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BOSON STARS IN THE BRANS-DICKE GRAVITATIONAL THEORIES

by

Mark Gunderson

A Thesis

submitted to the Graduate Faculty

of the

University of North Dakota

in partial fulfillment of the requirements

for the degree of

Master of Science

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This thesis, submitted by Mark A. Gunderson in partial fulfillment of the requirements for the Degree of Master of Science from the University of North Dakota, has been read by the Faculty Advisory Committee under whom the work has been done and is hereby approved.

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This thesis meets the standards for appearance, conforms to the style and format requirements of the Graduate School of the University of North Dakota, and is hereby approved.

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ABSTRACT

Boson stars consist of a system of self gravitating scalar fields (bosons) which form a macroscopic quantum state and is one of several dark matter candidates. When considering cosmological models based on the Brans-Dicke gravitational theories, one question that must be addressed is whether certain types of dark matter candidates, possibly boson stars, can exist in the Brans-Dicke gravitational theories. We show in this paper that at least one type of dark matter – boson stars – can exist in these gravitational theories.

For purposes of comparison, we introduce a boson star model in general relativity with a quartic coupling interaction in the boson field and give some of the solutions to the equations of motion.

To derive the equations of motion of the Brans-Dicke model which also contains a quartic coupling interaction in the boson scalar field, we introduce an action similar to that of general relativity with the exception of the introduction of the Brans-Dicke scalar field. Once we have the equations of motion, we then solve them using a Runge-Kutta method. We consider only nonsingular, finite-mass, zero-node solutions to the equations of motion for $\omega = 6$.

When we obtain the solutions from the Brans-Dicke model for different values of the coupling interaction strength Λ , we notice a slight decrease in the maximum mass of the star which is due to the Brans-Dicke scalar field interaction with the matter density of the universe. We also consider the equations of motion in the limit of large values of Λ , and when the solutions of these equations are compared with their general relativistic counterparts, we again notice a slight decrease in the maximum mass. We also note that the maximum mass of the solution to these equations, which have a $\Lambda^{\frac{1}{2}}$ dependency, tend to agree fairly well with values for large Λ , which is expected from the type of modification used. We also note that the maximum mass decreases as ω decreases.

By finding solutions to this model, we have shown that boson stars can exist as a form of dark matter in the Brans-Dicke gravitational theories with a slight decrease in the maximum allowed mass of the star.

Chapter 1

INTRODUCTION

A very important problem in astrophysics and cosmology is the existence and nature of non-luminous matter or dark matter in the universe. The existence of dark matter is closely related to many issues regarding the evolution of the universe, including the earliest history of the universe, the formation of galaxies and galaxy clusters, and also in the ultimate fate of the universe. Evidence of the existence of dark matter is seen in some astronomical observations, including a measurement of the rotation curves of spiral galaxies [1]. A rotation curve is the measurement of orbital velocities of stars and gas clouds at different radial distances from the center of the spiral galaxies and depends on the amount of matter outside the orbit. The measurement of the rotation curves shows that the galaxies are three to ten times more massive than the amount of luminous matter present suggests, leading to the conclusion that some form of dark matter must exist and is much more abundant than luminous matter.

A possible explanation for the existence of dark matter is seen in the inflationary “Big Bang” models, which have been recently supported by the discovery of minute fluctuations in the cosmic background radiation [1]. These models predict that the

matter density of the universe should be equal to the critical density needed for the universe to eventually collapse. However, recent observations have shown that the density of luminous matter in the universe is only approximately one one-hundreth of the critical density [1]. Therefore, it is quite possible that the majority of the matter in the universe is dark matter. These models also suggest that this matter does not consist of baryons, which include protons and neutrons [1].

There are several possible forms of dark matter which have been suggested, including massive neutrinos, solitons, axions, and exotic bosons [1]. In this paper, we concentrate on spinless exotic bosons in the form of boson stars, as these objects are somewhat easier to interpret physically. Boson stars are gravitationally bound macroscopic quantum states of a system of scalar bosons, which basically means that all of the bosons in the star can exist in the same state at the same time [2]. Boson stars are similar in many aspects to neutron stars. The main difference between these stars is that neutron stars are supported from collapse through neutron degeneracy by the exclusion principle while boson stars are supported from collapse by the Heisenberg uncertainty principle.

In the late 60's, Ruffini and Bonazzola did the first detailed study of equilibrium configurations of noninteracting scalar bosons (boson stars) [2]. They first considered the Newtonian treatment for a sphere of self-gravitating bosons of mass m interacting only gravitationally using a Hartree-like mean-field approximation. The Hartree-like

mean-field approximation begins by replacing the sum of the gravitational potentials between each of the identical particles in the system

$$-\sum_{i \neq j} \frac{Gm^2}{|\vec{r}_i - \vec{r}_j|}, \quad (1.1)$$

where G is the gravitational constant, by a mean approximation of the gravitational potential $V(r)$ of the system which only depends on the quantity r instead of the distance between the particles $|\vec{r}_i - \vec{r}_j|$, thereby simplifying the problem. Along with this approximation, the Newtonian boson star model also uses Schrödinger's equation for a particle of mass m in the presence of the mean potential $V(r)$. This mean gravitational potential, in turn, satisfies the Poisson equation, and the wavefunction of the single particle is normalized to unity. In this paper, we quote some of the numerical results for the binding energy and for the mass of the star from this system of equations that were first found by Ruffini and Bonazzola [2]. More details on this calculation are given in Section 2.1.

Ruffini and Bonazzola also considered a general relativistic treatment of this problem by describing the noninteracting scalar bosons by the curved space Klein-Gordon equation. Since boson stars are in a macroscopic quantum state, they assumed that all of the bosons are in the ground state and used second quantization, a method of quantizing the modes of the system, to derive the equations of motion. More details on this derivation and calculation are given in Section 2.3.

Boson stars can also be understood from a simple dimensional argument [3, 4]. Because all bosons in the star can exist in the same state at the same time, namely the ground state, the gravitationally bound states of bosons of mass m have a scale radius R given by

$$\frac{MG}{R} \sim 1 \quad (1.2)$$

which is similar to the Schwarzschild radius. (Note that $\hbar \equiv 1 \equiv c$ in these equations.) M is the mass of the bound state, G is the gravitational constant, and the Planck mass M_{Pl} is given by

$$M_{Pl} = \frac{1}{\sqrt{G}}. \quad (1.3)$$

A typical boson of characteristic size R has the momentum

$$p \sim \frac{1}{R}. \quad (1.4)$$

Thus, in a moderately relativistic boson star,

$$p \sim m \quad (1.5)$$

and therefore

$$R \sim \frac{1}{m}. \quad (1.6)$$

Using equations (1.3) and (1.6) in equation (1.2), we find that

$$M \sim \frac{M_{Pl}^2}{m} \quad (1.7)$$

for marginally relativistic boson stars. This mass tends to be much smaller than the Chandrasekhar mass

$$M_{Ch} \sim \frac{M_{Pl}^3}{m^2} \quad (1.8)$$

of marginally relativistic fermion stars.

Although Ruffini and Bonazzola have explicitly demonstrated that boson stars could exist, the masses that they obtained were much too small to make a significant contribution to the existence of dark matter [2]. However, Colpi, Shapiro, and Wasserman, by introducing a coupling effect mediated by a coupling constant λ into the general relativistic model, showed that boson stars under the effect of even a very weak coupling have quite a different form from stars where $\lambda = 0$, obtaining masses on the order of the Chandrasekhar mass [4]. By finding solutions with this mass, they have shown that boson stars can contribute significantly to the existence of dark matter. In their calculation, they also treated the boson star classically rather than quantum mechanically and showed that for $\lambda = 0$, their results agree completely with those of Ruffini and Bonazzola. More details on this model are given in Section 2.4.

In our research, we also consider a system of self-gravitating scalar bosons with the same coupling interaction as the general relativistic model. However, in place of general relativity, we use the Brans-Dicke gravitational theories. The study of boson stars in the Brans-Dicke theory is of great interest because of the success of inflationary cosmology based on the Brans-Dicke gravitational theories, more commonly

known as the “extended inflation” models. These models, like most inflationary models, suggest the existence of dark matter, possibly in the form of boson stars.

The Brans-Dicke gravitational theory has its origin in Mach’s Principle [5]. According to Mach’s Principle, the only meaningful motion is that relative to other matter in the universe and that the inertial forces observed locally in an accelerated laboratory may be interpreted as gravitational effects caused by distant matter accelerated relative to the laboratory. If this is the case, then one might expect that the locally observed value of the inertial mass of a test body is dependent on the distribution of matter about the test body. As was pointed out by Brans and Dicke, this statement suggests that for a plausible gravitational theory to hold in the context of Mach’s Principle, either the mass of the test body in question or the gravitational constant G or both must be able to vary with position. This is quite different from general relativity, in which the fundamental physical properties of a particle are the same no matter what the location is. Therefore, while general relativity supports the “strong principle of equivalence,” Mach’s Principle and the Brans-Dicke gravitational theory, which will be based on Mach’s Principle, only support the “weak principle of equivalence.” The “strong principle of equivalence” basically states that it is possible to choose at every point in space-time a “locally inertial coordinate system” such that, within a sufficiently small region of space, the laws of nature are the same as in an unaccelerated “Cartesian coordinate system” in the absence of a gravitational

field. The “weak principle of equivalence,” simply put, states that gravitational and inertial effects are equivalent.

In the theory presented by Brans and Dicke, the universal gravitational constant is replaced by a scalar field which is sometimes referred to as a cosmic field [5]. As indicated above, this scalar field or “variable gravitational constant” mediates locally observed inertial forces due to the mass distribution of the universe and depends on the radial distance of the observation point from the center of the object of study as well as the mass distribution of the universe. Since the Brans-Dicke gravitational theory should agree with general relativity at large radial distances from the star ($r \rightarrow \infty$), as is seen from the weak field approximation performed by Brans and Dicke, the scalar field is then related to the universal gravitational constant by [5]

$$\Phi(r \rightarrow \infty) = \left(\frac{2\omega + 4}{2\omega + 3} \right) \frac{1}{G} \quad (1.9)$$

The quantity ω regulates the strength of the inertial forces due to the matter distribution of the universe.

As will be shown in Sections 3.2 and 3.3, we were able to derive and to numerically solve the equations of motion of a boson star in the Brans-Dicke gravitational theories and thus prove that it is possible for boson stars to exist in these gravitational theories. We also found masses of the boson star that were on the order of the Chandrasekhar mass, thereby allowing Brans-Dicke boson stars to possibly have a significant contribution to the amount of dark matter which must be present. How-

ever, as will be discussed in Section 3.3 and Chapter 4, there was a slight decrease in the mass of the Brans-Dicke boson star when compared to the general relativistic model.

In Chapter 2, we begin by reviewing some of the earlier models of the boson star as they are somewhat easier to physically interpret. Our intention is also to lay out the foundation on which the Brans-Dicke boson star model will be based. First, we consider the Newtonian approximation presented by Ruffini and Bonazzola in Section 2.1. Then, we look at some analytic estimates of the Newtonian approach in Section 2.2 in order to confirm the results found by the approximation. Next, in Section 2.3, we introduce the concept of a scalar field and use this field in a general relativistic model which was also presented by Ruffini and Bonazzola. This section supplies the foundation for the rest of the models considered in this paper. The last of the previously studied models to be discussed in Chapter 2, which was introduced by Colpi, Shapiro, and Wasserman, is a general relativistic model in which a coupling interaction is introduced into the theory. In Section 2.4, we derive the equations of motion of this model, solve them numerically, and compare the results to those found by Ruffini and Bonazzola.

In Section 3.1, we begin by giving a review of the Brans-Dicke gravitational theory along with some of the initial starting conditions. In Section 3.2, we derive the equations of motion of the Brans-Dicke boson star with a coupling effect and explain the function of the quantity ω . In Section 3.3, we solve the equations of motion,

analyze their solutions in a manner similar to that in the general relativistic coupling model, and then compare these results with the results of the general relativistic coupling model. Finally, in Chapter 4, we make some conclusions about the results and the comparisons between the Brans-Dicke and the general relativistic models with a coupling interaction.

Chapter 2

NEWTONIAN AND GENERAL RELATIVISTIC TREATMENT

In this chapter, we review some of the previous models of boson stars as these models are somewhat easier to physically interpret in comparison with the Brans-Dicke model. Also, these models are the foundation for the Brans-Dicke model. They include the Newtonian mean field approximation along with a few analytic estimates, a model in general relativity using a quantum approach, and also a general relativistic model in which a coupling interaction is introduced.

2.1 Newtonian Mean Field Approximation

We begin with the Newtonian model as it is the easiest model to understand. This Newtonian treatment of a sphere of self-gravitating bosons, which was presented by Ruffini and Bonazzola, uses a Hartree-like mean field approximation [2]. As was indicated in the introduction, the approximation begins by replacing the sum of the gravitational potential between each of the bosons by a mean approximation of the gravitational potential of the boson star. We also assume that the system of bosons is at absolute zero and that each boson is in the ground state, therefore implying

that the boson star itself is in the ground state. Also, the boson star contains no antiparticles as they would increase the energy of the boson star.

Now, the gravitational potential V of the boson star satisfies the Poisson equation

$$\nabla^2 V = -4\pi G\rho \quad (2.1)$$

where ρ is the matter density and G is Newton's gravitational constant [1, 2]. For a particle of mass m in the gravitational field of the boson star, Schrödinger's equation is

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E + mV)\psi = 0. \quad (2.2)$$

Now, if the assumption is made that all the bosons within the sphere occupy the same quantum state, the system then has the matter density

$$\rho = Nm\psi^*\psi \quad (2.3)$$

where ψ is the wavefunction of a single boson. This wavefunction is normalized by

$$\int \psi^*\psi d^3x = 1. \quad (2.4)$$

With the assumption that all of the bosons are in the spherically symmetric ground state, then the radial wavefunction satisfies the following system of equations:

$$\frac{1}{x} \frac{d^2}{dx^2}(x\phi) + (\hat{E} + \hat{V})\phi = 0 \quad (2.5)$$

$$\frac{1}{x} \frac{d^2}{dx^2}(x\hat{V}) = -\phi^*\phi \quad (2.6)$$

$$\int_0^\infty \phi^*\phi x^2 dx = 1 \quad (2.7)$$

where the dimensionless quantities $x, \phi, \hat{E}$, and $\hat{V}$ are given by

$$x = \frac{2m^3GN}{\hbar^2}r \quad (2.8)$$

$$\phi = \sqrt{4\pi} \left(\frac{\hbar^2}{2m^3GN} \right)^{\frac{3}{2}} \psi \quad (2.9)$$

$$\hat{V} = \frac{\hbar^2}{2G^2N^2m^4}V \quad (2.10)$$

$$\hat{E} = \frac{\hbar^2}{2G^2N^2m^5}E \quad (2.11)$$

$\hat{V}$ and $\hat{E}$ correspond to the one particle potential and the one particle energy, respectively. The boundary conditions require that $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$. Also, since the boson star is in the ground state, it is required that $\phi(r)$ have no nodes.

By solving equations (2.5), (2.6), and (2.7) numerically for $\hat{E}$, letting $E = E_0$ for the binding energy of a boson in the ground state within the star, and multiplying this by the number of bosons in the star N , the binding energy of the boson star is then [1, 2]

$$E_0 = -0.05426 \frac{N^3G^2m^5}{\hbar^2}. \quad (2.12)$$

Thus, the total mass M of the boson star becomes

$$M = Nm - 0.05426 \frac{N^3G^2m^5}{\hbar^2c^2} \quad (2.13)$$

where Nm is the rest mass of the N bosons and the last term is the mass equivalent of the binding energy which is lost when the boson star forms. This result is also obtained by Membrado, where the system of equations above was written and solved

Figure 1: This plot shows the dimensionless mass quantity $\mathcal{M}$ as a function of the particle number quantity $\mathcal{N}$.

in terms of a variational problem for the particle density [6]

$$n(r) = \frac{\rho(r)}{m}. \quad (2.14)$$

In Figure 1, the total mass $\mathcal{M}$ is plotted has a function of $\mathcal{N}$. Note that the maximum mass occurs when

$$\mathcal{N} = 2.48 \quad (2.15)$$

where

$$\mathcal{N} = N \frac{m^2}{M_{Pl}^2}. \quad (2.16)$$

At this value of $\mathcal{N}$, we find that

$$\mathcal{M}_{max} = 1.652 \quad (2.17)$$

where

$$\mathcal{M}_{max} = M_{max} \frac{m}{M_{Pl}^2}. \quad (2.18)$$

Note from Figure 1 that the mass decreases after reaching its maximum value and even becomes negative for large values of $\mathcal{N}$. Also, note that at the maximum mass, the gravitational binding energy of one boson reaches one-third of its rest mass energy. Thus, the Newtonian approximation breaks down long before the maximum mass is reached. One other problem with this model is that even as the number of particles increases, the mass of the star decreases, which is physically impossible.

