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Geometry Of Quivers

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GEOMETRY OF QUIVERS

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This thesis, submitted by Patrick A. Durkin in partial fulfillment of the requirements for the Degree of Master of Science from the University of North Dakota, has been read by the Faculty Advisory Committee under whom the work has been done and is hereby approved.

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Patrick A. Durkin
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ABSTRACT

A quiver is a directed graph, but the term usually implies such a graph is being considered along with representations. These representations consist of vector spaces and linear transformations. We explore some the connections between quivers and geometric structures. To begin, we consider a theorem that says every projective variety can be considered as a quiver Grassmannian. The reasoning of the proof is demonstrated by example. We then prove the existence of a countable quiver containing every finite quiver as a subquiver. Following this we consider some properties of its category of representations. Finally, we give an overview of quiver varieties, which have been well-studied in geometric representation theory.

CHAPTER I

INTRODUCTION

The goal here is to present the many ways in which certain fairly simple mathematical structures, called quivers, are connected to geometry. First we draw out a strong connection between quivers and algebraic geometry. Second, we follow this with an exploration of the properties of a certain quiver representation category. These categorical properties have geometric applications and interpretations, but for the most part the explicit details of those relationships will be omitted. Lastly, we turn to the growing theory of quiver varieties and geometric representation theory, in which quivers are used to give a new twist to the usual representation theory. The final section sketches some very preliminary ideas which will hopefully eventually apply themselves to the study of the group law on plane elliptic curves. The purpose of this thesis is simply to explore these connections, and *not* to imply that using quivers to study these topics would be a good idea–though hopefully it could be!

The material is aimed at a reader familiar with basic graduate-level topology and algebra, as well as familiarity with the language of category theory. In particular, readers familiar with the texts of Munkres [12] (topology), Hungerford [9] (algebra), and Mac Lane [11] (category theory), or similar texts, will be suitably prepared. With the algebra some strength in linear algebra is assumed, but Hoffman and Kunze [8] is another excellent reference for what is needed here. We begin by introducing other fundamentals: the general theory of quivers and some basics of algebraic geometry. For those needing a refresher on category theory, several definitions are given in Appendix A. The remaining prerequisite information is introduced as needed for each chapter.

Quivers and their Representations

In this section we give an overview of the theory of quiver representations. The name quiver is meant to bring to mind a collection of arrows, but mathematical quivers come equipped with additional information on the source and target of each arrow. Another accessible source with this information and more is [3].

Definition I.1. A **quiver** Q is a pair of finite sets (Q_0, Q_1), together with a pair of functions $s, t : Q_1 \to Q_0$.

In more intuitive terms, a quiver is simply a multi-digraph, possibly with loops[1]. The finiteness condition on the sets is a simplifying assumption, but we will occasionally be interested in infinite quivers. Infinite quivers turn out to be capable of capturing a great deal of information, such as the structure of any small category.

The set Q_0 above is understood as the set of vertices and Q_1 the set of arrows (or directed edges) between them. For each arrow, the functions s and t indicate the *source* and *target*, respectively. In practice we omit the parentheses for applying the source and target functions, so $t(a)$ is written ta. The question is often raised: why give a new name to an old object? In fact, the term quiver is reserved primarily for these sorts of directed graphs equipped with a particular kind of representation.

Definition I.2. Let Q be a quiver and k a field. A **representation** V of Q is a pair of sets $V_0 := \{V_x \mid x \in Q_0\}$ and $V_1 := \{V^a : V_{sa} \to V_{ta} \mid a \in Q_1\}$, where each V_x is a finite-dimensional k-vector space and each V^a is a k-linear transformation. If V is

[1] Unfortunately, the verbage of graph theory is inconsistent. This verbose description should cover all possibilities.

a representation of Q, then the *dimension vector of* V is the function $d_V : Q_0 \to \mathbb{N}$ defined by $d_V(x) = \dim_k(V_x)$.

If we fix a quiver Q and a field k, we would like to consider the collection of all quiver representations of Q (over k). Once we consider this as a collection of objects, it becomes natural to ask what sort of maps we would like to consider between them. If we have two representations V and W of the quiver Q, then for any vertex $x \in Q_0$ a map from V to W should take V_x to W_x. We then need to carry each map V^a to a map $W_{sa} \to W_{ta}$ in a way that preserves all the necessary structure. This leads to the following definition of quiver morphisms, and a category of quiver representations.

