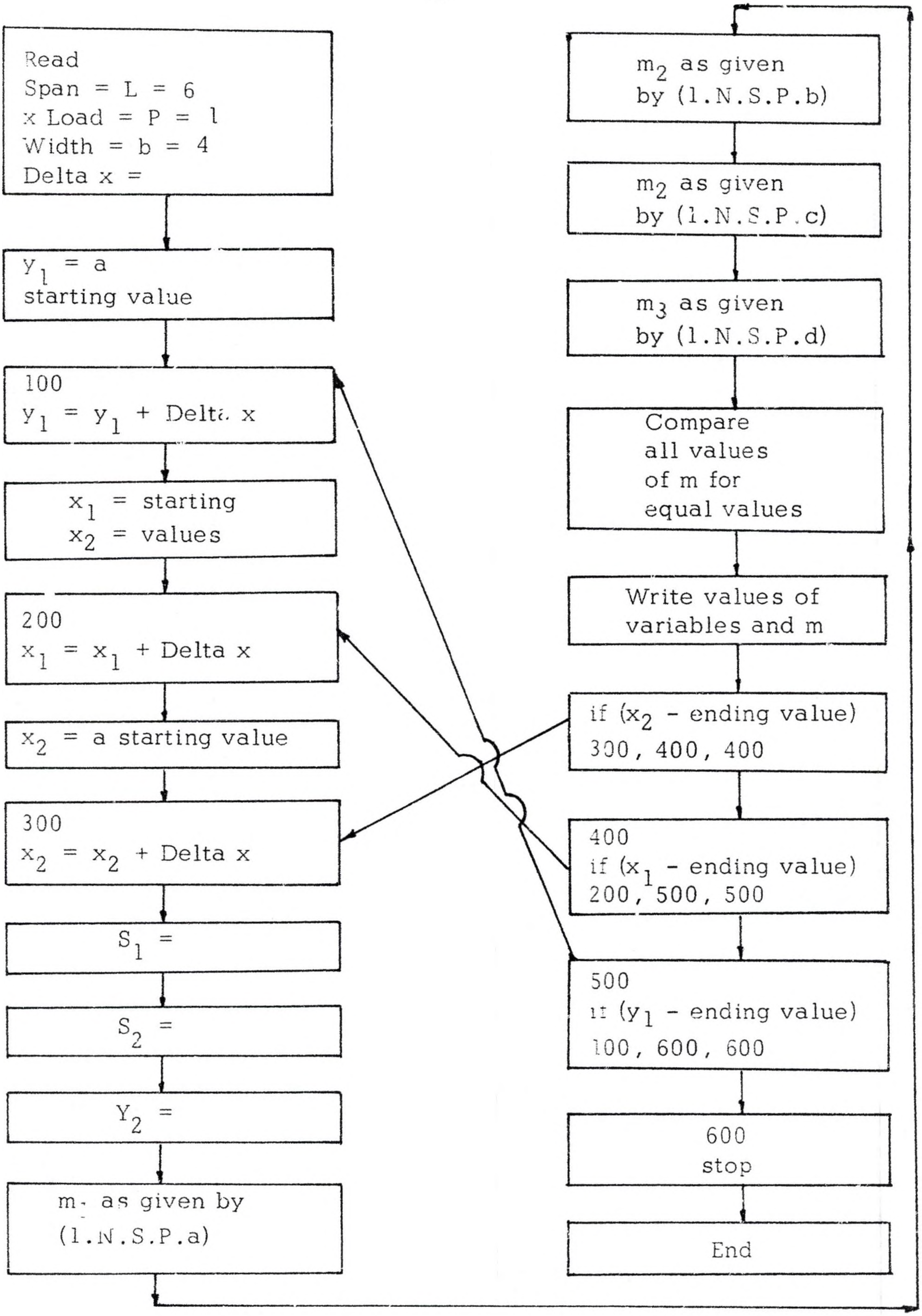
y_2 = 2.27 ft.

m_1 = 2.19 ft. - kip/ft.

m_2 = 2.15 ft. - kip/ft.; m_2 = 2.15 ft. - kip/ft.

m_3 = 2.23 ft. - kip/ft.

The following flow chart was used to program this problem on the computer:



Read
Span = L = 6
x Load = P = 1
Width = b = 4
Delta x =
y1 = a
starting value
100
y1 = y1 + Delta x
x1 = starting
x2 = values
200
x1 = x1 + Delta x
x2 = a starting value
300
x2 = x2 + Delta x
S1 =
S2 =
Y2 =
m as given by
(1.N.S.P.a)
m2 as given
by (1.N.S.P.b)
m2 as given
by (1.N.S.P.c)
m3 as given
by (1.N.S.P.d)
Compare
all values
of m for
equal values
Write values of
variables and m
if (x2 - ending value)
300, 400, 400
400
if (x1 - ending value)
200, 500, 500
500
if (y1 - ending value)
100, 600, 600
600
stop
End

CONCLUSIONS

The following conclusions can be stated about the proposed methods.

(1) The values of moments, computed from each region, change considerably with a small change in the location of a yielding pattern.

(2) The thin membrane analogy is rather limited in its applications.

(3) The mathematical method provided an accuracy of 3.5% in the case of non-symmetrical plates while early investigators were satisfied with an accuracy of 25%.

(4) A very elementary knowledge of computer programming is sufficient for applying the mathematical method.

(5) The mathematical method consumes a considerable amount of computer time when there are many variables.

(6) In applying the semi-graphical solution to non-symmetrical plates, an accuracy of 30% is considered to be adequate.

(7) The proposed methods do not involve any mathematical complications.

To apply the mathematical method efficiently, a computer program should be run with large increments in the independent variables. The values of the moments obtained will not only confine the yielding pattern within limits but will also enable the programmer to set boundaries on the value of the yielding moment. Having confined the yielding pattern within limits, small increments in the variables are chosen, and the exact location of the yielding pattern is determined.

LIST OF NOTATIONS AND SYMBOLS

m	Ultimate moment of a plate per unit length
$m_1, m_2, m_3, \ldots$	Ultimate moments per unit length along some specified lines
p	Load per unit area
P	Total load
Q	Nodal force
Q_{Δ_A}	Nodal force acting on a segment Δ_A
U	Overload factor
X, Y, Z	Coordinate axes
$\alpha \beta \gamma$	Angles between yield lines
Δ	Magnitudes of defections
Δ_A	Segment of region A
θ	Angle of region rotation
ϕ	Undercapacity factor
——— — ———	Axis
——— – – ——— – –	Axis of rotation
	Fixed edge support
⊗	Force acting downward
⊙	Force acting upward

Free edge of a plate

Moment

Simple column support

Simple edge support

Yield line

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