2.2 Analytic Estimates of the Ground State Energy of a Boson Star

Several papers have been written on analytic bounds to the ground-state energy of a system of N self-gravitating bosons in the Newtonian approximation [7, 8]. As we shall see in this section, these analytic estimates seem to support the results found from the Hartree approximation.

Let's begin by considering a system of N identical bosons of mass m interacting in pairs through Newton's potential [7]

$$V(\mathbf{r}_i - \mathbf{r}_j) = \frac{-Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2.19)$$

The total Hamiltonian of the system, therefore, is

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} - \sum_{1 \leq i < j \leq N} \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2.20)$$

However, this Hamiltonian can be rewritten as the sum of $\frac{N(N-1)}{2}$ non-commuting two-body Hamiltonians h_{ij}

$$H = \sum_{i < j} h_{ij}. \quad (2.21)$$

Note that each two-body Hamiltonian

$$h_{ij} = \frac{p_i^2}{2m(N-1)} + \frac{p_j^2}{2m(N-1)} - \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (2.22)$$

has a hydrogen-like form with an effective mass of $m(N-1)$. To minimize the ground state energy of the N -particle boson sphere, we can first minimize each two-body Hamiltonian. Because of the hydrogen-like form of the two-body Hamiltonians, we immediately see that their expectation values $\langle h_{ij} \rangle$ in the ground state satisfies the inequality

$$\langle h_{ij} \rangle \geq -\frac{(N-1)}{4} \frac{G^2 m^5}{\hbar^2} \quad (2.23)$$

which can be seen from the ground state energy of the electron in a Coulomb potential of a hydrogen-like system

$$E_1 = -\frac{me^4}{2\hbar^2}. \quad (2.24)$$

Multiplying this by $\frac{N(N-1)}{2}$, the lower bound for the ground state energy of the system is

$$E_o \geq -\frac{N(N-1)^2}{8} \frac{G^2 m^5}{\hbar^2}. \quad (2.25)$$

For large N , we find that

$$E_o \geq -0.125 \frac{N^3 G^2 m^5}{\hbar^2}. \quad (2.26)$$

A better estimate would be found if we subtracted the contribution due to motions of the center-of-mass since global motions of the system are of no interest here [8]. To separate out the center-of-mass kinetic energy, we can use the following identity:

$$\sum_{i=1}^N p_i^2 = \frac{1}{N} \left[\left(\sum_i p_i \right)^2 + \sum_{i < j} (p_i - p_j)^2 \right] \quad (2.27)$$

where $\sum_i p_i = \mathbf{P}$ is the total momentum of the system. The relative motion of the N bosons in the center of mass system ($\mathbf{P} = 0$) is described by

$$H_{rel} = \sum_{i < j} \hat{h}_{ij} \quad (2.28)$$

where

$$\hat{h}_{ij} = \frac{4\pi_{ij}^2}{2mN} - \frac{Gm^2}{|\mathbf{Q}_{ij}|} \quad (2.29)$$

after using the canonical coordinates $\mathbf{Q}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\pi_{ij} = \frac{\mathbf{p}_i - \mathbf{p}_j}{2}$. Due to the hydrogen-like form of the Hamiltonian, the expectation value of the two-body Hamiltonian has the form

$$\langle \hat{h}_{ij} \rangle \geq -\frac{N}{8} \frac{G^2 m^5}{\hbar^2}. \quad (2.30)$$

Multiplying this by $\frac{N(N-1)}{2}$, the ground state energy is bounded by

$$E_o \geq -\frac{N^2(N-1)}{16} \frac{G^2 m^5}{\hbar^2}. \quad (2.31)$$

For large values of N ,

$$E_o \geq -0.0625 \frac{N^3 G^2 m^5}{\hbar^2}. \quad (2.32)$$

These results could be improved somewhat by considering the system as three- or four-body Hamiltonians. However, the calculations would become much more difficult.

These analytic arguments give results for the binding energy which are within fifteen percent of the value found by the Hartree approximation. Therefore, the analytic results seem to support this approximation fairly well and suggest that, in the context of the Newtonian approach, the Hartree method gives fairly reasonable results.

2.3 General Relativistic Treatment Using a Quantum Boson Field

We will start this section by deriving the normal modes and their corresponding energies for a system of two coupled harmonic oscillators in Section 3.2.1 and generalize to a system of N coupled harmonic oscillators in order to gain a better physical interpretation of the modes and their relationships to particles and antiparticles in the quantum approach which will be introduced in Section 2.3.3.

In Section 2.3.2, we will introduce the classical scalar field through the continuous analogue of the coupled oscillator problem in Section 3.2.1 because the concept of a field will play an important role in the derivation of the equations of motion of a boson star. In this section, we will also introduce the Lagrangian associated with the Klein-Gordon equation as the Klein-Gordon equation represents the wave equation of the scalar field in the boson star. The scalar field satisfying the Klein-Gordon equation will also be expanded in terms of the eigenmodes of the system.

In Section 2.3.3, we will discuss the procedure of passing from classical field theory to quantum field theory and will also discuss the procedure of subtracting the zero-point energy from the energy levels of the system. The quantum approach will have the advantage of being able to relate the modes of a system to the presence of particles and antiparticles.

In Section 2.3.4, we will introduce the complex scalar field as a compact method of interpreting boson-antiboson systems and will show that the complex field results in the conservation of the difference between the number of bosons and antibosons in the system.

Finally, in Section 2.3.5, we will derive the Einstein field equations and will use the results of Sections 2.3.3 and 2.3.4 to derive the equations of motion of the boson star. Then we will give a brief discussion on these equations, which were found by Ruffini and Bonazzola.

2.3.1 *Modes of a System of Coupled Oscillators as a Preface to Quantum Mode Operators*

As a preface to quantum mode operators, which will be studied in Section 2.3.3, we begin by studying the properties of modes. Therefore, we consider the problem of finding the normal modes of two coupled harmonic oscillators of mass m where the oscillators are each attached to a separate wall and attached to each other by strings which have the same tension [9]. These oscillators undergo small transverse vibrations where $q_r(t)$ is the transverse displacement. In this problem, we first state the kinetic energy of the system and find the potential energy. Then, we find the equations of motion, which will most likely be coupled. But, by defining two new coordinates $Q_1(t)$ and $Q_2(t)$ in terms of the oscillator coordinates $q_1(t)$ and $q_2(t)$, we find that in terms of the new coordinates, the equations of motion are now uncoupled. By defining these coordinates, we also find that the motion of the system is actually a superposition of two modes, where a mode is defined as the movement of the system with a definite frequency. Since $Q_1(t)$ and $Q_2(t)$ each define a specific mode of the system, we call these normal mode coordinates. Some of the details of this calculation are given in Appendix A.1.

One of the results of this calculation is that the modes of the system do not interact because the equations of motion in terms of the normal mode coordinates are uncoupled. However, this is true only because we have ignored the anharmonic

terms in the potential energy. This noninteraction of modes is also seen in the total energy of the system, which is

$$E = \left[\frac{1}{2}m \left(\frac{dQ_1}{dt} \right)^2 + \frac{1}{2}m\omega_1^2 Q_1^2 \right] + \left[\frac{1}{2}m \left(\frac{dQ_2}{dt} \right)^2 + \frac{1}{2}m\omega_2^2 Q_2^2 \right]. \quad (2.33)$$

This energy has the appropriate form for two non-interacting modes each executing simple harmonic motion.

One important thing to notice here is that each mode corresponds to a degree of freedom. If we consider a system of N coupled oscillators which has N degrees of freedom, we will then find N different modes [9]. In the N oscillator case, we also ignore the anharmonic terms in the potential and define a linear superposition of the coordinates to give the normal coordinates

$$Q_r = \sum_{s=1}^N a_{rs} q_s \quad (2.34)$$

Therefore, it is possible to write E as the sum of the energies of all of the N modes

$$E = \sum_{r=1}^N \left[\frac{1}{2}m \left(\frac{dQ_r}{dt} \right)^2 + \frac{1}{2}m\omega_r^2 Q_r^2 \right] \quad (2.35)$$

as none of the modes interact. Thus, we can say that the N -oscillator system or N -atom system behaves as if there are N separate and free harmonic oscillators where each mode oscillator and degree of freedom correspond to a mode.

In Sections 2.3.3 and 2.3.4, we will show how modes correspond to the destruction and creation operators of particles and antiparticles.

2.3.2 Introduction of the Scalar Field

In this section, we will begin by considering the continuous analogue of the N -oscillator system and introduce the concept of a scalar field. We will use Hamilton's principle of least action, in which the action of a system is minimized with respect to the field, to introduce Lagrange's equations of motion from which the wave equation of the continuous string of oscillators can be found [10, 11]. Then, we will give a brief discussion on the Hamiltonian and the canonical momentum and introduce the momentum field conjugate to the field introduced in this section. We will also discuss the behavior of the modes in the continuum case. Finally, we will introduce the Klein-Gordon field as this field serves as the scalar boson field in the next boson star model to be discussed.

To move from the one-dimensional discrete oscillator problem to the continuum case of a string of oscillators, we allow the number of oscillators to increase from N to infinity while decreasing the separation between the oscillators to zero in such a manner that the product Na remains finite. In this limit, we now have a continuous string of oscillators where $Na = L$ is the length of the string. The transverse vibrations of the string are now described by a scalar field $\phi(x, t)$ which measures the displacement from equilibrium of a small element of string around the point x at time t . Therefore, we can say that in the continuum limit,

$$q_r(t) \rightarrow \phi(x, t). \quad (2.36)$$

Now, we want to derive the wave equation for the field $\phi(x, t)$. To derive the wave equation, we begin by using Hamilton's principle of least action [10, 11]. The principle of least action is based on the postulate that out of all of the possible paths that the particle can follow in moving from one location to another, it follows the path for which the action I is minimized.

The action I is defined as the time integral over the Lagrangian

$$I = \int dt L \quad (2.37)$$

where the Lagrangian is given by the between the kinetic and potential energies

$$L = T + V. \quad (2.38)$$

However, in the continuum case, we can define the Lagrangian density in the one-dimensional case by the integral

$$L = \int dx \mathcal{L}. \quad (2.39)$$

Therefore, the action becomes

$$I = \int dt \int dx \mathcal{L}. \quad (2.40)$$

To minimize the action, we make use of the calculus of variations. The Lagrangian density $\mathcal{L}$, in the continuous case, is a functional of the field $\phi(x, t)$, the time derivative

of the field $\frac{d\phi}{dt}$, and the space derivative of the field $\frac{d\phi}{dx}$. In the calculus of variations, we consider a small variation of the field $\delta\phi(x, t)$. At the minimum, the change in the action δI corresponding to a change in the field $\delta\phi(x, t)$ must vanish. This change in the action is given by

$$\delta I = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial x}\right)} \delta \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial t}\right)} \delta \left(\frac{\partial \phi}{\partial t} \right) \right]. \quad (2.41)$$

Now, if we integrate the last two terms by parts and discard the surface terms, we find that

$$\delta I = \int dt \int dx \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial x}\right)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial t}\right)} \right) \right]. \quad (2.42)$$

At the minimum, $\delta I = 0$. Also, since the variation $\delta\phi$ is arbitrary, the term in brackets must be equal to zero. Therefore, we end up with Lagrange's equations of motion

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial x}\right)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial t}\right)} \right). \quad (2.43)$$

We should also introduce the Hamiltonian as it will play an important role when we change from classical field theory to quantum field theory. In classical particle mechanics, we define the Hamiltonian by

$$H(p, q) = p \frac{dq}{dt} - L. \quad (2.44)$$

However, if we work in the continuous case, we can calculate the Hamiltonian from

$$H = \int dx \mathcal{H} \quad (2.45)$$

where $\mathcal{H}$ is the Hamiltonian density. Now, in particle mechanics, the conjugate momentum of the coordinate q is defined by

$$p \equiv \frac{\partial L}{\partial \left(\frac{dq}{dt} \right)}. \quad (2.46)$$

Likewise, in the continuous case, $\phi(x, t)$ has a conjugate “momentum field” defined by

$$\pi(x, t) = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi(x, t)}{\partial t} \right)}. \quad (2.47)$$

Therefore, in a similar manner to the Hamiltonian, we define the Hamiltonian density by

$$\mathcal{H}(\phi, \pi) = \pi(x, t) \frac{\partial \phi(x, t)}{\partial t} - \mathcal{L}. \quad (2.48)$$

All of the aspects of fields and field theory that we have introduced above will become very useful when we quantize and apply the field to boson stars.

Now, we consider the Lagrangian density of the continuous string of oscillators

$$\mathcal{L} = \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \rho c^2 \left(\frac{\partial \phi}{\partial x} \right)^2 \quad (2.49)$$

where ρ is the linear mass density and c is the velocity of propagation of a wave [10].

The quantity ρc^2 is sometimes referred to as Young’s modulus Y . Substituting this into Lagrange’s equation (2.43), we obtain the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^2} = 0. \quad (2.50)$$

In Appendix A.2, we assume a sinusoidal solution to this equation

$$\phi_r(x, t) = A_r(t) \sin \frac{r\pi x}{L} \quad : r = 1, 2, 3, \dots \quad (2.51)$$

In showing that this solution satisfies the wave equation, we find that the amplitude of the sine wave corresponds to a specific mode. We also notice that in an analogous manner to the discrete case the general motion of this system is described by the superposition of the mode functions which is given by

$$\phi(x, t) = \sum_{r=1}^{\infty} A_r(t) \sin \frac{r\pi x}{L}. \quad (2.52)$$

We also find an energy relation analogous to the energy relation in the discrete case.

This energy is given by

$$E = \left(\frac{L}{2}\right) \sum_{r=1}^{\infty} \left[\frac{1}{2} \rho \left(\frac{\partial A_r(t)}{\partial t} \right)^2 + \frac{1}{2} \rho \omega_r^2 A_r^2 \right], \quad (2.53)$$

which is the sum of the individual mode energies just as in the discrete case. Note that A_r , which is actually a Fourier component, acts as a normal coordinate. Therefore, we see that even though there are an infinite number of oscillators, the infinite number of modes all behave as independent harmonic oscillators.

$A_r(t)$, which is the normal mode coordinate in the continuous case, will play an important role when we quantize the field in the next section. In quantum field theory, the quantity analogous to $A_r(t)$ will be known as a quantum mode operator, and it will act as either a creator or annihilator of particles or antiparticles.

We should also introduce the Klein-Gordon field as it will be the foundation of the scalar boson field of the boson star [9]. Therefore, we begin by generalizing Lagrange's equation (2.43) to three dimensions and by writing it in a relativistically invariant form

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \quad (2.54)$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. With $\rho = c = 1$, the Lagrangian density of the continuous string (2.49) becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \quad (2.55)$$

where

$$\partial^\mu \phi = \eta^{\mu\nu} \partial_\nu \phi \quad (2.56)$$

and $\eta^{\mu\nu}$ is the inverse of the flat space Minkowski metric $\eta_{\mu\nu}$

$$\eta_{00} = 1 \quad (2.57)$$

$$\eta_{11} = \eta_{22} = \eta_{33} = -1 \quad (2.58)$$

Using the Lagrangian density (2.55) in Lagrange's equation of motion, we find the wave equation

$$\partial_\mu \partial^\mu \phi = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi = 0. \quad (2.59)$$

Note that except for ϕ , all of these relations will remain unchanged when we quantize the field.