Definition I.3. Let V and W be representations of a quiver Q (both over the same field k). A **morphism of quivers** $\Phi : V \to W$ is a family of k-linear transformations $\{\varphi_x : V_x \to W_x \mid x \in Q_0\}$ such that for every arrow $a \in Q_1$, we have $W^a \circ \varphi_{sa} = \varphi_{ta} \circ V^a$. That is, for each $a \in Q_1$ the diagram

$$\begin{array}{ccc} V_{sa} & \xrightarrow{\varphi_{sa}} & W_{sa} \\ {\scriptstyle V^a}\downarrow & & \downarrow{\scriptstyle W^a} \\ V_{ta} & \xrightarrow[\varphi_{ta}]{} & W_{ta} \end{array}$$

commutes. An **isomorphism** is a morphism $\Phi : V \to W$ such that for all $x \in Q_0$, the associated map φ_x is an isomorphism of vector spaces. We define the category $\textbf{Rep}_k(Q)$ of representations of Q and quiver morphisms between them.

We can make a few basic remarks about $\textbf{Rep}_k(Q)$ to see this is indeed a category. Clearly the identity morphism on a representation V is the identity on the maps V_a. Composition of quiver morphisms follows by gluing together two copies of the diagram above, and associativity by gluing three copies.

One of the goals of the study of quivers is to classify the isomorphism classes of representations. As an easy start, we may note that the categories $\textbf{Rep}_k(Q)$ each have an initial object: the representation consisting of all 0-dimensional k-vector spaces and their trivial maps. Such a representation carries with it no more structure than the quiver Q. This is clearly the only member of its isomorphism class, and we refer to this as the trivial representation. For more complicated representations the following definition becomes useful, and simplifies the classification problem.

Definition I.4. Let V and W be objects of $\textbf{Rep}_k(Q)$. The **direct sum** $V \oplus W$ is given by taking $(V \oplus W)_x := V_x \oplus W_x$ for all $x \in Q_0$ and

$$(V \oplus W)^a := \begin{pmatrix} V^a & 0 \\ 0 & W^a \end{pmatrix}$$

for all $a \in Q_1$.

Note that this indeed means for all $a \in Q_1$ we have $(V \oplus W)^a : V_{sa} \oplus W_{sa} \to V_{ta} \oplus W_{ta}$, with V^a and W^a applied component-wise. This definition allows for the following.

Definition I.5. Let V be a quiver representation. If there exist nontrivial representations W and Z such that $V \cong W \oplus Z$, then V is **decomposable**. We say V is **indecomposable** if there are no such W and Z.

If V is a quiver representation with $V \cong W \oplus Z$, then we expect W and Z to be isomorphic to subrepesentations of V (for a suitable definition of subrepresentation). We will frequently be interested in subrepresentations, both in decompositions and for their key role in the definition of quiver Grassmannians.

Definition I.6. Let V be a representation of a quiver Q. We say a representation W is a **subrepresentation of** V if:

(i) W_x is a subspace of V_x for each $x \in Q_0$, and

(ii) $W^a = V^a|_{W_{sa}}$ for each $a \in Q_1$.

Since a subrepresentation is itself a representation, we have the same notion of dimension vector. In fact, we will be interested in considering all possible subrepresentations *in a specified dimension vector*. This is the notion of a quiver Grassmannian, to be defined precisely later.

Algebraic Geometry Fundamentals

In this section we give an overview of some of the basic definitions and theorems of algebraic geometry. The terminology will be indispensable in the sequel as we consider the relationships of quivers to these ideas. The most important concept is that of a variety, and the following builds to that. A succinct–and enjoyable–account of this information and more may be found in [10].

Definition I.7. Let k be an algebraically closed field. The n-**dimensional affine space** $\mathbb{A}^n$ is the set k^n.

Observe $\mathbb{A}^n$ has a natural vector space structure. Each function f in the polynomial ring $k[x_1, \ldots, x_n]$ gives a map $\mathbb{A}^n \to k$ by evaluation.

Definition I.8. Let k be an algebraically closed field. If $S \subset k[x_1, \ldots, x_n]$, define the set $V(S) := \{X \in \mathbb{A}^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S\}$. An **affine variety** is a set $W \subset \mathbb{A}^n$ such that $W = V(S)$ for some $S \subset k[x_1, \ldots, x_n]$, with (S) a prime ideal of $k[x_1, \ldots, x_n]$. If W is an affine variety defined by prime ideal I, then the **coordinate chart of** W is the quotient ring $k[x_1, \ldots, x_n]/I$.