The invariant Lagrangian density above describes a massless particle. For a particle of mass m , the Lagrangian density takes the form (2.55)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (2.60)$$

where we have added the kinetic term for a particle of mass m . Substituting this Lagrangian density into Lagrange's equation (2.54) gives the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0. \quad (2.61)$$

In a spherically symmetric system, the Klein-Gordon field, in a similar manner to the field of the continuous string, can be expanded in terms of the eigenmodes of the system

$$\phi(\mathbf{r}, t) = \sum_{nlm} \psi_{nlm}(\mathbf{r}, t) \quad (2.62)$$

where

$$\psi_{nlm}(\mathbf{r}, t) = R_{nl}(r) Y_l^m(\theta, \varphi) e^{-i\omega_{nl}t}. \quad (2.63)$$

Therefore, the most general form of the Klein-Gordon field is given by

$$\phi(\mathbf{r}, t) = \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} [a_{nlm} \psi_{nlm}(\mathbf{r}, t) + a_{nlm}^* \psi_{nlm}^*(\mathbf{r}, t)]. \quad (2.64)$$

Note that the Klein-Gordon field is real and that the coefficients a_{nlm}^* and a_{nlm} behave as the normal mode coordinates for the system described by this field. We have also introduced the normalization factor $\frac{1}{\sqrt{2\omega_{nl}}}$ as it will be convenient for calculations in Section 2.4.

2.3.3 Quantization of the Klein-Gordon Field

As we are interested in the Klein-Gordon quantum field, we now review the procedure in passing from classical to quantum field theory. In this procedure, the Heisenberg formulation of quantum mechanics is very useful [12].

We also follow the same procedure as in Appendix A.3 when going from classical to quantum field theory [9]. Therefore, we replace the field $\phi(x, t)$ and its conjugate momentum field $\pi(x, t)$ by their corresponding quantum operators $\hat{\phi}$ and $\hat{\pi}$, where

$$\hat{\pi} = \frac{\partial \hat{\mathcal{L}}}{\partial \left(\frac{\partial \hat{\phi}}{\partial t} \right)}. \quad (2.65)$$

We expect that these operators have the same form of commutator as their counterparts in particle mechanics. Therefore, generalizing to the continuum case,

$$[\hat{\phi}(x, t), \hat{\pi}(y, t)] = i\delta(x - y) \quad (2.66)$$

where we have replaced δ_{rs} by its corresponding continuum counterpart, the Dirac delta function $\delta(x - y)$. The generalization of this commutator to three dimensions gives the following relation:

$$[\hat{\phi}(\mathbf{r}, t), \hat{\pi}(\mathbf{r}', t)] = i\delta^3(\mathbf{r} - \mathbf{r}'). \quad (2.67)$$

When we quantize the Klein-Gordon field (2.64), the coefficients or mode coordinates a_{nlm} and a_{nlm}^* , which are actually the quantities in the field that describe the behavior of the system through the modes, become the quantum mode operators $\hat{a}_{nlm}$

and $\hat{a}_{nlm}^\dagger$. Therefore,

$$\hat{\phi}(\mathbf{r}, t) = \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} [\hat{a}_{nlm} \psi_{nlm}(\mathbf{r}, t) + \hat{a}_{nlm}^\dagger \psi_{nlm}^*(\mathbf{r}, t)]. \quad (2.68)$$

By using the corresponding operator of the Lagrangian density (2.60), the definition of the momentum field density operator (2.65), and the field (2.64), the momentum field density becomes

$$\hat{\pi}(\mathbf{r}, t) = -i \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} \omega_{nl} [\hat{a}_{nlm} \psi_{nlm}(\mathbf{r}, t) - \hat{a}_{nlm}^\dagger \psi_{nlm}^*(\mathbf{r}, t)]. \quad (2.69)$$

Now that we have the operators $\hat{\phi}$ and $\hat{\pi}$, we can substitute them into the commutator (2.67) and use the closure relation

$$\sum_{nlm} \psi_{nlm}(\mathbf{r}, t) \psi_{nlm}^*(\mathbf{r}', t) = \delta^3(\mathbf{r} - \mathbf{r}') \quad (2.70)$$

to find the commutator relations between the mode operators, which is given by

$$[\hat{a}_{nlm}, \hat{a}_{n'l'm'}^\dagger] = \delta_{n'n} \delta_{l'l} \delta_{m'm}, \quad (2.71)$$

with all others vanishing. Note that this commutator has the same form as the commutator of the creation and annihilation operators of the harmonic oscillator [9]

$$[\hat{a}_r, \hat{a}_s^\dagger] = \delta_{rs} \quad (2.72)$$

In order to clarify the behavior of these operators, let's consider a state ket of the form

$$|\dots, N_{nlm}, N_{n'l'm'}, \dots\rangle \quad (2.73)$$

where each N_{nlm} corresponds to the occupancy of a given state ψ_{nlm} . Now, the operators $\hat{a}_{nlm}$ and $\hat{a}_{nlm}^\dagger$, in a similar manner to the creation and annihilation operators of the harmonic oscillator, will act on this ket. However, each of the $\hat{a}_{nlm}$ will act only on the state specified by the operator, as is shown in the following equations

$$\hat{a}_{nlm} |\dots, N_{nlm}, N_{n'l'm'}, \dots\rangle = \sqrt{N_{nlm}} |\dots, N_{nlm} - 1, N_{n'l'm'}, \dots\rangle \quad (2.74)$$

$$\hat{a}_{nlm}^\dagger |\dots, N_{nlm}, N_{n'l'm'}, \dots\rangle = \sqrt{N_{nlm} + 1} |\dots, N_{nlm} + 1, N_{n'l'm'}, \dots\rangle \quad (2.75)$$

These equations can be shown from the commutator (2.71). These operators thus act as annihilation and creation operators of the quanta associated with the quantized Klein-Gordon field.

We can further support this argument by finding the Hamiltonian associated with the Klein-Gordon field in terms of the mode operators [9]. From equations (2.45), (2.48), and (2.60), we find that

$$\hat{H}_{KG} = \frac{1}{2} \int d\mathbf{r} \left[\left(\frac{\partial \hat{\phi}}{\partial t} \right)^2 + \nabla \hat{\phi} \cdot \nabla \hat{\phi} + m^2 \hat{\phi}^2 \right]. \quad (2.76)$$

If we integrate the second term by parts, drop the surface term, and use the Klein-Gordon equation (2.61), then the Hamiltonian becomes

$$\hat{H}_{KG} = \frac{1}{2} \int d\mathbf{r} \left[\left(\frac{\partial \hat{\phi}}{\partial t} \right)^2 - \frac{\partial^2 \hat{\phi}}{\partial t^2} \hat{\phi} \right]. \quad (2.77)$$

If we substitute in the Klein-Gordon field operator (2.64), the Hamiltonian then has

the form

$$\begin{aligned} \hat{H}_{KG} = & \frac{1}{2} \sum_{nlm} \sum_{n'l'm'} \sqrt{\omega_{nl}\omega_{n'l'}} \int d\mathbf{r} [\hat{a}_{nlm}^\dagger \hat{a}_{n'l'm'} \psi_{nlm}^*(\mathbf{r}, t) \psi_{n'l'm'}(\mathbf{r}, t) \\ & + \hat{a}_{nlm} \hat{a}_{n'l'm'}^\dagger \psi_{nlm}(\mathbf{r}, t) \psi_{n'l'm'}^*(\mathbf{r}, t)] \end{aligned} \quad (2.78)$$

where $\psi_{nlm}(\mathbf{r}, t)$ is again given by (2.63). At this point, it is useful to think of the system in question as being in a large spherical box. If we impose the boundary condition on the radial part of the field, that it must vanish at the edge of the box and then allow the radius of the box to become infinite, each radial solution then has a discrete energy eigenvalue associated with it. Therefore, the radial solutions form a complete, orthogonal set. Because the spherical harmonics also forms a complete, orthogonal set, we then have the normalization condition

$$\int d\mathbf{r} \psi_{nlm}(\mathbf{r}, t) \psi_{n'l'm'}^*(\mathbf{r}, t) = \delta_{n'n} \delta_{l'l} \delta_{m'm} \quad (2.79)$$

and the Hamiltonian takes the form

$$\hat{H}_{KG} = \sum_{nlm} \frac{1}{2} (\hat{a}_{nlm}^\dagger \hat{a}_{nlm} + \hat{a}_{nlm} \hat{a}_{nlm}^\dagger) \omega_{nl}. \quad (2.80)$$

If we use the commutator relation of the mode operators (2.71) with $n = n'$, $l = l'$, and $m = m'$, then

$$\hat{H}_{KG} = \sum_{nlm} \hat{a}_{nlm}^\dagger \hat{a}_{nlm} \omega_{nl} + \frac{1}{2} \sum_{nlm} \omega_{nl}. \quad (2.81)$$

The form of the Hamiltonian is very similar to that of the harmonic oscillator Hamiltonian [9]

$$\hat{H} = \hat{a}^\dagger \hat{a} \omega + \frac{1}{2} \omega \quad (2.82)$$

which also supports the interpretation of the mode operators as the creation and annihilation operators of the Klein-Gordon quanta.

If we assume that the mode operators behave in this manner, then the expectation value of $\hat{H}_{KG}$ in the vacuum state is

$$\langle 0 | \hat{H}_{KG} | 0 \rangle = \frac{1}{2} \sum_{nlm} \omega_{nl} \quad (2.83)$$

This term therefore corresponds to the analogue of the zero point energy of the oscillator and is infinite. However, since we are interested only in energy differences, we then define the vacuum state to be at zero energy by discarding the infinite “zero point” energy. Therefore, the final form of the Hamiltonian is

$$\hat{H}_{KG} = \sum_{nlm} \hat{a}_{nlm}^\dagger \hat{a}_{nlm} \omega_{nl}. \quad (2.84)$$

It is thus likely that the mode operators $\hat{a}_{nlm}$ and $\hat{a}_{nlm}^\dagger$ are the creation and annihilation operators, respectively, of the quanta associated with the Klein-Gordon field.

2.3.4 Introduction of the Complex Scalar Field

The introduction of complex fields at this point is very important because of their usefulness in the boson star model that will be introduced in the next section. The complex field is a natural way of describing the two distinct states of most types of matter, namely the particle and antiparticle states. Another important property of the complex field that a real field does not have is a conserved symmetry current

associated with the quantity of the number of particle states minus the number of antiparticle states. The boson star must have both of these properties.

The Klein-Gordon field (2.64) is known as a real scalar field because it is real and because it has no spin degrees of freedom. We also note that the Klein-Gordon field only allows one type of state, the particle state in our case, to exist for the quanta of mass m . However, it is likely that a given mass m has two distinct forms, particles and antiparticles. To describe this situation, we have two choices [9]: We either introduce two real scalar fields, $\hat{\phi}_1$ and $\hat{\phi}_2$ with the Lagrangian density

$$\hat{\mathcal{L}} = \frac{1}{2}\partial_\mu\hat{\phi}_1\partial^\mu\hat{\phi}_1 - \frac{1}{2}m^2\hat{\phi}_1^2 + \frac{1}{2}\partial_\mu\hat{\phi}_2\partial^\mu\hat{\phi}_2 - \frac{1}{2}m^2\hat{\phi}_2^2; \quad (2.85)$$

or equivalently, we introduce a complex scalar field

$$\hat{\phi} = \hat{\phi}_1 - i\hat{\phi}_2 \quad (2.86)$$

$$\hat{\phi}^\dagger = \hat{\phi}_1 + i\hat{\phi}_2 \quad (2.87)$$

which has the corresponding Lagrangian density

$$\hat{\mathcal{L}} = \frac{1}{2}\partial_\mu\hat{\phi}^\dagger\partial^\mu\hat{\phi} - \frac{1}{2}m^2|\hat{\phi}|^2. \quad (2.88)$$

We choose to work with the complex field as it is a compact and convenient form to use.

Since $\hat{\phi}_1$ and $\hat{\phi}_2$ are two real Klein-Gordon fields with the form given by equation (2.64), then the complex scalar field (2.86) has the form

$$\hat{\phi}(\mathbf{r}, t) = \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} [\hat{a}_{nlm} \psi_{nlm}(\mathbf{r}, t) + \hat{b}_{nlm}^\dagger \psi_{nlm}^*(\mathbf{r}, t)] \quad (2.89)$$

where $\psi_{nlm}(\mathbf{r}, t)$ is given by equation (2.63) and

$$\hat{a}_{nlm} = \hat{a}_{(1)nlm} - i\hat{a}_{(2)nlm} \quad (2.90)$$

$$\hat{b}_{nlm}^\dagger = \hat{a}_{(1)nlm}^\dagger - i\hat{a}_{(2)nlm}^\dagger. \quad (2.91)$$

By using the commutators for the $\hat{a}_{(i)nlm}$ operators (2.71), the operators $\hat{a}_{nlm}$, $\hat{a}_{nlm}^\dagger$, $\hat{b}_{nlm}$, and $\hat{b}_{nlm}^\dagger$ now obey the commutator relations

$$[\hat{a}_{nlm}, \hat{a}_{n'l'm'}^\dagger] = \delta_{n'n} \delta_{l'l} \delta_{m'm} \quad (2.92)$$

$$[\hat{b}_{nlm}, \hat{b}_{n'l'm'}^\dagger] = \delta_{n'n} \delta_{l'l} \delta_{m'm}. \quad (2.93)$$

with all other commutators equal to zero. Note that we now have two distinct mode operators.

Now, by finding the Hamiltonian in terms of these new mode operators, we intend to show that $\hat{a}_{nlm}$ and $\hat{b}_{nlm}$ act as the annihilation operators for particles and antiparticles, respectively, and that $\hat{a}_{nlm}^\dagger$ and $\hat{b}_{nlm}^\dagger$ act as the creation operators for particles and antiparticles, respectively, of the quanta of the complex field. If we follow a similar procedure as in the case of a real field (2.82) and substitute in the

complex field (2.89), the Hamiltonian, after dropping the zero-point energy terms, becomes [9]

$$\hat{H} = \sum_{nlm} (\hat{a}_{nlm}^\dagger \hat{a}_{nlm} + \hat{b}_{nlm}^\dagger \hat{b}_{nlm}) \omega_{nl}. \quad (2.94)$$

Since both terms in this Hamiltonian are of the form of the Hamiltonian in the case of the real scalar field, this supports the statement given above about the new mode operators.

The next task is to calculate the symmetry current for the complex field [9]. In order for the symmetry current to exist and to be conserved, the Lagrangian density must be globally invariant under $U(1)$ group transformations. Real fields do not have the $U(1)$ symmetry and thus are not desirable for use in boson stars. A $U(1)$ transformation is simply a unitary one-dimensional transformation matrix and has the form

$$\hat{\phi}' = e^{-i\alpha} \hat{\phi}. \quad (2.95)$$

Under this transformation for a complex field, the change in the Lagrangian density is zero, $\delta\hat{\mathcal{L}} = 0$. Now, since $\hat{\mathcal{L}}$ is an explicit function of $\hat{\phi}$, $\partial_\mu \hat{\phi}$, and their Hermitean conjugates, then the change in the Lagrangian density is

$$0 = \frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu \hat{\phi})} \delta(\partial_\mu \hat{\phi}) + \frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu \hat{\phi}^\dagger)} \delta(\partial_\mu \hat{\phi}^\dagger) + \frac{\partial\hat{\mathcal{L}}}{\partial\hat{\phi}} \delta\hat{\phi} + \frac{\partial\hat{\mathcal{L}}}{\partial\hat{\phi}^\dagger} \delta\hat{\phi}^\dagger. \quad (2.96)$$

However, if we use Lagrange's equations (2.54) and use the $U(1)$ transformation symmetry of the complex field, we then find that

$$0 = \alpha \partial_\mu [i(\partial^\mu \hat{\phi}) \hat{\phi}^\dagger - i(\partial^\mu \hat{\phi}^\dagger) \hat{\phi}]. \quad (2.97)$$

But, because the quantity α is arbitrary, then

$$J^\mu = i[\hat{\phi}^\dagger(\partial^\mu \hat{\phi}) - \hat{\phi}(\partial^\mu \hat{\phi}^\dagger)] \quad (2.98)$$

is conserved:

$$\partial_\mu J^\mu = 0. \quad (2.99)$$

The quantity J^μ is the symmetry current stated above. The symmetry operator $\hat{N}$, which is also known as the number operator, is given by [9]

$$\hat{N} = \int d\mathbf{r} \hat{J}^0. \quad (2.100)$$

If we use equation (2.98), the normalization condition (2.79), and substitute in the complex field (2.89), the number operator becomes [9]

$$\hat{N} = \sum_{nlm} (\hat{a}_{nlm}^\dagger \hat{a}_{nlm} - \hat{b}_{nlm}^\dagger \hat{b}_{nlm}). \quad (2.101)$$

This calculation is somewhat similar to the calculation of the Hamiltonian. Note the similarity between this operator and the number operator of the harmonic oscillator [9]

$$\hat{N} = \hat{a}^\dagger \hat{a}. \quad (2.102)$$

In the harmonic oscillator, the number operator counts the number of quanta or particles present. Therefore, this leads to the interpretation in equation (2.101) of $\hat{a}_{nlm}^\dagger \hat{a}_{nlm}$ as the operator which keeps track of the number of particles in the state ψ_{nlm} and of $\hat{b}_{nlm}^\dagger \hat{b}_{nlm}$ as the operator which keeps track of the number of antiparticles

in the state ψ_{nlm} . We interpret $\hat{b}_{nlm}^\dagger \hat{b}_{nlm}$ as the antiparticle number operator because it has the opposite effect, noted by the minus sign, as the particle number operator on the total number operator.

Due to that minus sign, we see that the conserved quantity is the number of particles minus the number of antiparticles. The boson star model which will be introduced in the next section must have this property. Therefore, we intend to use a complex scalar boson field.

2.3.5 Derivation of the Equations of Motion

In this section, we derive the equations of motion of the boson star in general relativity which were first considered by Ruffini and Bonazzola [2]. The bosons within the star are self-gravitating and exhibit no coupling interaction. Therefore, we are able to describe the scalar bosons by the curved-space Klein-Gordon equation. We must note that in the curved space of general relativity, we replace the Minkowski metric $\eta_{\mu\nu}$ by the curved-space metric tensor $g_{\mu\nu}$. $g_{\mu\nu}$ depends on the symmetry of the system we are concerned with.

We will begin by describing the system in terms of an action similar to that in Section 2.3.2. We will also introduce the action of the gravitational field. By considering a variation of these actions in a similar procedure to that in Section 2.3.2, we will then derive the equations motions for the system, including the Klein-Gordon equation. Next, we will use a procedure analogous to the procedure in flat space

given in Section 2.3.3. We will write out the analogous curved-space quantities and will show that the mode operators have the same behavior in curved and flat space. Then, we will use these analogous curved-space quantities to derive the explicit form of the equations of motion. Finally, we will discuss some of the results found by Ruffini and Bonazzola.

Since we are working with a spherically symmetric, isotropic system, we must use a specific form for the metric and the metric tensor of the system. A metric

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \quad (2.103)$$

and the metric tensor $g_{\mu\nu}$ basically depend the properties of the space in which our system is located. The metric of a spherically symmetric space is known as the Schwarzschild metric [13]

$$d\tau^2 = B(r) dt^2 - A(r) dr^2 - r^2 d\Omega^2 \quad (2.104)$$

where

$$g_{00} = -B(r)$$

$$g_{11} = A(r)$$

$$g_{22} = r^2$$

$$g_{33} = r^2 \sin^2 \theta$$

and g , which we shall define as minus the determinant of the metric tensor, is given

by

$$g = -\text{Det } g_{\mu\nu} = A(r)B(r)r^4 \sin^2 \theta. \quad (2.105)$$

Since we are now working in curved space associated with general relativity, the Lagrangian density (2.60) now takes on the form [2]

$$\mathcal{L} = -\frac{1}{2}\sqrt{g} (g^{\mu\nu} \phi_{;\mu}^* \phi_{;\nu} + m^2 |\phi|^2). \quad (2.106)$$

Note also that the Lagrangian density contains $\sqrt{g}$.

We begin by calculating the equations of motion and the Klein-Gordon equation from the action

$$I = I_M + I_G \quad (2.107)$$

where

$$I_M = \int d^4x \mathcal{L} \quad (2.108)$$

is the matter action and

$$I_G = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R \quad (2.109)$$

is the gravitational action [13]. The quantity R , which is called the curvature scalar, is defined by

$$R = g^{\mu\nu} R_{\mu\nu} \quad (2.110)$$

where the quantity $R_{\mu\nu}$ is the Ricci tensor [13]

$$R_{\mu\nu} = \partial_\nu \Gamma_{\mu\lambda}^\lambda - \partial_\lambda \Gamma_{\mu\nu}^\lambda + \Gamma_{\mu\lambda}^\eta \Gamma_{\nu\eta}^\lambda - \Gamma_{\mu\nu}^\eta \Gamma_{\lambda\eta}^\lambda. \quad (2.111)$$

The quantity $\Gamma_{\mu\nu}^\lambda$ is known as the affine connection and is related to $g_{\mu\nu}$ by [13]

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}). \quad (2.112)$$

Now, we must consider a variation of the action δI . A variation of the gravitational action δI_G gives

$$\delta(\sqrt{g}R) = \sqrt{g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{g} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}. \quad (2.113)$$

The quantities $\delta g^{\mu\nu}$, $\delta\sqrt{g}$, and $\delta R_{\mu\nu}$ are calculated in reference [13] and turn out to be

$$\delta R_{\mu\nu} = 0 \quad (2.114)$$

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \quad (2.115)$$

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}. \quad (2.116)$$

Therefore, if we substitute these into the variation of the integrand (2.113), the gravitational action then becomes

$$\delta I_G = \frac{1}{16\pi G} \int d^4x \sqrt{g}\delta g_{\mu\nu} [R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R]. \quad (2.117)$$

Now, let's consider a variation of the matter action

$$\delta I_M = \int d^4x \delta(\mathcal{L}) \quad (2.118)$$

By using equations (2.106), (2.115), (2.116), and integration by parts, the variation of the matter action becomes

$$\begin{aligned} \delta I_M = & \int d^4x \sqrt{g} \delta g_{\mu\nu} \left[\frac{1}{4} (\partial^\mu \phi^* \partial^\nu \phi + \partial^\nu \phi^* \partial^\mu \phi) - \frac{1}{4} g^{\mu\nu} (g^{\rho\kappa} \partial_\rho \phi^* \partial_\kappa \phi \right. \\ & \left. + m^2 |\phi|^2) \right] + \frac{1}{2} \int d^4x \sqrt{g} \delta \phi^* [\phi_{;\mu}{}^{;\mu} - m^2 \phi] \\ & + \frac{1}{2} \int d^4x \sqrt{g} \delta \phi [\phi_{;\mu}^*{}^{;\mu} - m^2 \phi^*]. \end{aligned} \quad (2.119)$$

According to the principle of least action, the variation of the total action of this system must be zero. Therefore,

$$\delta I = \delta I_M + \delta I_G = 0. \quad (2.120)$$

However, since the variations $\delta g_{\mu\nu}$, $\delta \phi$, and $\delta \phi^*$ are arbitrary, the quantity in brackets associated with each of these variations must be equal to zero. Therefore, we read off the equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \quad (2.121)$$

$$\phi_{;\mu}{}^{;\mu} - m^2 \phi = 0 \quad (2.122)$$

where

$$T_{\mu\nu} = \frac{1}{2} (\partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi) - \frac{1}{2} g_{\mu\nu} (g^{\rho\kappa} \partial_\rho \phi^* \partial_\kappa \phi + m^2 |\phi|^2) \quad (2.123)$$

is the energy-momentum tensor. Equation (2.121) is known as the Einstein gravitational equation and equation (2.122) is the Klein-Gordon equation in curved space

where we have used the definition

$$\phi_{;\mu}{}^{;\mu} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi). \quad (2.124)$$

In quantum field theory, all of the observables in the quantities above are replaced by their corresponding quantum operators.

Even though we are now considering a system in curved space described by the curved-space Klein-Gordon equation, the form of the complex field given by equation (2.89) remains unchanged

$$\hat{\phi}(\mathbf{r}, t) = \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} [\hat{a}_{nlm} \psi_{nlm}(\mathbf{r}, t) + \hat{b}_{nlm}^\dagger \psi_{nlm}^*(\mathbf{r}, t)]. \quad (2.125)$$

The eigenmodes ψ_{nlm} of the system are defined by

$$\psi_{nlm}(\mathbf{r}, t) = R_{nl}(r) Y_l^m(\theta, \varphi) e^{-i\omega_{nl}t} \quad (2.126)$$

just as in the case of flat space.

Now, let's consider some of the differences between the quantities in flat space and quantities in curved space. One difference that we note is the introduction of $\sqrt{g}$ along with the replacement of $\eta_{\mu\nu}$ by $g_{\mu\nu}$ in the Lagrangian density (2.106). Because of this change in the Lagrangian density, we also note that the momentum density

$$\hat{\pi} = \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_0 \hat{\phi})} \quad (2.127)$$

is now given by

$$\hat{\pi}(\mathbf{r}, t) = i\sqrt{g} (-g^{00}) \sum_{nlm} \frac{1}{\sqrt{2\omega_{nl}}} \omega_{nl} [\hat{a}_{nlm}^\dagger \psi_{nlm}^*(\mathbf{r}, t) - \hat{b}_{nlm} \psi_{nlm}(\mathbf{r}, t)]. \quad (2.128)$$

In a similar manner to flat space, we find that the commutator given by equation (2.67) must still hold. If we choose to use the commutators in equations (2.92) and (2.93), then the closure relation must become

$$\sum_{nlm} \sqrt{g} (-g^{00}) \psi_{nlm}(\mathbf{r}, t) \psi_{nlm}^*(\mathbf{r}', t) = \delta^3(\mathbf{r} - \mathbf{r}'). \quad (2.129)$$

Now, the form of the Hamiltonian in terms of the mode operators $\hat{a}_{nlm}$, $\hat{a}_{nlm}^\dagger$, $\hat{b}_{nlm}$, and $\hat{b}_{nlm}^\dagger$ in curved space is the same as in flat space for complex fields

$$\hat{H} = \sum_{nlm} (\hat{a}_{nlm}^\dagger \hat{a}_{nlm} + \hat{b}_{nlm}^\dagger \hat{b}_{nlm}) \omega_{nl} \quad (2.130)$$

as well as real scalar fields if we follow the same procedure laid out in Section 2.3.3. However, because of the occurrence of the factor $\sqrt{g} g^{00}$ in the momentum field density, the orthogonality condition will become

$$\int d\mathbf{r} \sqrt{g} (-g^{00}) \psi_{nlm}(\mathbf{r}, t) \psi_{n'l'm'}^*(\mathbf{r}, t) = \delta_{n'n} \delta_{l'l} \delta_{m'm}. \quad (2.131)$$

We also must consider the symmetry current. Using the same type of argument in this space as in flat space, we find that

$$J^\mu = i[\hat{\phi}(\partial^\mu \hat{\phi}^\dagger) - \hat{\phi}^\dagger(\partial^\mu \hat{\phi})]. \quad (2.132)$$

Note that there is an overall change in sign which is due to the sign convention of the metric in this problem [13]. However, the number operator has the same form as in flat space

$$\hat{N} = \sum_{nlm} (\hat{a}_{nlm}^\dagger \hat{a}_{nlm} - \hat{b}_{nlm}^\dagger \hat{b}_{nlm}). \quad (2.133)$$

Therefore, this suggests that the mode operators $\hat{a}_{nlm}$ and $\hat{a}_{nlm}^\dagger$ act as annihilation and creation operators for particles, respectively, while $\hat{b}_{nlm}$ and $\hat{b}_{nlm}^\dagger$ act in a similar manner for antiparticles just as they had in flat space.

We are now ready to apply this material to the boson star model. Since the eigenmodes (2.126) are also solutions to the Klein-Gordon equation (2.122) in curved space, we substitute the eigenmodes into the equation and find that [2]

$$\frac{1}{m^2} R''_{nl} + \frac{1}{m^2} \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{r} \right) R'_{nl} + A \left(\frac{\omega_{nl}^2}{m^2 B} - 1 - \frac{l(l+1)}{Ar^2} \right) R_{nl} = 0 \quad (2.134)$$

where the prime denotes differentiation with respect to r . This will represent the wave equation of the boson scalar field, which is now given by (2.125).

Since the boson star in this model is considered to be in a spherically symmetric ground state, then $n = 1$, $l = 0$, and $m = 0$. A boson star consisting of a fixed number of particles N , which must be conserved, is only in the ground state when the star does not contain a mixture of particles and antiparticles. This is obvious from equation (2.130). Therefore, we choose a boson star that is made up purely of bosons. The basis ket we are using to represent the ground state, which comes from equation (2.73), is given by

$$|Q\rangle = |N, 0, 0, \dots\rangle \quad (2.135)$$

where the N bosons of the star are all in the ground state. The zeros correspond to the higher energy states, all of which are unoccupied. Since we are considering a star

without any antibosons, the annihilation operator acting on the ket gives

$$b_{nlm}|Q\rangle = 0. \quad (2.136)$$

Also, because $\hat{a}_{nlm}$ and $\hat{a}_{nlm}^\dagger$ are very similar to the harmonic oscillator annihilation and creation operators, we expect that

$$\hat{a}_{100}|N, 0, 0, \dots\rangle = \sqrt{N} |N-1, 0, 0, \dots\rangle \quad (2.137)$$

$$\hat{a}_{100}^\dagger|N, 0, 0, \dots\rangle = \sqrt{N+1} |N+1, 0, 0, \dots\rangle. \quad (2.138)$$

This is also seen from equations (2.74) and (2.75).

We are now ready to calculate the expectation values of the energy-momentum tensor $\hat{T}_{\mu\nu}$ and the symmetry operator $\hat{J}^0$. By substituting the boson field into equations (2.123) and (2.132), using the metric tensor $g_{\mu\nu}$ and g , and also using the behavior of the mode operators on the ket $|Q\rangle$, we find that

$$\langle Q|T_{00}|Q\rangle = \frac{NB}{16\pi\omega_{10}} \left[\left(\frac{\omega_{10}^2}{B} + m^2 \right) R_{10}^2(r) + \frac{R_{10}'^2(r)}{A} \right] \quad (2.139)$$

$$\langle Q|T_{11}|Q\rangle = \frac{NA}{16\pi\omega_{10}} \left[\left(\frac{\omega_{10}^2}{B} - m^2 \right) R_{10}^2(r) + \frac{R_{10}'^2(r)}{A} \right] \quad (2.140)$$

$$\langle Q|J^0|Q\rangle = \frac{NR_{10}^2(r)}{8\pi B}. \quad (2.141)$$

The Einstein tensors, which are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (2.142)$$

must now be calculated. Using the metric tensor elements $g_{\mu\nu}$ and equations (2.110),

(2.111), and (2.112), G_{00} and G_{11} become

$$G_{00} = -\frac{B'}{rB} + \frac{A}{r^2}\left(1 - \frac{1}{A}\right) \quad (2.143)$$

$$G_{11} = -\frac{BA'}{rA^2} + \frac{B}{r^2}\left(1 - \frac{1}{A}\right). \quad (2.144)$$

The other two Einstein tensors and the corresponding Einstein equations are not considered here.