The coordinate ring of an affine variety W is exactly the morphisms with domain W in the category of algebraic varieties.

Definition I.9. Let $W \subset \mathbb{A}^n$ and $Z \subset \mathbb{A}^m$ be affine varieties. If W is defined by the ideal $I \subset k[x_1, \ldots, x_n]$, then a **regular map** from W to Z is a function of the form $f = (f_1, \ldots, f_m)$, where $f_i \in k[x_1, \ldots, x_n]/I$ for $1 \leq i \leq m$ and such that $f(W) \subset Z$. A regular map with regular inverse is called **biregular**.

Of course if regular maps are the morphisms, then the biregular maps are isomorphisms in the category of algebraic varieties.

It turns out that the study of geometry on affine spaces lacks a certain je ne sais quoi. By constructing a different-but-related ambient space we can enjoy the catharsis of a more elegant theory. That space is *projective space*.

Definition I.10. Let k be an algebraically closed field. Then n**-dimensional projective space** $k\mathbb{P}^n$ is the set of $(n-1)$-dimensional subspaces of $k\mathbb{A}^n$. Points of $k\mathbb{P}^n$ are given by **homogeneous coordinates** in $n+1$ elements of k (not all zero). That is, a point of $k\mathbb{P}^n$ is $[X_0 : X_1 : \cdots : X_n]$ for $X_i \in k$ and some $X_i \neq 0$. By homogeneous, we mean $[X_0 : X_1 : \cdots : X_n]$ and $[Y_0 : Y_1 : \cdots : Y_n]$ describe the same point if and only if there is $\lambda \in k$ such that $X_i = \lambda Y_i$, $i = 0, 1, \ldots, n$.

We identify the points for which $X_n \neq 0$ with the affine space $k\mathbb{A}^n$. Also, we will typically omit the reference to the field k in the notation. To define a *projective* variety, we now need the notion of a homogeneous polynomial.

Definition I.11. A **homogeneous polynomial** of degree n (in k variables) is one in which every monomial has degree n.

Recall the degree of a monomial involving several variables is the sum of the exponents of each variable. Given any polynomial in n variables, we can always "homogenize" it by introducing an additional variable.

Example I.12. The polynomial expression $xy + x^3 - y$ is defined on $\mathbb{A}^2$ but is not of interest in any projective space as it is not homogeneous. Its homogenezation is obtained by realizing its degree is 3, so we introduce a third variable in such a way as to end up with a homogeneous polynomial of degree 3. The result is $xyz + x^3 - yz^2$. This expression is now homogeneous and therefore its zeros are of interest in $\mathbb{P}^2$. Note by setting $z = 1$ we get the original polynomial. Therefore we refer to $C : xy + x^3 - y = 0$ (a variety in $\mathbb{A}^2$) as the *affine portion* of the projective variety defined by the homogeneous polynomial.

CHAPTER II

QUIVER GRASSMANNIANS AS PROJECTIVE VARIETIES

For our purposes, a Grassmannian is a concept from linear algebra (it can be treated as a more general functor on different sorts of categories). As with many concepts concerning vector spaces, we can extend the notion to quivers, which are essentially *systems* of vector spaces. Here we develop some of the main ideas surrounding Grassmannians.

Definition II.1. Let V be a vector space of finite dimension n. For any nonnegative r, we define the **Grassmannian** $\mathrm{Gr}_r(V)$ to be the set of all r-dimensional subspaces of V (note if $r > n$ then $\mathrm{Gr}_r(V) = \emptyset$). In fact, Gr_r is a functor Gr_r : **FinVect** $\to$ **Set**.

We expand the notion to quiver representations in the obvious way.

Definition II.2. Let Q be a quiver and V a representation of Q. Let $m : Q_0 \to \mathbb{N}$. Then a **quiver Grassmannian** is the set of all subresentations of V with dimension vector m.

One of the most important motivations for us to consider the Grassmannian is that projective space *is* a Grassmannian (by our choice of definition). This leads to the natural question of just how much projective geometry can be captured by Grassmannians. In the next section we will see that *every* projective variety is a Grassmannian of a quiver representation, thereby fully answering the question.

Quiver Grassmannians and Projective Varieties

In early 2012, Markus Reineke posted a note on arXiv titled "Every projective variety is a quiver Grassmannian" (later published. See [16]). Rather than discuss the proof in detail here (though it is only two pages), we will illustrate the argument by example below. Recently, Pieter Belmans wrote an algorithm in SAGE which can take the defining polynomials for any projective variety and return a description of the isomorphic quiver Grassmannian. The code can be found on his website [2], but is also included in Appendix B for convenience. We will verify our example using Belmans's algorithm.