If we now substitute the Einstein tensor elements (2.143) and (2.144) and the corresponding energy-momentum tensor elements (2.139) and (2.140) into the Einstein field equations (2.121), divide by m^2 , and define the dimensionless quantities

$\Omega^2 \equiv \frac{\omega_{10}^2}{m^2}$, $x \equiv mr$, and $\sigma \equiv \sqrt{\frac{NG}{2\omega_{10}}} R_{10} = \sqrt{\frac{N}{2\omega_{10}}} \frac{R_{10}}{M_{Pl}}$, we then obtain

$$\frac{A'}{xA^2} + \frac{1}{x^2}\left(1 - \frac{1}{A}\right) = \left[\left(\frac{\Omega^2}{B} + 1\right)\sigma^2(x) + \frac{\sigma'^2(x)}{A}\right] \quad (2.145)$$

$$\frac{B'}{xAB} - \frac{1}{x^2}\left(1 - \frac{1}{A}\right) = \left[\left(\frac{\Omega^2}{B} - 1\right)\sigma^2(x) + \frac{\sigma'^2(x)}{A}\right]. \quad (2.146)$$

Using the dimensionless quantities in equation (2.134), we also have

$$\sigma'' + \left(\frac{B'}{2B} + \frac{A'}{2A} + \frac{2}{x}\right)\sigma' + A\left(\frac{\Omega^2}{B} - 1\right)\sigma = 0. \quad (2.147)$$

We must also consider the normalization condition [2]

$$\int d^3x \sqrt{g} < Q | J^0 | Q > = N. \quad (2.148)$$

If we substitute in the expectation value of the current (2.141) and use the dimensionless quantities given above in writing the equations of motion, then

$$\frac{M_{pl}^2}{m^2} \int_0^\infty dx \sqrt{A} \left(\frac{\Omega^2}{B}\right)^{\frac{1}{2}} \sigma^2 x^2 = N. \quad (2.149)$$

If we define a new quantity representing the particle number

$$\mathcal{N} = N \frac{m^2}{M_{Pl}^2} \quad (2.150)$$

then we will find that

$$\mathcal{N} = \int_0^\infty dx \sqrt{A} \left(\frac{\Omega^2}{B} \right)^{\frac{1}{2}} \sigma^2 x^2. \quad (2.151)$$

By performing a numerical calculation using a Runge-Kutta method, Ruffini and Bonazzola calculated the mass of the boson star for different values of the central density of the star [2]. They found that for one specific value of the central density, the mass reached a maximum value, and then decreased monotonically for an arbitrary increase in the central density, as is seen in Figure 2. They also showed that the number of particles in the star increased with an increase in mass and decreased with a decrease in mass, as should be expected. Therefore, they found that the model in general relativity corrected the problem of the Newtonian model where the mass of the star decreases even though the number of particles increases, as is shown in Figure 1.

From Figure 2, we see that the maximum mass of the star is

$$M_{max} \sim 0.633 \frac{M_{Pl}^2}{m}. \quad (2.152)$$

The maximum mass can also be considered as the critical mass as stars with larger masses would be unstable, as is indicated by the solutions. The central density of the

Figure 2: This plot shows the the mass as a function of σ_c , which corresponds to the central density of the star.

star is given by

$$\rho_{crit} \sim \sigma^2(0)m^2 M_{Pl}^2 = \sigma_c^2 m^2 M_{Pl}^2 \quad (2.153)$$

As an example, let's consider bosons with a mass of $m = 1.67 \times 10^{-24}$ g. Since $M_{Pl}^2 = 4.72 \times 10^{-10}$ g², then the maximum mass of the boson star is

$$M_{max} \approx 1.8 \times 10^{14} g \approx 9 \times 10^{-20} M_{\odot} \quad (2.154)$$

where $M_{\odot}$ is the solar mass. Stars of this mass are not likely to contribute significantly

to the dark matter content of the universe predicted by the inflationary model. To have a mass on the order of the Chandrasekhar mass ($M_{Ch} = 1.44 M_{\odot}$), the boson would need a mass $m \sim 1 \times 10^{-43}$ g, which is a very small mass indeed!

Is it possible to increase the mass of the boson star? As we shall see in the next section, we can introduce a coupling interaction between the bosons that will dramatically increase the mass of the boson star.

2.4 General Relativistic Treatment with a Self-Interaction Potential

Although the previous model shows that boson stars are possible in general relativity, they tend to have a very small mass in comparison with the Chandrasekhar mass limit. The maximum mass of the boson star would need to be increased significantly to play an important role as a dark matter candidate. The model presented in this section solves the problem of the insignificant mass seen in the previous model.

Recently, a variation of the model presented by Ruffini and Bonazzola [2] was introduced by Colpi, Shapiro, and Wasserman [4] where a self-interaction potential was added to the Lagrangian describing the system. By adding this interaction, they were able to significantly increase the maximum allowed mass of the boson star.

This model begins with the addition of an interaction potential of the form

$$V(\phi) = \frac{1}{4}\lambda|\phi|^4 \quad (2.155)$$

where λ is a dimensionless coupling constant. Even with this small coupling constant, the effect of this interaction on the properties of the boson star is dramatic. With

$\lambda = 0$, the characteristic energy density of the boson star is

$$\rho \sim m^2 M_{Pl}^2. \quad (2.156)$$

However, for non-interacting bosons in relativistic stars,

$$\rho \sim m^2 |\phi|^2 \quad (2.157)$$

and thus $|\phi| \sim M_{Pl}$. If λ were increased slightly above zero, then the scalar boson tends to approximately keep the same mass m , and the equilibrium value of $|\phi|$ remains unchanged. Therefore, the importance of the self-interaction is given by the ratio of the interaction potential to the kinetic energy:

$$\frac{V(\phi)}{m^2 |\phi|^2}. \quad (2.158)$$

If λ is sufficiently small, then this ratio becomes

$$\lambda \frac{M_{Pl}^2}{m^2}. \quad (2.159)$$

Thus, the self-interactions can only be ignored if

$$\lambda \ll \frac{m^2}{M_{Pl}^2}. \quad (2.160)$$

In the equilibrium state, the ratio has the form

$$\Lambda = \frac{\lambda M_{Pl}^2}{4\pi m^2} \quad (2.161)$$

and it will be shown that for $\Lambda \gg 1$,

$$M \sim \Lambda^{\frac{1}{2}} \frac{M_{Pl}^2}{m} \sim \lambda^{\frac{1}{2}} \frac{M_{Pl}^3}{m^2} \sim \lambda^{\frac{1}{2}} M_{Ch}. \quad (2.162)$$

Therefore, for $\lambda \sim 1$, the mass of the boson star is comparable to the mass of its fermion counterpart.

With the addition of the interaction potential, the Lagrangian density becomes

$$\mathcal{L}' = \mathcal{L} - \frac{1}{4} \sqrt{g} \lambda |\phi|^4 \quad (2.163)$$

where

$$\mathcal{L} = -\frac{1}{2} \sqrt{g} (g^{\mu\nu} \phi_{;\mu}^* \phi_{;\nu} - m^2 |\phi|^2). \quad (2.164)$$

With this Lagrangian density, the action of the system is given by

$$I = I_G + I_M \quad (2.165)$$

where I_G is given by equation (2.109) and

$$I_M = \int d^4x \mathcal{L} - \frac{1}{4} \lambda \int d^4x \sqrt{g} |\phi|^4. \quad (2.166)$$

Next, we must consider a variation of the total action. However, the variations δI_G and $\delta \int d^4x \mathcal{L}$ are already calculated and given by equations (2.117) and (2.119).

Thus, we must consider the variation of the last integral in the matter action. The variation of the integrand, with the use of equation (2.115), is

$$\delta(\sqrt{g} |\phi|^4) = \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} |\phi|^4 + \sqrt{g} \phi \phi^{*2} \delta \phi + \sqrt{g} \phi^* \phi^2 \delta \phi^*. \quad (2.167)$$

Thus, δI_M is

$$\begin{aligned} \delta I_M = & \int d^4x \sqrt{g} \delta g_{\mu\nu} \left[\frac{1}{4} (\partial^\mu \phi^* \partial^\nu \phi + \partial^\nu \phi^* \partial^\mu \phi) - \frac{1}{4} g^{\mu\nu} (g^{\rho\kappa} \partial_\rho \phi^* \partial_\kappa \phi \right. \\ & \left. + m^2 |\phi|^2 + \frac{1}{2} \lambda |\phi|^4) \right] + \frac{1}{2} \int d^4x \sqrt{g} \delta \phi^* [\phi_{;\mu}{}^{;\mu} - m^2 \phi - \lambda \phi^2 \phi^*] \\ & + \frac{1}{2} \int d^4x \sqrt{g} \delta \phi [\phi^*_{;\mu}{}^{;\mu} - m^2 \phi^* - \lambda \phi^{*2} \phi]. \end{aligned} \quad (2.168)$$

From the variation of the total action,

$$\delta I = \delta I_G + \delta I_M \quad (2.169)$$

we then find the Einstein field equations

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \quad (2.170)$$

where the energy-momentum tensor is now given by

$$T_{\mu\nu} = \frac{1}{2} (\partial_\mu \phi^* \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi^*) - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\kappa} \partial_\rho \phi^* \partial_\kappa \phi + m^2 |\phi|^2 + \frac{1}{2} \lambda |\phi|^4 \right) \quad (2.171)$$

the Einstein tensors G_{00} and G_{11} are given by equations (2.143) and (2.144), and the scalar wave equation is

$$\phi_{;\mu}{}^{;\mu} - m^2 |\phi|^2 - \lambda \phi^2 \phi^* = 0. \quad (2.172)$$

Again, since we are considering spherically symmetric, time independent solutions to the system, we can use the Schwarzschild metric (2.104). But, how do we treat the scalar field ϕ ? As in the last model, we can use a quantum scalar field. However,

a classical scalar field can also be used and, in fact, will give the exact same solutions as before. Thus, we use the solution

$$\phi(r, t) = \Phi(r)e^{-i\omega t} \quad (2.173)$$

where $\Phi(r)$ is a real function and ω corresponds to the frequency and the energy of an individual boson. (We could equally well consider “antiboson stars” with the solution $\phi(r, t) = \Phi(r)e^{i\omega t}$.)

By using the classical scalar field in equation (2.171), the energy-momentum tensor elements of interest are

$$T_{00} = \frac{B}{2} \left[\left(\frac{\omega^2}{B} + m^2 \right) \Phi^2(r) + \frac{\Phi'^2(r)}{A} + \frac{1}{2} \lambda \Phi^4(r) \right] \quad (2.174)$$

$$T_{11} = \frac{A}{2} \left[\left(\frac{\omega^2}{B} - m^2 \right) \Phi^2(r) + \frac{\Phi'^2(r)}{A} - \frac{1}{2} \lambda \Phi^4(r) \right]. \quad (2.175)$$

Thus, the Einstein field equations

$$G_{11} = -8\pi G T_{11} \quad (2.176)$$

$$G_{00} = -8\pi G T_{00} \quad (2.177)$$

and the scalar wave equation become

$$\frac{A'}{xA^2} + \frac{1}{x^2} \left(1 - \frac{1}{A} \right) = \left(\frac{\Omega^2}{B} + 1 \right) \sigma^2 + \frac{\Lambda}{2} \sigma^4 + \frac{\sigma'^2}{A} \quad (2.178)$$

$$\frac{B'}{xAB} - \frac{1}{x^2} \left(1 - \frac{1}{A} \right) = \left(\frac{\Omega^2}{B} - 1 \right) \sigma^2 - \frac{\Lambda}{2} \sigma^4 + \frac{\sigma'^2}{A} \quad (2.179)$$

$$\sigma'' + \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{x} \right) \sigma' + A \left[\left(\frac{\Omega^2}{B} - 1 \right) \sigma - \Lambda \sigma^3 \right] = 0 \quad (2.180)$$

where we again use the dimensionless quantities σ , Ω , and x . However, if we define the quantity $\mathcal{M}$, which represents the mass of the system, so that [13]

$$A(x) = \left(1 - \frac{2\mathcal{M}(x)}{x}\right)^{-1} \quad (2.181)$$

it can be shown that

$$\frac{A'}{xA^2} + \frac{1}{x^2} \left(1 - \frac{1}{A}\right) = \frac{2\mathcal{M}'(x)}{x^2}. \quad (2.182)$$

Therefore, equation (2.178) becomes

$$\mathcal{M}'(x) = \frac{x^2}{2} \left[\left(\frac{\Omega^2}{B} + 1 \right) \sigma^2 + \frac{\Lambda}{2} \sigma^4 + \frac{\sigma'^2}{A} \right]. \quad (2.183)$$

The equations are now in the form where numerical calculations can be made. Since we are considering a spherically symmetric system in a ground state level, we must look for nonsingular, finite-mass, zero-node solutions to these equations. These solutions tend to be the lowest-energy bound states because they minimize the effect of the term containing the derivative of the scalar field.

To solve these equations, we use a Runge-Kutta solving routine in FORTRAN similar to that in Appendix A.4 and choose the following boundary conditions:

$$\mathcal{M}(0) = 0 \quad (2.184)$$

$$\sigma(0) = \sigma_c \quad (2.185)$$

$$\sigma'(0) = 0. \quad (2.186)$$

We also require that $B(\infty) = 1$. However, instead of using this boundary condition directly, we notice that in the equations of motion, B only occurs in the fractions $\frac{\Omega^2}{B}$

and $\frac{B'}{B}$. Also, we note that Ω occurs only in the fraction $\frac{\Omega^2}{B}$. Due to this property, we found that we could choose an arbitrary value of Ω and search for the corresponding value of B that gives a solution with no nodes in $\sigma(x)$. After we find the solution, then we rescale Ω and B so that $B(\infty) = 1$ and thus obtain the correct energy eigenvalue without affecting the solution. Therefore, for each solution, the total mass is given by

$$M = \mathcal{M}(\infty) \frac{M_{Pl}^2}{m}. \quad (2.187)$$

The numerical results for $\mathcal{M}(\infty)$ as a function of σ_c that we found are shown in Figure 3 for different values of Λ . They are in excellent agreement with the results of Colpi, Shapiro, and Wasserman. The $\Lambda = 0$ case is also in excellent agreement with the results of Ruffini and Bonazzola.

It is apparent from Figure 3 that for each value of Λ , a characteristic maximum mass is obtained for a specific value of σ_c , where σ_c corresponds to the central density of the boson star. Table 1 in Section 3.3 contains the value of the maximum mass and its corresponding value of σ_c for each value of Λ . The initial values of B are also given, and in each case, we set $\Omega = 0.8842$. As stated earlier, it is possible to choose an arbitrary value for Ω as it will give a proper solution even though $B(\infty)$ has not been scaled to unity.

To look at this trend in a more quantitative manner, we plot the maximum boson-star mass as a function of Λ in Figure 4.

Figure 3: These are plots of the mass as a function of σ_c for different values of Λ , which corresponds to the strength of the coupling interaction.

Note that for $\Lambda \gg 1$, the maximum mass follows the curve given by

$$M_{max} \approx 0.218 \Lambda^{\frac{1}{2}} \frac{M_{Pl}^2}{m}. \quad (2.188)$$

To derive this relation, we set out by looking at plots of $\sigma(x)$ versus x with $\sigma_c = 0.055$ for $\Lambda = 0$ and $\Lambda = 300$. The curve for the large value of Λ takes on a noticeably different structure from the $\Lambda = 0$ curve. From Figure 5, we see that for small Λ , $\sigma(x)$ falls off smoothly and eventually exponentially to zero in a length $\sim \frac{1}{m}$.

Table 1: This table contains the maximum mass for each value of the coupling Λ found in Figure 3 and the corresponding values of $B(0)$ and $\sigma(0)$.

Λ	$B(0)$	σ_c	$M\left(\frac{m}{M_{Pl}^2}\right)$
0	0.5020	0.275	0.633
10	0.4341	0.225	0.921
30	0.4358	0.150	1.334
100	0.4225	0.090	2.254
200	0.3900	0.070	3.131
300	0.4066	0.055	3.814
400	0.4296	0.045	4.379

However, for large Λ , we find that $\sigma(x)$ falls off rather slowly out to about the radius

$$\Lambda^{\frac{1}{2}} \frac{1}{m} \tag{2.189}$$

with an exponential decline beyond this radius. Because of the relationship between $\sigma(x)$ and $\mathcal{M}(x)$, which can be seen in Figures 6 and 7, we can see that most of the mass in the large Λ case is found in the region of slow decline. And, as Λ increases, the region of slow decline becomes increasingly dominant, as can be seen in Figure 5.

These results support the new nondimensional quantities

$$\psi = \sigma \Lambda^{\frac{1}{2}} \tag{2.190}$$

Figure 4: This graph plots the maximum mass as a function of the coupling interaction strength Λ . The solid curve represents the results found from the equations of motion modified for large values of Λ and the points represent the exact solutions found.

$$y = \frac{x}{\Lambda^{\frac{1}{2}}} \quad (2.191)$$

$$\mathcal{M}(y) = \frac{\mathcal{M}(x)}{\Lambda^{\frac{1}{2}}}. \quad (2.192)$$

By using these dimensionless quantities and ignoring terms of order $\frac{1}{\Lambda}$, the scalar wave equation (2.180) becomes

$$\psi = \left(\frac{\Omega^2}{B} - 1 \right)^{\frac{1}{2}} \quad (2.193)$$

Figure 5: This plots show the exact values of σ as a function of the distance x for $\Lambda = 0$ and $\Lambda = 300$ and $\sigma_c = 0.055$. The points represent the values σ from the modified equations of motion for $\Lambda = 300$.

and by substituting this into the rescaled forms of equations (2.178) and (2.179), we have

$$\mathcal{M}'(y) = \frac{y^2}{4} \left(\frac{3\Omega^2}{B} + 1 \right) \left(\frac{\Omega^2}{B} - 1 \right) \quad (2.194)$$

$$\frac{B'}{yAB} - \frac{1}{y^2} \left(1 - \frac{1}{A} \right) = \frac{1}{2} \left(\frac{\Omega^2}{B} - 1 \right)^2. \quad (2.195)$$

Figure 6: This is a plot of σ versus the distance x for $\Lambda = 0$.