It is well-known that the Grassmannian of any vector space is a projective variety. The usual way of showing this involves the wedge product and a map known as the Plücker embedding. It does not seem that this construction is readily extended to quivers. An alternative construction, which does extend nicely, is given in [5]. The details are too involved to be discussed here.[2]

The proof that every projective variety may be viewed as a quiver Gassmannian works by using a map known as the Veronese embedding, and does not require the wedge product. This is because we are free to pick a sufficiently nice quiver and dimension vector, allowing us to skirt the more involved constructions. The goal in our example will be to start with a homogenenous polynomial, embed the variety it generates into a larger projective space, then show that a certain quiver Grassmannian satisfies appropriate relational equations.

Since conics have no secrets, elliptic curves are the natural examples to look to. We will consider an elliptic curve given by $X^3 + X^2Z + 2XZ^2 - Y^2Z = 0$. We

[2]The idea is to view the Grassmannian as a collection of open sets modulo some linear relations. The extension comes readily by noting the maps of a quiver, being linear maps, simply give additional relational information. Some care must be taken to be sure this is actually a variety and not just a scheme.

offer a word of caution before we begin: the process is purely constructive and the motivations at each step may be unclear. By the end of the whole process the connections from step to step should be more apparent, but the reader should consult [16]. The affine portion of our chosen variety is pictured below.

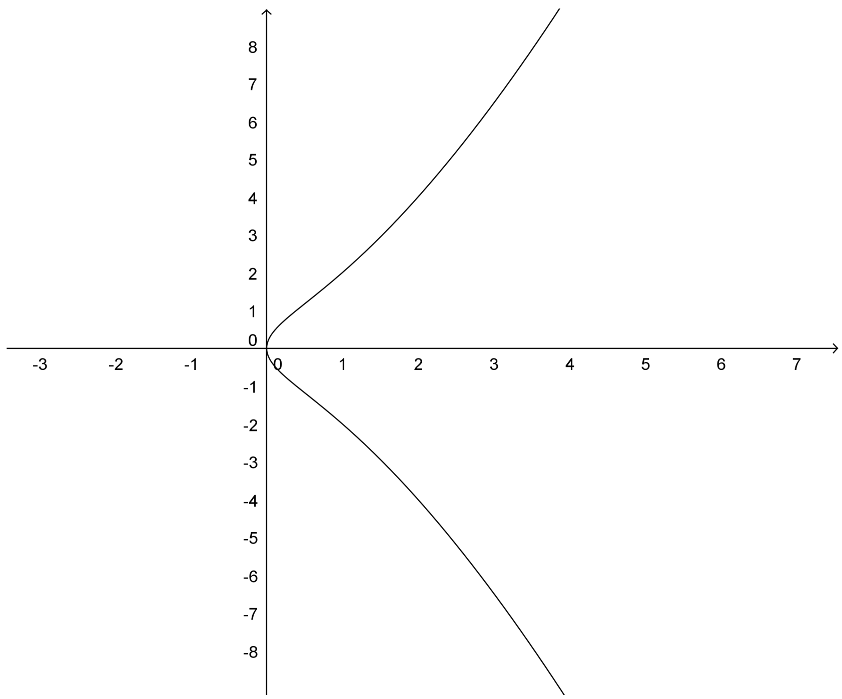


Figure 1: Affine Portion of $C : X^3 + X^2Z + 2XZ^2 - Y^2Z$

Our goal now is to find a quiver Q, a representation V of Q, and a dimension vector m such that the quiver Grassmannian $Gr_m(V)$ is isomorphic to the variety C in $\mathbb{P}^2$. The first tool we will need in this process is the *Veronese embedding*. For our purposes, we only need the Veronese map of degree 3 in 3 variables. The reason will become apparent.

Definition II.3. The **Veronese embedding of degree d in $n+1$ variables** is the function $\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ taking a point $[x_0 : \cdots : x_n]$ to the point determined by evaluation of all possible monomials of degree d in n variables. In particular, the Veronese map of degree 3 in 3 variables is the mapping $\nu_3 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ given by

$[x : y : z] \mapsto [x^3 : x^2y : x^2z : xy^2 : xyz : xz^2 : y^3 : y^2z : yz^2 : z^3]$.

Note that changing the order of the monomials in the target gives an automorphism of $\mathbb{P}^m$, so there is no need to be concerned about the order. In the general version of the definition above, it is clear m may be determined by a combinatorial argument given d and n. In particular, $m = \binom{n+d}{d} - 1$. Now if we associate the target coordinates in the mapping ν_3 above with the standard projective coordinates $[X_0 : \cdots : X_9]$ for $\mathbb{P}^9$, we have the following.

Proposition II.4. *Let $C : X^3 + X^2Z + 2XZ^2 - Y^2Z$. Then $\nu_3(C) = \nu_3(\mathbb{P}^2) \cap V(X_1 + X_3 + 2X_6 - X_8)$. In other words, C becomes the intersection of the image of $\mathbb{P}^2$ with the hyperplane $V(X_0 + X_2 + 2X_5 - X_7)$ in $\mathbb{P}^9$.*

Proof. Recall the correspondence between monomials and projective coordinates given above. We need only note that $\nu_3(C) = V(X_0 + X_2 + 2X_5 - X_7)$ under this correspondence. Moreover, it is clear we are only interested in the maximal subset of $\nu_3(C)$ contained in $\nu_3(\mathbb{P}^2)$ to obtain an isomorphism onto this image. □

Now we want to take all possible monomials of degree 2 in the variables x, y, and z: namely $x^2, xy, xz, y^2, yz, z^2$, and consider the matrix

$$M = \begin{bmatrix} x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x^3 & x^2y & x^2z \\ x^2y & xy^2 & xyz \\ x^2z & xyz & xz^2 \\ xy^2 & y^3 & y^2z \\ xyz & y^2z & yz^2 \\ xz^2 & yz^2 & z^3 \end{bmatrix}.$$

Note that all 2×2 minors of M vanish–i.e. by deleting four rows and one column, the determinant of the remaining entries is zero. In particular, this means the rank of M is 1.

For simplicity, relabel M according to our standard $\mathbb{P}^9$ coordinates so we have

$$M = \begin{bmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \\ X_3 & X_6 & X_7 \\ X_4 & X_7 & X_8 \\ X_5 & X_8 & X_9 \end{bmatrix}$$

and for $p \in \mathbb{P}^9$, set $\varphi(p) := X_0 + X_2 + 2X_5 - X_7$. Since $M(p)$ is of rank 1 if (and in fact only if) p is the image of a point in $\mathbb{P}^2$ under ν_3, by Proposition II.4 we may recover the an isomorphic copy of C by taking the set of points $p \in \mathbb{P}^9$ such that $M(p)$ has rank 1 and $\varphi(p) = 0$ (the second condition being p lies on the hyperplane as in Proposition II.4).

Now it is time to construct our quiver. Reineke's result is actually quite a bit stronger than we have let on, in part because the quiver required needs just three vertices. Call these vertices A, B, and C. We now require a single arrow from B to A corresponding to the hyperplane discussed above. We also require 3 arrows from B to C corresponding to the three variables defining coordinates of $\mathbb{P}^2$. This means our underlying digraph is determined completely by the the dimension of the ambient space containing our variety. It should be noted Reineke's construction also works for varieties defined by non-principal ideals, and so may require more than one arrow from B to C if more hypersurfaces are needed to describe the image in the larger space.

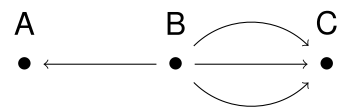


Figure 2: The Underlying Digraph Q

Now call the digraph we have constructed so far Q. We must now select a representation for Q. To do so, we will take as our ground field $\mathbb{C}$ and select the vector spaces as follows: let V_A be one-dimensional so $V_A = \mathbb{C}$, let V_B be 10-dimensional so $V_B = \mathbb{C}^{10}$ (this comes from the number of homogeneous coordinates defining the space $\mathbb{P}^9$), and let V_C be 6-dimensional so $V_C = \mathbb{C}^6$. The reason for this "6" is combinatorial and will not be discussed in detail here, but we will say it comes from the number of 3-tuples of natural numbers with entries summing to 2: $(2,0,0),(0,2,0),(0,0,2),(0,1,1),(1,0,1),(1,1,0)$. The reasoning for this is obscured by appealing to the "rank 1" argument we used earlier. See [16] for one-and-a-half alternative lines of reasoning which may shed light on the choices of dimension.