Note that these equations essentially become exact when $\Lambda \rightarrow \infty$ and are reasonably accurate for large values of Λ ($\Lambda > 50$). Since these equations do not depend explicitly on Λ , we can use these equations to determine the rescaled mass $\mathcal{M}(y)$ as a function of $\frac{\Omega^2}{B(0)}$. The Runge-Kutta method is also used to solve these equations. Figure 5 compares the values of $\sigma(x)$ from equations (2.194) and (2.195) with the exact results for $\sigma_c = 0.055$, and the value of the maximum mass from these equations is shown in Figure 8.

Figure 7: This is a plot of the mass as a function of the distance x for $\Lambda = 0$.

Note that the value of the maximum mass read from this plot

$$\mathcal{M}_{max}(y) \approx 0.218 \quad (2.196)$$

is the same value used in equation (2.188), and the peak is located at $\frac{\Omega^2}{B(0)} = 0.287$.

Therefore, by introducing the interaction potential

$$V(\phi) = \frac{1}{4}\lambda|\phi|^4, \quad (2.197)$$

Figure 8: This is a plot of the mass values from the equations of motion modified for large values of Λ .

Colpi, Shapiro, and Wasserman found that [4]

$$M_{max} \approx 0.218\Lambda^{\frac{1}{2}} \frac{M_{Pl}^2}{m}. \quad (2.198)$$

When Λ is substituted into this relation,

$$M_{max} \approx 0.0615\lambda^{\frac{1}{2}} \frac{M_{Pl}^3}{m^2} \quad (2.199)$$

which is comparable to the Chandrasekhar mass for fermions of mass

$$m_{fermion} = \lambda^{-\frac{1}{4}} m_{boson}. \quad (2.200)$$

Thus, it is possible that boson stars with a mass approaching that of equation (2.199) could have formed in the early Universe from the gravitational collapse of dark bosonic matter, and if so, they may have a significant contribution to the amount of dark matter predicted by the inflationary model.

Chapter 3

BOSON STARS IN BRANS-DICKE GRAVITY

Working through the self-interaction coupling model of the boson star presented by Colpi, Shapiro, and Wasserman [5], we have confirmed the results that they have found for the maximum mass limits of the boson star in general relativity. Now, following a similar procedure, we have worked out a model for boson stars in the Brans-Dicke gravitational theory. The Brans-Dicke gravitational theory leads to some very interesting results when introduced into inflationary cosmology.

To begin, we will go through a review of the Brans-Dicke theory. Then, we will apply the results of this theory to boson stars.

3.1 The Brans-Dicke Gravitational Theory

The Brans-Dicke gravitational theory [5] is essentially a generalization of general relativity. The gravitational constant G is replaced by a scalar field of infinite range, and thus the gravitational effects are due partly to the geometry of space-time and due partly to a scalar field interaction between the test body and the matter distribution of the universe.

What is this scalar field? Is it a quantity we are already familiar with? At first glance, it seems that the scalar curvature R may be a good candidate. However, for the scalar quantity to be consistent with the proposed theory, it must have essentially an infinite range. The scalar curvature, on the other hand, falls off more rapidly than $\frac{1}{r}$ from the mass source and thus depends only on local matter distributions. Therefore, we must introduce a new scalar quantity.

We start with the general relativistic action [5]

$$I_{GR} = \int d^4x \sqrt{g} \left(-R + \frac{16\pi}{\sqrt{g}} G\mathcal{L} \right) \quad (3.1)$$

where $\mathcal{L}$ is the Lagrangian density of matter including all nongravitational fields. To generalize this action and develop the Brans-Dicke theory, we must replace $\frac{1}{g}$ by the Brans-Dicke scalar ϕ_{BD} . Also, we must consider the Lagrangian density of ϕ_{BD} [5]

$$\omega \frac{g^{\mu\nu} \partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}} \quad (3.2)$$

which behaves as the kinetic term for the field. Note that this term has been divided by ϕ_{BD} so that ω will be a dimensionless quantity. As it will be shown, ω mediates the strength of the scalar interaction. Therefore, the action becomes [5]

$$I_{BD} = I_G + I_M + I_{\phi_{BD}} = \int d^4x \sqrt{g} \left(-\phi_{BD} g^{\mu\nu} R_{\mu\nu} + \frac{16\pi}{\sqrt{g}} \mathcal{L} - \omega \frac{g^{\mu\nu} \partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}} \right) \quad (3.3)$$

Note that the Lagrangian density term has not been affected by this change. Thus, the equations of motion for matter located in a metric field which is caused by a source

outside the system are the same as those found in general relativity. The difference between the theories lies in the gravitational field equations which determine $g^{\mu\nu}$.

We must also address the boundary conditions that must be imposed on the Brans-Dicke scalar field. By working through a weak field approximation similar to that in the paper by Brans and Dicke, it can be shown that [5]

$$\phi_{BD}(\infty) = \frac{1}{G} \left(\frac{2\omega + 4}{2\omega + 3} \right) \quad (3.4)$$

One important thing to notice from this boundary condition is that as ω becomes very large, the Brans-Dicke gravitational theory becomes equivalent to general relativity because the scalar field, in this limit, is the inverse of the gravitational constant, or as $\omega \rightarrow \infty$,

$$\phi_{BD}(\infty) = \frac{1}{G} \quad (3.5)$$

This behavior can also be seen in the equations of motion derived from the Brans-Dicke theory. One other aspect of the scalar field is that it should have a regular behavior at the origin. Therefore, we expect that its first derivative at the origin of the test body should be zero

$$\phi'_{BD}(0) = 0 \quad (3.6)$$

3.2 Application of the Brans-Dicke Theory to Boson Stars

To derive the equations of motion for the boson star in the Brans-Dicke gravitational theory with a self-interaction coupling term, we must now substitute the

Lagrangian density describing this boson star into the action. Since we are also considering quartic self-interaction between the boson fields, we can use the same Lagrangian density as was used in the general relativistic treatment. Therefore, the Lagrangian, as is given by equation (2.163), is

$$\mathcal{L} = -\frac{1}{2} \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m^2 |\phi|^2 + \frac{1}{2} \lambda |\phi|^4 \right) \quad (3.7)$$

In a similar manner to the general relativistic theory, we must take a variation of the action. Thus, we have

$$\delta I_{BD} = \delta I_G + \delta I_M + \delta I_{\phi BD} = 0. \quad (3.8)$$

However, since δI_M is the same as it was in general relativity, we already know its value from equation (2.119). Next, we must consider δI_G . Varying the action

$$I_G = \int d^4x \sqrt{g} (-\phi_{BD} g^{\mu\nu} R_{\mu\nu}), \quad (3.9)$$

we have

$$\begin{aligned} \delta I_G = & - \int d^4x (\phi_{BD} R \delta \sqrt{g} + \phi_{BD} \sqrt{g} R_{\mu\nu} \delta g^{\mu\nu}) - \int d^4x \sqrt{g} R \delta \phi_{BD} \\ & - \int d^4x \phi_{BD} \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} \end{aligned} \quad (3.10)$$

or, by using equations (2.116) and (2.115),

$$\begin{aligned} \delta I_G = & \int d^4x \sqrt{g} \delta g_{\mu\nu} \phi_{BD} (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) - \int d^4x \sqrt{g} R \delta \phi_{BD} \\ & - \int d^4x \sqrt{g} \phi_{BD} g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \quad (3.11)$$

The last integral in this equation must also be simplified. By using the Palatini identity [13]

$$\delta R_{\mu\nu} = (\delta\Gamma_{\mu\lambda}^{\lambda})_{;\nu} - (\delta\Gamma_{\mu\nu}^{\lambda})_{;\lambda}, \quad (3.12)$$

we find that

$$\int d^4x \sqrt{g} \phi_{BD} g^{\mu\nu} R_{\mu\nu} = \int d^4x \sqrt{g} \phi_{BD} [(g^{\mu\nu} \delta\Gamma_{\mu\lambda}^{\lambda})_{;\nu} - (g^{\mu\nu} \delta\Gamma_{\mu\nu}^{\lambda})_{;\lambda}]. \quad (3.13)$$

However, if we use the definition

$$V^{\mu}_{;\mu} = \frac{1}{\sqrt{g}} \partial_{\mu}(\sqrt{g} V^{\mu}), \quad (3.14)$$

integrate by parts, and use the divergence theorem, then the integral is given by

$$\int d^4x \sqrt{g} \phi_{BD} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{g} g^{\mu\nu} [\partial_{\lambda}(\phi_{BD}) \delta\Gamma_{\mu\nu}^{\lambda} - \partial_{\nu}(\phi_{BD}) \delta\Gamma_{\mu\lambda}^{\lambda}]. \quad (3.15)$$

Finally, if we use the following identities,

$$\delta\Gamma_{\mu\lambda}^{\lambda} = \frac{1}{2} g^{\lambda\rho} \partial_{\mu}(\delta g_{\rho\lambda}) - g^{\lambda\rho} \Gamma_{\mu\lambda}^{\sigma} \delta g_{\rho\sigma} = \frac{1}{2} \partial_{\mu}(g^{\rho\sigma} \delta g_{\rho\sigma}) \quad (3.16)$$

$$\delta\Gamma_{\mu\nu}^{\lambda} = g^{\lambda\rho} \partial_{\mu}(\delta g_{\rho\nu}) - \frac{1}{2} g^{\lambda\rho} \partial_{\rho}(\delta g_{\mu\nu}) - g^{\lambda\rho} \Gamma_{\mu\nu}^{\sigma} \delta g_{\rho\sigma} \quad (3.17)$$

$$g^{\mu\nu} \Gamma_{\mu\nu}^{\sigma} = -\frac{1}{\sqrt{g}} \partial_{\mu}(\sqrt{g} g^{\mu\sigma}) \quad (3.18)$$

integrate by parts, and use the divergence theorem, then

$$\int d^4x \sqrt{g} \phi_{BD} g^{\mu\nu} \delta R_{\mu\nu} = - \int d^4x \sqrt{g} \delta g_{\mu\nu} (\phi_{BD}^{;\mu}{}^{;\nu} - g^{\mu\nu} \phi_{BD}{}^{;\lambda}{}_{;\lambda}) \quad (3.19)$$

and the variation of the Brans-Dicke gravitational action becomes

$$\begin{aligned} \delta I_G = & \int d^4x \sqrt{g} \delta g_{\mu\nu} \left[\phi_{BD} (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) + (\phi_{BD}^{;\mu;\nu} - g^{\mu\nu} \phi_{BD;\lambda}^{;\lambda}) \right] \\ & - \int d^4x \sqrt{g} R \delta \phi_{BD}. \end{aligned} \quad (3.20)$$

The action due to the Brans-Dicke scalar field is

$$I_{\phi_{BD}} = -\omega \int d^4x \sqrt{g} \frac{g^{\mu\nu} \partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}}. \quad (3.21)$$

The variation of this action is given by

$$\begin{aligned} \delta I_{\phi_{BD}} = & -\omega \int d^4x \left[\delta(\sqrt{g}) \frac{g^{\mu\nu} \partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}} + \sqrt{g} \delta(g^{\mu\nu}) \frac{\partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}} \right. \\ & \left. + \sqrt{g} g^{\mu\nu} \delta \left(\frac{\partial_\mu \phi_{BD} \partial_\nu \phi_{BD}}{\phi_{BD}} \right) \right]. \end{aligned} \quad (3.22)$$

By using equations (2.116) and (2.115) and expanding out the last term and integrating it by parts, the variation of the action becomes

$$\begin{aligned} \delta I_{\phi_{BD}} = & \omega \int d^4x \sqrt{g} \delta g_{\mu\nu} \left[\frac{\partial^\mu \phi_{BD} \partial^\nu \phi_{BD}}{\phi_{BD}} - \frac{1}{2} g^{\mu\nu} \frac{\partial_\lambda \phi_{BD} \partial^\lambda \phi_{BD}}{\phi_{BD}} \right] \\ & + \omega \int d^4x \sqrt{g} \delta \phi_{BD} \left[\frac{2}{\phi_{BD}} \phi_{BD;\lambda}^{;\lambda} - \frac{\partial^\lambda \phi_{BD} \partial_\lambda \phi_{BD}}{\phi_{BD}^2} \right]. \end{aligned} \quad (3.23)$$

Using δI_M from equation (2.119), δI_G , and $\delta I_{\phi_{BD}}$, then the variation of the total action is

$$\begin{aligned} \delta I_{BD} = & \int d^4x \sqrt{g} \delta g_{\mu\nu} \left[\phi_{BD} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) + 8\pi T^{\mu\nu} + \omega \left(\frac{\partial^\mu \phi_{BD} \partial^\nu \phi_{BD}}{\phi_{BD}} \right. \right. \\ & \left. \left. - \frac{1}{2} g^{\mu\nu} \frac{\partial_\lambda \phi_{BD} \partial^\lambda \phi_{BD}}{\phi_{BD}} \right) + (\phi_{BD}^{;\mu;\nu} - g^{\mu\nu} \phi_{BD;\lambda}^{;\lambda}) \right] \end{aligned}$$

$$\begin{aligned}
& +8\pi \int d^4x \sqrt{g} \delta\phi^* [\phi_{;\lambda}^{;\lambda} - m^2\phi - \lambda\phi^2\phi^*] \\
& +8\pi \int d^4x \sqrt{g} \delta\phi [\phi_{;\lambda}^{*\lambda} - m^2\phi^* - \lambda\phi^{*2}\phi] \\
& + \int d^4x \sqrt{g} \delta\phi_{BD} \left(\frac{2\omega}{\phi_{BD}} \phi_{BD;\lambda}^{;\lambda} - \omega \frac{\partial^\lambda \phi_{BD} \partial_\lambda \phi_{BD}}{\phi_{BD}^2} - R \right). \tag{3.24}
\end{aligned}$$

Thus, we can read off the following equations with lowered indices:

$$\begin{aligned}
G_{\mu\nu} = & -\frac{8\pi}{\phi_{BD}} T_{\mu\nu} - \frac{\omega}{\phi_{BD}^2} \left(\partial_\mu \phi_{BD} \partial_\nu \phi_{BD} - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi_{BD} \partial^\lambda \phi_{BD} \right) \\
& - \frac{1}{\phi_{BD}} (\phi_{BD;\mu;\nu} - g_{\mu\nu} \phi_{BD;\lambda}^{;\lambda}) \tag{3.25}
\end{aligned}$$

$$0 = \phi_{;\lambda}^{;\lambda} - m^2\phi - \lambda\phi^2\phi^* \tag{3.26}$$

$$0 = \frac{2\omega}{\phi_{BD}} \phi_{BD;\lambda}^{;\lambda} - \omega \frac{\partial^\lambda \phi_{BD} \partial_\lambda \phi_{BD}}{\phi_{BD}^2} - R. \tag{3.27}$$

At this point, we have a choice of using either a classical or quantum boson field. Since the classical field gives the same results as the quantum field, we will choose the classical boson field

$$\phi(r, t) = \Phi(r) e^{-i\omega_E t} \tag{3.28}$$

where ω_E corresponds to the frequency, and thus the energy, of a boson.