With our choice of vector spaces finished, we simply need to choose linear maps and a dimension vector. The linear maps are easy. Define them using the columns of the 6×3 matrix M:

$$r : V_B \to V_C \qquad [X_0 : \cdots : X_9] \mapsto [X_0 : X_1 : X_2 : X_3 : X_4 : X_5],$$

$$s : V_B \to V_C \qquad [X_0 : \cdots : X_9] \mapsto [X_1 : X_3 : X_4 : X_6 : X_7 : X_8],$$

$$t : V_B \to V_C \qquad [X_0 : \cdots : X_9] \mapsto [X_2 : X_4 : X_5 : X_7 : X_8 : X_9].$$

The purpose of the map $V_2 \to V_1$ is to encode the data about the hyperplane induced by the defining equation for C in $\mathbb{P}^2$. That is, define

$$p : V_B \to V_A \qquad\qquad [X_0 : \cdots : X_9] \mapsto X_0 + X_2 + 2X_5 - X_7.$$

Now it may seem quite a puzzler to determine a suitable dimension vector for our Grassmannian. Fortunately for us, we get to use one of the simplest possible: define $m = (0, 1, 1)$. This is yet another reason the proof is stronger than you have been led to believe. Recall this means we will be considering all (the only) zero-dimensional subspace of V_A, all 1-dimensional subspaces of V_B, and all 1-dimensional subspaces of V_C. Note that if we looked at any other variety in $\mathbb{P}^2$, the only difference to this point would be the hyperplane in $\mathbb{P}^9$ (and thereby the map p as well). Recall for higher-dimensions we need additional maps $V_B \to V_C$, but in general the dimension vector is always $(0, 1, 1)$.

We now take our quiver Grassmannian $\mathrm{Gr}_m(V)$. Let $W \in \mathrm{Gr}_m(V)$, so W_B is a 1-dimensional subspace of V_B and let v be a vector of W_B. By choice of dimension vector, $p(v) = 0$ since p must be the trivial map. This means v determines a point on our hyperplane. Likewise, $r(v), s(v), t(v) \to W_C$ describe the column space of $M(v)$, and $\dim W_C = 1$. Thus our matrix $M(v)$ has rank 1, hence $v \in \nu_3(\mathbb{P}^2)$. With both of these conditions met, we conclude $v \in \nu_3(C) \cap V(X_0 + X_2 + 2X_5 - X_7)$. On the other hand, it is not hard to see all p on C determine a subrepresentation of dimension $(0, 1, 1)$ (consider the form of Q, in which all arrows have source B). So $\mathrm{Gr}_m(V) \cong \nu(\mathbb{P}^2) \cap V(X_0 + X_2 + 2X_5 - X_7) \cong C$, as desired.

We can check this is consistent with Belmans's algorithm. The output is given on the following page.

Reineke's proof goes on to show the representation V has the property of being *Schurian*, but this is beyond the scope of this paper. We end with one definition.

```
Considering the projective variety of dimension 1 in PG(2, Q) defined by
    x^3 + x^2*z - y^2*z + 2*x*z^2
The same equations, all of the same degree (d = 3)
    x^3 + x^2*z - y^2*z + 2*x*z^2
The dimension vector is (1, 10, 6)
The 1 morphism(s) defining the variety (i.e. the maps 2->1) are
    x0 + x2 + 2*x5 - x7
The 3 morphisms defining the d-uple embedding (i.e. the maps 2->3) are described by
    (x0, x1, x2, x3, x4, x5)
    (x1, x3, x4, x6, x7, x8)
    (x2, x4, x5, x7, x8, x9)
```

Figure 3: Algorithmic Verification of the Computation

Definition II.5. A **Reineke (k, n)-quiver** is a quiver of the form

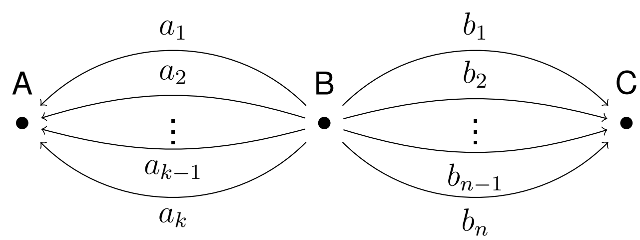


Figure 4: The Reineke (k, n)-quiver

with k arrows $B \to A$ and n arrows $B \to C$.

The proof that every projective variety is a quiver Grassmannian demonstrates every variety $V \subset \mathbb{P}^n$ can be represented using some Reineke $(k, n+1)$-quiver. Since every quiver Grassmannian is also a projective variety, such representations are not unique.

There are several ways to extend this line of thinking:

1. One goal would be to categorify the construction and make it functorial. There are a few challenges to this. It is tempting to think of a representation as a functor $Q \to$ **FinVect**, but Q is not actually a category. We will see later that

we can actually view a (bare) quiver as a functor, but still not a category itself. Ideally we could get a version which gives a full embedding into the category of algebraic varieties, possibly including quasi-projective varieties.

2. The non-uniqueness of representations of projective varieties begs for some sort of classification criterion. Something to ask may be: given a quiver and dimension vector, is there a canonical way to associate the associated Grassmannian projective variety with a representation on a Reineke (k, n)-quiver?

CHAPTER III

THE RANDOM QUIVER AND ITS REPRESENTATIONS

In this chapter we give another way of thinking of quivers geometrically. This time "geometrically" is in terms of certain categorical properties that arise frequently in geometry. We begin by describing a canonical way in which *any* category can be viewed topologically. Then we demonstrate the existence of a particularly nice quiver, the *random quiver*, and move on to consider some properties of its representation category. The reason for considering the random quiver is that it is capable of capturing *all finite structure* that can exist in digraphs. At each step we will investigate the relevance of the categorical properties in question.

Simplicial Sets and Geometric Realization

The study of simplicial sets arose from combinatorial topology and the study of spaces that can be represented, up to homotopy, in terms of their simplices. A nice aspect of the simplicial set approach (depending on your viewpoint, of course!) is that the definition does not involve any topology. Similar structures, such as the broader notion of CW complex, carry a topology. A great deal of introductory-level information on simplicial sets can be found in [17]. We begin with a purely combinatorial definition of simplicial sets, but will quickly demonstrate a slick-albeit-opaque category-theoretic definition. This secondary definition allows us to conveniently define **sSet**, the category of simplicial sets. The reason for beginning this chapter with this section is to demonstrate a minimal way in which quivers and their representation categories may be related to homotopy.

Definition III.1. A *simplicial set* is a set $X = \{X_0, X_1, X_2, \dots\}$. The elements of X_n are called n-*simplices*. Additionally, for each n, functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for all $0 \leq i \leq n$ satisfying:

(i) $d_i d_j = d_{j-1} d_i$ whenever $i < j$,

(ii) $s_i s_j = s_{j+1} s_i$ whenever $i \leq j$, and

(iii) $d_i s_j = \begin{cases} \mathrm{id}, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1. \end{cases}$

The maps d_i are called *face maps* and the maps s_i are called *degeneracy maps*.

This definition will, undoubtedly, take some time to parse. Even if one understands all the *properties* inside and out, the use and interpretation are not at all obvious. After giving our alternative definition, we will describe ways to think about n-simplices, face maps, and degeneracy maps.

Before giving the category-theoretic definition of simplicial sets, we will need some notation. For each $n \in \mathbb{N}$ we will denote the $(n+1)$-element linearly-ordered set by $[n] = \{0, 1, \dots, n\}$. The fact that for us $[n]$ has $n+1$ elements (and in fact *is* the ordinal $n+1$) may be uncomfortable, but it is in keeping with conventions from topology. We denote the category of all non-empty finite ordinals, together with order-preserving morphisms, by $\mathbf{\Delta}$. This notation is sometimes used in other contexts to denote the category of *all* finite ordinals with order-preserving maps (alternatively denoted **FinOrd**).

Definition III.2. A *simplicial set* X is a contravariant functor $X : \mathbf{\Delta} \to \mathbf{Set}$. We denote $X[n]$ by X_n, and its elements are called n-*simplices*. The category of simplicial sets is denoted **sSet**.

Clearly **sSet** is just another name for the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$, so a map between simplicial sets $\varphi : X \to Y$ is a natural transformation. That is, φ consists of maps of n-simplices $X_n \to Y_n$. These maps also commute with the face and degeneracy maps addressed above. This definition is extremely slick, but not particularly informative to anyone not extremely familiar with categorical thinking. This definition encapsulates all of the information of our first definition by way of the structure inherent to categories and the fact that the order-preserving maps of $\mathbf{\Delta}$ have a generating set of morphisms. Note the duality in the domains of the following maps, with respect to Definition III.1.

Proposition III.3. *Every morphism in* $\mathbf{\Delta}$ *can be written as a composition of maps of the form*

$$d_i : [n-1] \to [n], \qquad k \mapsto \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases}$$

$$s_i : [n+1] \to [n], \qquad k \mapsto \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases}$$

Moreover, these maps satisfy conditions dual to those of Definition III.1.