The values of G_{00} , G_{11} , T_{00} , and T_{11} are given by equations (2.143), (2.144), (2.139), and (2.140), respectively. Now, the quantities

$$\phi_{BD;\lambda}^{;\lambda} = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g} g^{\lambda\rho} \partial_\rho \phi_{BD}) \tag{3.29}$$

$$\phi_{BD;0;0} = \partial_0 \partial_0 \phi_{BD} - \Gamma_{00}^\lambda \partial_\lambda \phi_{BD} \tag{3.30}$$

$$\phi_{BD;1;1} = \partial_1 \partial_1 \phi_{BD} - \Gamma_{11}^\lambda \partial_\lambda \phi_{BD} \tag{3.31}$$

must be calculated explicitly. Γ_{00}^λ and Γ_{11}^λ are given by equation (2.112). Since ϕ_{BD} is a function of r only, then

$$\phi_{BD ; \lambda}^{; \lambda} = \frac{\phi_{BD}''}{A} + \left(\frac{B'}{2AB} - \frac{A'}{2A^2} + \frac{2}{rA} \right) \phi_{BD}' \quad (3.32)$$

$$\phi_{BD ; 0 ; 0} = -\frac{B'}{2A} \phi_{BD}' \quad (3.33)$$

$$\phi_{BD ; 1 ; 1} = \phi_{BD}'' - \frac{A'}{2A} \phi_{BD}'. \quad (3.34)$$

Also, if we substitute the trace of equation (3.25)

$$R = \frac{8\pi}{\phi_{BD}} T - \frac{\omega}{\phi_{BD}^2} \partial^\lambda \phi_{BD} \partial_\lambda \phi_{BD} - \frac{3}{\phi_{BD}} \phi_{BD ; \lambda}^{; \lambda} \quad (3.35)$$

into equation (3.27), then we find that

$$\phi_{BD ; \lambda}^{; \lambda} = \frac{8\pi}{2\omega + 3} T \quad (3.36)$$

or

$$\phi_{BD ; \lambda}^{; \lambda} = \phi_{BD}'' + \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{r} \right) \phi_{BD}' = \frac{8\pi}{2\omega + 3} T \quad (3.37)$$

where T is the trace of the energy momentum tensor and is given by

$$T = \left(\frac{\omega_E^2}{B} - 2m^2 \right) \Phi^2(r) - \frac{\Phi'^2(r)}{A} - \lambda \Phi^4(r) \quad (3.38)$$

From the field equations (3.25), the boson scalar wave equation (3.26), and the wave equation for the Brans-Dicke scalar field (3.37), the equations of motion for the boson star become

$$\mathcal{M}'(x) = \frac{x^2}{2\Phi_{BD}} \left(\frac{2\omega + 3}{2\omega + 4} \right) \left[\left(\frac{\Omega^2}{B} + 1 \right) \sigma^2 + \frac{\sigma'^2}{A} + \frac{\Lambda}{2} \sigma^4 \right] + \frac{\omega x^2}{4A} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right)^2$$

$$+ \frac{B'x^2}{4AB} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right) + \frac{x^2}{\Phi_{BD}} \left(\frac{1}{2\omega+4} \right) \left[\left(\frac{\Omega^2}{B} - 2 \right) \sigma^2 - \frac{\sigma'^2}{A} - \Lambda \sigma^4 \right] \quad (3.39)$$

$$\begin{aligned} B'(x) = & \frac{AB}{x} - \frac{B}{x} + \frac{xAB}{\Phi_{BD}} \left(\frac{2\omega+3}{2\omega+4} \right) \left[\left(\frac{\Omega^2}{B} - 1 \right) \sigma^2 + \frac{\sigma'^2}{A} - \frac{\Lambda}{2} \sigma^4 \right] \\ & + \frac{\omega x B}{2} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right)^2 + \frac{x B}{\Phi_{BD}} \left(\Phi''_{BD} - \frac{A'}{2A} \Phi'_{BD} \right) \\ & - \frac{2xAB}{\Phi_{BD}} \left(\frac{1}{2\omega+4} \right) \left[\left(\frac{\Omega^2}{B} - 2 \right) \sigma^2 - \frac{\sigma'^2}{A} - \Lambda \sigma^4 \right] \end{aligned} \quad (3.40)$$

$$0 = \sigma'' + \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{x} \right) \sigma' + A \left[\left(\frac{\Omega^2}{B} - 1 \right) \sigma - \Lambda \sigma^3 \right] \quad (3.41)$$

$$\begin{aligned} 0 = & \Phi''_{BD} + \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{x} \right) \Phi'_{BD} - \left(\frac{2A}{2\omega+4} \right) \left[\left(\frac{\Omega^2}{B} - 2 \right) \sigma^2 \right. \\ & \left. - \frac{\sigma'^2}{A} - \Lambda \sigma^4 \right] \end{aligned} \quad (3.42)$$

where x and Ω have been used again along with the other dimensionless quantities

$$\sigma = \sqrt{4\pi} \frac{\Phi(r)}{M_{Pl}} \quad (3.43)$$

$$\Phi_{BD} = \left(\frac{2\omega+3}{2\omega+4} \right) \frac{\phi_{BD}}{M_{Pl}^2} \quad (3.44)$$

$$\Lambda = \frac{\lambda}{4\pi} \left(\frac{M_{Pl}}{m} \right)^2 \quad (3.45)$$

and $\mathcal{M}'(x)$ is given by equation (2.182).

Let's examine in more detail the role of ω in the Brans-Dicke theory. From equation (3.36), it is quite clear that as ω becomes much larger than unity, then

$$\phi'_{BD} \sim O\left(\frac{1}{\omega}\right) \quad (3.46)$$

Thus, as ω becomes larger, the terms containing the derivatives of the Brans-Dicke field in the equations of motion become smaller, and from equation (1.9), it is also seen that the Brans-Dicke field approaches the value $\frac{1}{G}$. It therefore seems that the value of ω controls the interaction strength between the test body and the matter distribution of the surrounding universe through the Brans-Dicke scalar field. If $\omega \rightarrow \infty$, the terms containing the derivatives of the Brans-Dicke field go to zero, $\phi_{BD}(\infty) = \frac{1}{G}$, and the Brans-Dicke field equations revert to the field equations in general relativity.

3.3 Results and Comparisons to the General Relativistic Model

In a similar manner as in the model presented by Colpi, Shapiro, and Wasserman, we have also sought to obtain nonsingular, finite-mass, zero-node solutions to the equations of motion of a boson star in the Brans-Dicke gravitational theory for the $\omega = 6$ case. Therefore, a Runge-Kutta solving routine in FORTRAN, which is given in Appendix A.4, is used to solve the equations subject to the following boundary conditions:

$$\mathcal{M}(0) = 0$$

$$\sigma(0) = \sigma_c$$

$$\sigma'(0) = 0$$

$$B(\infty) = 1$$

$$\Phi_{BD}(\infty) = 1$$

$$\Phi'_{BD}(0) = 0$$

where the dimensionless quantity Φ_{BD} representing the Brans-Dicke field has its boundary condition because of its definition. Also, the derivative of the Brans-Dicke field is set to zero at the center of the star as we expect that the field should have a regular behavior there.

In solving these equations, the same technique that was applied in solving the general relativistic equations of motion are used here as well. Note the special way in which B and Ω occur in the equations of motion. As was described earlier, an arbitrary value of Ω is chosen even if B does not go to one at large distances. Then, once the solutions have been found, the quantities B and Ω are rescaled so that $B(\infty) = 1$ without affecting the solution. However, since a similar method cannot be used for Φ_{BD} , we must carefully choose its value at the center of the star so that $\Phi_{BD}(\infty) = 1$.

The results of the numerical analysis of these equations are shown in Figure 9.

Note that the behavior of the mass versus σ_c is similar to that seen in the general relativistic results with a mass peak at a different value of σ_c for each value of Λ studied. Table 2 gives the value of the mass peak at its corresponding value of σ_c for each Λ . The corresponding initial values for B and Φ_{BD} are also provided and in

Figure 9: This is a plot of the mass as a function of σ_c for different values of Λ in the Brans-Dicke model.

all cases, $\Omega = 0.8842$. Again, the mass solution does not depend specifically on this value.

When the maximum masses obtained in the Brans-Dicke theory, as given in Table 2, are compared with those obtained in the general relativistic theory, which are found in Table 1, a slight decrease in the peak masses is found along with a shift of each of the peaks to a larger value of σ_c corresponding to a higher density at the center of the boson star. However, when these maximum masses are plotted as a

Table 2: This table contains the maximum mass for each value of the coupling Λ found in Figure 9 with the corresponding values of $B(0)$, $\sigma(0)$, and Φ_{BD} .

Λ	$B(0)$	σ_c	$\Phi_{BD}(0)$	$M\left(\frac{m}{M_{Pl}^2}\right)$
0	0.4637	0.325	1.0289	0.600
10	0.4001	0.250	1.0365	0.894
30	0.3896	0.170	1.0390	1.310
100	0.3820	0.100	1.0430	2.254
200	0.3904	0.070	1.0400	3.100
300	0.3728	0.060	1.0400	3.814
400	0.3886	0.050	1.0400	4.379

function of Λ , as in Figure 10, they seem to have a $\Lambda^{\frac{1}{2}}$ dependency for large values of Λ just as they do in the general relativistic model.

Following a procedure similar to that of Colpi, Shapiro, and Wasserman, two plots of $\sigma(x)$ are made for $\sigma_c = 0.06$ but for different values of Λ . These are shown in Figure 11.

One plot has $\Lambda = 0$ and the other plot has $\Lambda = 300$. However, since the behavior of $\sigma(x)$ in the Brans-Dicke theory and the general relativistic theory are similar, $\sigma(x)$, for large values of Λ , falls off slowly to the approximate radius

$$\frac{\Lambda^{\frac{1}{2}}}{m}. \quad (3.47)$$

Figure 10: This is a plot of the maximum masses as a function of Λ in the Brans-Dicke model. The solid curve corresponds to the approximate solutions and the points correspond to the exact solutions seen in the previous figure.

With these results, the alternative nondimensional quantities given by equations (2.190), (2.191), and (2.192) are used here as well. When these quantities are introduced into the equations of motion given by (3.39), (3.40), (3.41), (3.42), and the terms $O\left(\frac{1}{\Lambda}\right)$ are dropped for large Λ , then the equations of motion take on the form

$$\mathcal{M}'(y) = \frac{y^2}{4\Phi_{BD}} \left(\frac{2\omega + 3}{2\omega + 4} \right) \left(\frac{3\Omega^2}{B} + 1 \right) \left(\frac{\Omega^2}{B} - 1 \right) + \frac{\omega y^2}{4A} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right)^2$$

Figure 11: This graph plots σ versus the distance x for $\Lambda = 0$ and $\Lambda = 300$ and for $\sigma_c = 0.06$ in the Brans-Dicke model. The points correspond to the results of the equations of motion modified for large values of Λ .

$$-\frac{B'y^2}{4AB} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right) - \frac{y^2}{\Phi_{BD}} \left(\frac{1}{2\omega + 4} \right) \left(\frac{\Omega^2}{B} - 1 \right) \quad (3.48)$$

$$\begin{aligned} B'(y) = & \frac{AB}{y} - \frac{B}{y} + \frac{AB y}{2\Phi_{BD}} \left(\frac{2\omega + 3}{2\omega + 4} \right) \left(\frac{\Omega^2}{B} - 1 \right)^2 + \frac{\omega B y}{2} \left(\frac{\Phi'_{BD}}{\Phi_{BD}} \right)^2 \\ & + \frac{B y}{\Phi_{BD}} \left(\Phi''_{BD} - \frac{A'}{2A} \Phi'_{BD} \right) + \frac{2AB y}{\Phi_{BD}} \left(\frac{1}{2\omega + 4} \right) \left(\frac{\Omega^2}{B} - 1 \right) \end{aligned} \quad (3.49)$$

$$0 = \Phi_{BD}'' + \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{2}{x} \right) \Phi_{BD}' + \left(\frac{2A}{2\omega + 4} \right) \left(\frac{\Omega^2}{B} - 1 \right). \quad (3.50)$$

The same type of rescaling of the Brans-Dicke field Φ_{BD} , however, is not warranted as it does not exhibit the same type of dependency on Λ as does $\sigma(x)$. Therefore, these equations are essentially as simple as we can make them. However, even in this simplified form, this system of equations is still very difficult to solve analytically.

These equations are now used to generate some approximate solutions for large Λ . In Figure 11, we compare the values of $\sigma(x)$ found from these equations with the exact values of $\sigma(x)$ for $\Lambda = 300$ and $\sigma_c = 0.06$. As was seen in the general relativistic model, the approximate and exact values agree except at radii larger than $\frac{\Lambda^{\frac{1}{2}}}{m}$.

The rescaled mass

$$\mathcal{M}(y) = \frac{\mathcal{M}(x)}{\Lambda^{\frac{1}{2}}} \quad (3.51)$$

is determined as a function of the parameter $\frac{\Omega^2}{B(0)}$ with the additional boundary condition $\Phi_{BD}(\infty) = 1$. The results of the calculation are shown in Figure 12.

When these results are compared with the results of the approximation in the general relativistic theory, a slight decrease in the maximum mass similar to the decrease seen in the exact values is observed. The peak value of the mass in Figure 12 is

$$\mathcal{M}_{BD \max} \approx 0.213 \quad (3.52)$$

and the peak is located at $\frac{\Omega^2}{B(0)} = 0.312$. Therefore, for large values of Λ , the exact

Figure 12: This plot corresponds to the results of the mass found from the modified equations of motion.

values of M_{max} should follow the curve

$$M_{max} = 0.213\Lambda^{\frac{1}{2}} \frac{M_{Pl}^2}{m}. \quad (3.53)$$

This curve is plotted on Figure 10, which contains the exact values of the maximum mass. Note that for large Λ , the exact results agree with this curve. The curve given by the general relativistic model is given by

$$M_{max} = 0.218\Lambda^{\frac{1}{2}} \frac{M_{Pl}^2}{m}. \quad (3.54)$$

Figure 13: This is a plot of the maximum mass found from the modified equations of motion as a function of the Brans-Dicke constant ω .

In general, the Brans-Dicke model of the boson star gives a maximum mass which is slightly less than the general relativistic model, as noticed in equations (3.53) and (3.54). Therefore, it seems that as the value of ω is decreased, the mass of the boson star also decreases. To observe this effect in more detail, the approximate equations of motion for large Λ are solved for decreasing values of ω , and the results are shown in Figure 13.

When ω is large, the maximum masses found in the general relativistic model are obtained. However, as ω is decreased, the values of the maximum masses also decrease, and the decrease becomes more dramatic as $\omega < 10$.

Why does the value of ω effect the mass in this way? It turns out that as ω decreases, the strength of the scalar interaction between the test body and the surrounding matter distribution of the universe increases. This effect is seen in Figure 14, in which plots of the Brans-Dicke field for $\omega = 10^6$ (General Relativity) and $\omega = 6$ for $\Lambda = 300$ are shown as a function of the radial distance from the center of the star.

When $\omega = 6$, $\Phi_{BD} > 1$, and a value of Φ_{BD} above one corresponds to a smaller effective gravitational constant as $\Phi_{BD} \sim \frac{1}{G}$. Therefore, it seems that the scalar interaction actually opposes the effect of the gravitational field, or the gravitational field appears to weaken as ω is decreased in value. Due to the overall decrease in the effects of the gravitational field, the mass of the boson star decreases, as the results show.

If the definition of Λ is substituted into the maximum mass equation, then

$$M_{max} = 0.06\lambda^{\frac{1}{2}} \frac{M_{Pl}^3}{m^2} \quad (3.55)$$

which is comparable to the Chandrasekhar mass for fermions of mass $\frac{m}{\lambda^{\frac{1}{4}}}$. Therefore, just as was observed in general relativity, it is possible that fairly massive boson stars could have formed in the early universe under the influence of Brans-Dicke gravity in the presence of even a small coupling effect ($\lambda \ll 1$).

Figure 14: This is a plot of the Brans-Dicke scalar field as a function of distance x for $\omega = 6$ (Brans-Dicke) and $\omega = 10^6$ (general relativity).

Chapter 4

CONCLUSIONS

Unlike previously existing models, which predict boson stars of negligible mass, the general relativistic model with the introduction of a quartic interaction potential in the boson scalar field allows boson stars with cosmologically significant masses to possibly exist. In this paper, it has been shown that if general relativity is replaced by the Brans-Dicke gravitational theories in the quartic coupling model, boson stars with cosmologically significant masses are still possible as we have shown through the numerical calculations that solutions exist.

It has also been shown that a slight decrease in the mass of the boson star occurs if general relativity is replaced by the Brans-Dicke gravitational theories. By studying the effect of a decrease in ω on the Brans-Dicke scalar field and on the mass of the boson star, we have concluded that the interaction between the mass within the radius R and the surrounding matter distribution of the universe through the scalar field actually opposes the effect of the gravitational field, thereby decreasing the gravitational effect on the mass within the radius R and giving rise to a slight decrease in mass. This decrease has also been observed in a study done on neutron stars in the Brans-Dicke gravitational theories.

Thus, it is possible, in a universe under the influence of the Brans-Dicke gravitational theories, that boson stars, if they exist, can contribute significantly to the amount of the dark matter present. This, in turn, supports some inflationary models which make use of the Brans-Dicke gravitational theories and the possible existence of dark matter.

In this paper, we have given a fairly detailed argument as to why boson stars can exist in the Brans-Dicke theories of gravity. However, there are some important questions that we have not covered here.

When looking at solutions of stars, in general, stability is an important factor. The stability determines whether the star will be able to remain in an equilibrium state for an extended period of time. Therefore, unstable solutions are not desirable for the formation or the existence of boson stars.

One method of determining whether a solution is stable or not is to look at analytical solutions of the equations of motion. By analyzing the effect of perturbations on the analytical solutions, we are then able to determine whether the solution has a stable or unstable behavior.

For these reasons, a further investigation into the stability of this model, with an emphasis on finding analytical solutions to the equations of motion, is warranted.

Appendix A

DERIVATIONS

A.1 Normal Modes of Two Coupled Oscillators

Consider two coupled harmonic oscillators of mass m where the oscillators are each attached to a separate wall and attached to each other by strings which have the same tension. These oscillators undergo small transverse vibrations where $q_r(t)$ is the transverse displacement. Each oscillator has the classical kinetic energy

$$T = \frac{1}{2}m \left(\frac{dq_r}{dt} \right)^2 \quad (\text{A.1})$$

and the potential energy, which can be shown by an analysis of the forces on each oscillator, has the form

$$V = k(q_1^2 + q_2^2 - q_1q_2). \quad (\text{A.2})$$

Therefore, the equations of motion for both oscillators are

$$m \frac{d^2q_1}{dt^2} = -k(2q_1 - q_2) \quad (\text{A.3})$$

$$m \frac{d^2q_2}{dt^2} = -k(2q_2 - q_1). \quad (\text{A.4})$$

However, these equations can be uncoupled by defining two new coordinates:

$$Q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2) \quad (\text{A.5})$$

$$Q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2). \quad (\text{A.6})$$

By adding and subtracting (A.4) from (A.3), we find that

$$m \frac{d^2 Q_1}{dt^2} = -k Q_1 \quad (\text{A.7})$$

$$m \frac{d^2 Q_2}{dt^2} = -k Q_2. \quad (\text{A.8})$$

Note that these equations of motion are uncoupled in terms of the new coordinates

Q_1 and Q_2 and have the general solutions

$$Q_1(t) = A \cos \omega_1 t + B \sin \omega_1 t \quad (\text{A.9})$$

$$Q_2(t) = C \cos \omega_2 t + D \sin \omega_2 t. \quad (\text{A.10})$$

If we choose an initial displacement of a and zero initial velocity for both oscillators, we will then find at all times t

$$Q_1(t) = \sqrt{2} a \cos \omega_1 t \quad (\text{A.11})$$

$$Q_2(t) = 0. \quad (\text{A.12})$$

Likewise, if we let the displacement of one oscillator be a and the displacement of the other oscillator to be $-a$, then we find at all times t

$$Q_1(t) = 0 \quad (\text{A.13})$$

$$Q_2(t) = \sqrt{2} a \cos \omega_2 t. \quad (\text{A.14})$$

Note that both of these motions constitute a “normal mode” as the whole system moves with a definite frequency. Therefore, Q_1 and Q_2 are known as normal mode coordinates. In terms of the normal coordinates, q_1 and q_2 have the form

$$q_1(t) = \frac{1}{\sqrt{2}}[Q_1(t) + Q_2(t)] \quad (\text{A.15})$$

$$q_2(t) = \frac{1}{\sqrt{2}}[Q_1(t) - Q_2(t)] \quad (\text{A.16})$$

and the system can be thought of as being in a “superposition of modes.”

Now, we see that the modes do not interact since their equations of motion are uncoupled. This can also be seen in the total energy of the system, which is

$$E = \left[\frac{1}{2}m \left(\frac{dQ_1}{dt} \right)^2 + \frac{1}{2}m\omega_1^2 Q_1^2 \right] + \left[\frac{1}{2}m \left(\frac{dQ_2}{dt} \right)^2 + \frac{1}{2}m\omega_2^2 Q_2^2 \right]. \quad (\text{A.17})$$

The energy has the appropriate form for two non-interacting modes, each mode executing simple harmonic motion. Again, this is true because the anharmonic terms in the potential are ignored.

A.2 Normal Modes on a String

Let's introduce the concept of normal modes into the continuous case of the vibrating string. Suppose that we stretch a string between the fixed points $x = 0$ and $x = L$. This constrains $\phi(x, t)$ to vanish at the ends of the string. A suitable form of $\phi(x, t)$ which satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^2} = 0 \quad (\text{A.18})$$

and the initial conditions above is

$$\phi_r(x, t) = A_r(t) \sin \frac{r\pi x}{L} \quad : r = 1, 2, 3, \dots \quad (\text{A.19})$$

which only allows half-wavelengths to fit onto the interval $(0, L)$. If we substitute $\phi_r(x, t)$ into the wave equation, we find

$$\frac{d^2 A_r}{dt^2} = -\omega_r^2 A_r \quad (\text{A.20})$$

where

$$\omega_r = \frac{r\pi c}{L}. \quad (\text{A.21})$$

Therefore, the amplitude $A_r(t)$ of the vibration described by $\phi_r(x, t)$ executes simple harmonic motion with a frequency ω_r , and each amplitude $A_r(t)$ corresponds to a mode where the amplitudes are actually the normal coordinates. The general motion of the string, which we saw in the discrete case, is a superposition of modes and is described by

$$\phi(x, t) = \sum_{r=1}^{\infty} A_r(t) \sin \frac{r\pi x}{L}. \quad (\text{A.22})$$

Now, the total energy in a classical, conservative, continuous system is given by the Hamiltonian (2.45). Therefore, from equations (2.45), (2.48), and (2.47), we then have

$$E = \int_0^L dx \left[\frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \rho c^2 \left(\frac{\partial \phi}{\partial x} \right)^2 \right] \quad (\text{A.23})$$

where the first term is the kinetic energy and the second term is the potential energy.

If we substitute $\phi(x, t)$ into E and use the orthogonality of the sine function on the

interval $(0, L)$, we find that

$$E = \left(\frac{L}{2}\right) \sum_{r=1}^{\infty} \left[\frac{1}{2} \rho \left(\frac{\partial A_r(t)}{\partial t} \right)^2 + \frac{1}{2} \rho \omega_r^2 A_r^2 \right] \quad (\text{A.24})$$

A.3 The Quantization Procedure

In the Heisenberg formulation, the operators representing dynamical quantities are time-dependent while the wavefunction is time-independent, unlike the more familiar Schrödinger formulation in which the reverse is true. Time-dependent operators are important as the generalized coordinate $q(t)$ and its conjugate momentum $p(t)$ in the Hamilton-Lagrange formulation of classical mechanics are time-dependent, just as in the Heisenberg formulation of quantum mechanics. This, in fact, suggests that there is a direct correspondence between the two theories.

When we go from classical theory to quantum theory, the general procedure is to replace observable quantities by their corresponding quantum operators. In quantum particle mechanics, the usual commutator between the coordinate operator and the momentum operator is

$$[\hat{q}_r(t), \hat{p}_s(t)] = i\delta_{rs} \quad (\text{A.25})$$

where the hat indicates an operator. The canonical momentum operator $\hat{p}$ is defined by

$$\hat{p} = \frac{\partial \hat{L}}{\partial \left(\frac{d\hat{q}}{dt} \right)} \quad (\text{A.26})$$

Also, in the Heisenberg formulation, Schrödinger's equation is replaced by the Heisenberg equation of motion

$$\frac{d\hat{A}}{dt} = -i[\hat{A}, \hat{H}] \quad (\text{A.27})$$

where the Hamiltonian operator is defined in terms of the Lagrangian operator by

$$\hat{H} = \hat{p} \frac{d\hat{q}}{dt} - \hat{L} \quad (\text{A.28})$$

and $\hat{A}$ is any observable quantity.

A.4 Fortran Program

This is the computer program that I used to calculate the results for the general relativistic model and the Brans-Dicke model. I used an existing subroutine called DIVPRK which uses a 5th-6th order Runge-Kutta-Verner solving method.

```

INTEGER NEQ, NPARAM, LOOP
PARAMETER (NEQ= 8, NPARAM=180)

CHARACTER DONE*1, REPL*1

INTEGER IDO, INORM, MXSTEP, CALC, RUNS, VAL, FLD
REAL FCN, PARAM(NPARAM), HMIN
DOUBLE PRECISION TOL, F(NEQ), X, XEND, IEND, STEP, OMEGA,
SIG

```

DOUBLE PRECISION PHIBD, BRANS, OMG, BOSMASS, STARMASS

EXTERNAL FCN, DIVPRK, SSET, UMACH

INITIALIZE

HMIN = 0.0

MXSTEP = 50000

INORM = 1.0

CALL SSET (NPARAM, 0.0, PARAM, 1)

PARAM(3) = HMIN

PARAM(7) = MXSTEP

PARAM(10) = INORM

PRINT *, 'HOW MANY VALUES OF SIG'

READ *, VAL

PRINT *, 'ENTER OMEGA AND THE FIELD'

READ *, OMEGA, PHIBD

CALC = 50

DO 50 RUNS = 1, VAL, 1

PRINT *, 'ENTER SIG(0)'

READ *, SIG

DO 20 LOOP = 1, CALC

TOL = 1.0D-13

STEP = 120.0D0

F(3) = SIG

F(5) = PHIBD

F(7) = OMEGA

PRINT *, 'ENTER B(0)'

READ *, F(2)

IDO = 1

X = 1.0D-12

*PRINT HEADING

WRITE (8, 99998)

PRINT 99998

*INTEGRATION

DO 10 IEND = 1.0D0 , STEP, 1.0D0

XEND = IEND

CALL DIVPRK (IDO, NEQ, FCN, X, XEND, TOL, PARAM, F)

PRINT 99999, X, F(1), F(2), F(3), F(4), F(5), F(6), F(8)

IF (F(3) .LT. 0.0D0 .OR. F(4) .GT. 0.0D0) THEN

GOTO 30


```
END IF

10  CONTINUE

30  CONTINUE

*FINNISH

IDO = 3

CALL DIVPRK (IDO, NEQ, FCN, X, XEND, TOL, PARAM, F)

99998  FORMAT (6X, 'X', 7X, 'M(X)', 5X, 'B(X)', 3X, 'SIG(X)', 4X
+ , 'SGP(X)', 3X, 'PHI(X)', 4X, 'PHP(X)')

99999  FORMAT (T1,F10.6,T12,F8.6,T21,F8.6,T30,F8.6,T39,F9.6,T49,F8.6,
+ T58,F9.6,T68,F9.6)

CLOSE (UNIT=8)

PRINT *, 'DO YOU WANT TO CHANGE THE FIELD'

READ 200, REPL

200  FORMAT(T1, A1)

IF (REPL.EQ.'Y') THEN

GOTO 160

END IF

20  CONTINUE

50  CONTINUE

END
```

*SUBROUTINE

SUBROUTINE FCN (NEQ, X, F, FPRIME)

INTEGER NEQ

DOUBLE PRECISION F(NEQ), FPRIME(NEQ), X, LMDA, OMEG

LMDA = 0.000000000000

OMEG = 1.0D6

IF (F(1) .EQ. 0.0D0) THEN

FPRIME(1)=0.0D0

FPRIME(2)=0.0D0

FPRIME(3)=0.0D0

FPRIME(4)=0.0D0

FPRIME(5)=0.0D0

FPRIME(6)=0.0D0

F(1)=0.0D0

F(4)=0.0D0

F(6)=0.0D0

END IF

FPRIME(1) = (X**2/(2.0D0*F(5)))*((2.0D0*OMEG+3.0D0)/(2.0D0*
 + OMEG+4.0D0))*((F(7)**2/F(2)+1.0D0)*F(3)**2
 + +(LMDA*F(3)**4 /2.0D0)+F(4)**2*(1.0D0-(2.0D0*F(1)/X)

$$\begin{aligned}
& +)) + (\text{OMEG} * X^{**2} * F(6)^{**2} * (1.0D0 - (2.0D0 * F(1)/X))) / (4.0D0 * F(5) \\
& + **2) - (F\text{PRIME}(2) * X^{**2} * F(6) * (1.0D0 - (2.0D0 * F(1)/X))) \\
& + / (4.0D0 * F(2) * F(5)) + (X^{**2} / (F(5) * (4.0D0 + 2.0D0 * \text{OMEG}))) * \\
& + ((F(7)^{**2} / F(2) - 2.0D0) * F(3)^{**2} - (1.0D0 - (2.0D0 * F(1)/X))) \\
& + * F(4)^{**2} - \text{LMDA} * F(3)^{**4})
\end{aligned}$$

$$\begin{aligned}
& F\text{PRIME}(2) = F(2) / (X * (1.0D0 - (2.0D0 * F(1)/X))) - F(2) / X + ((F(2) * X) / \\
& + (F(5) * (1.0D0 - (2.0D0 * F(1)/X)))) * ((2.0D0 * \text{OMEG} + 3.0D0) / (2.0D0 * \\
& + \text{OMEG} + 4.0D0)) * ((F(7)^{**2} / F(2) - 1.0D0) * F(3)^{**2} + \\
& + F(4)^{**2} * (1.0D0 - (2.0D0 * F(1)/X)) - \text{LMDA} * F(3)^{**4} / 2.0D0) + \\
& + \text{OMEG} * F(2) * X * F(6)^{**2} / (2.0D0 * F(5)^{**2}) + (F(2) * X / F(5)) * (\\
& + F\text{PRIME}(6) - F(6) * (F\text{PRIME}(1) * X - F(1))) / (X^{**2} * (1.0D0 - (2.0D0 * F(1) \\
& + / X))) - ((2.0D0 * F(2) * X) / (F(5) * (1.0D0 - (2.0D0 * F(1)/X))) * (\\
& + 4.0D0 + 2.0D0 * \text{OMEG})) * ((F(7)^{**2} / F(2) - 2.0D0) * F(3)^{**2} - (1.0D0 - \\
& + (2.0D0 * F(1)/X)) * F(4)^{**2} - \text{LMDA} * F(3)^{**4})
\end{aligned}$$

$$F\text{PRIME}(3) = F(4)$$

$$\begin{aligned}
& F\text{PRIME}(4) = -1.0D0 * ((F\text{PRIME}(2) / (2.0D0 * F(2))) - (F\text{PRIME}(1) * X - F(1))) / \\
& + (X^{**2} * (1.0D0 - (2.0D0 * F(1)/X))) + 2.0D0 / X * F(4) + (1.0D0 / (1.0D0 - \\
& + (2.0D0 * F(1)/X))) * ((F(7)^{**2} / F(2) - 1.0D0) * F(3) - \text{LMDA} * F(3)^{**3})
\end{aligned}$$

$$F\text{PRIME}(5) = F(6)$$

$$F\text{PRIME}(6) = -1.0D0 * ((F\text{PRIME}(2) / (2.0D0 * F(2))) - (F\text{PRIME}(1) * X - F(1))) /$$

```

+ (X**2*(1.0D0-(2.0D0*F(1)/X)))+2.0D0/X)*F(6)-(2.0D0/((1.0D0
+ -(2.0D0*
+ F(1)/X))*(4.0D0+2.0D0*OMEG)))*((F(7)**2/F(2)-2.0D0)*F(3)**2-
+ F(4)**2*(1.0D0-(2.0D0*F(1)/X)) - LMDA*F(3)**4))

```

```

FPRIME(7) = 0.0D0

```

```

FPRIME(8) = 0.0D0

```

```

F(8) = FPRIME(2)

```

```

RETURN

```

```

END

```


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