Proof. By dual conditions we mean the orders are reversed:

(i) $d_j d_i = d_i d_{j-1}$ whenever $i < j$,

(ii) $s_j s_i = s_i s_{j+1}$ whenever $i \leq j$, and

(iii) $s_jd_i = \begin{cases} \mathrm{id}, & i = j, j+1 \\ d_is_{j-1}, & i < j \\ d_{i-1}s_j, & i > j+1. \end{cases}$

Verification that d_i and s_i satisfy these is routine.

We prove the remainder of the theorem by induction on the length of the domain. Let $f : [0] \to [n]$ be an order-preserving map, let $k = f(0)$, and let $\iota : [k] \hookrightarrow [n]$ be the inclusion map. Clearly ι satisfies the conditions of the theorem. Now we have $f = \iota \circ d_0^k$.

Now let $f : [m] \to [n]$ be an order-preserving map, and suppose the theorem holds for any order-preserving $g : [m-1] \to [n]$. Then $f|_{[m-1]}$ may be written as a composition of s_i, d_i. Let $g : [m] \to [n]$ such that $g|_{[m-1]}$ satisfies the theorem and $f|_{[m-1]} = g|_{[m-1]}$. Now let $k = f(m)$, let $k' = g(m)$, and let $\iota : [k] \hookrightarrow [n]$ be the inclusion map. Now we either have $f = \iota \circ d_{k'}^{k-k'} \circ g$ (if $k \geq k'$) or $f = \iota \circ s_k^{k'-k} \circ g$ (if $k \leq k'$). $\square$

The maps above are called *coface* and *codegeneracy* maps, typically denoted d^i and s^i, respectively. The different choice of notation was made so as to not interfere with exponentiation in the proof. But why all the duality? Since simplicial sets were redefined as *contravariant* functors, the duality is "canceled out" when we move from $\mathbf{\Delta}^{\mathrm{op}}$ to **Set** to agree with Definition III.1.

In the previous proof we took a very small amount of care in ensuring domains and codomains agree, but there are still some subtleties to consider in order for this to make complete sense in a category; note that there is actually one d_i and one s_i for each possible domain. This is typically omitted to simplify notation, and the particular map should be clear from context. One possible resolution is to consider $\overline{\mathbf{\Delta}}$, with objects $\mathrm{Ob}(\boldsymbol{\Delta}) \cup \{\omega\}$ and order preserving maps. In this case, the combination of ω and the existence of restriction maps guarantees that everything can work out correctly. This level of care adds excessive pedantry to the proof.

Our reason for being interested in **sSet** is that its objects can be "geometrically realized" into a (compactly-generated Hausdorff) topological space. In order to make efficient use of geometric realization, it may be a good idea to reappraise our definition of quiver. We will make use of the following categorical formulation.

Definition III.4. First, define the category **free quiver**, denoted **Q**, as follows:

- $\mathrm{Ob}(\mathbf{Q}) = \{E, V\}$,
- $\hom(E, V) = \{s, t\}$,
- $\hom(V, E) = \emptyset$, and
- the only endomorphisms are the required identities.

That is, **Q** is simply

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$

Figure 5: The Free Quiver **Q**

Now we define **Quiv** to be the functor category $\mathbf{Set}^{\mathbf{Q}}$, so a **quiver** is a functor $Q : \mathbf{Q} \to \mathbf{Set}$.

This calls for a couple of remarks. First, notice the names of morphisms s and t coincide with the names of functions s and t from our original definition of quiver (Definition I.1). If we have a quiver $Q \in \mathrm{Ob}(\mathbf{Quiv}) = \mathrm{Ob}(\mathbf{Set}^{\mathbf{Q}})$, then for every $a \in QE$ (QE a set) we have $sa, ta \in QV$. That is, the particular functor Q takes each of V

and E to a set, and for each element of QE (the edges) we get a source and target in QV (the vertices).

The second fact to note is that if in our original definition of quiver morphism (Definition I.3) we omit the references to vector spaces and linear transformations and consider the definition only in the context of categories and commutative diagrams, then what we have is precisely what is required to make quiver morphisms natural transformations in **Quiv**.

Now before we can geometrically realize a quiver, we need to look on the other side of geometric realization: the notion of a "nerve." There are nerve functors into various kinds of categories, but we will be interested in the following:

Definition III.5. Let $i : \mathbf{\Delta} \to$ **Quiv**. The **nerve functor** N_i : **Quiv** $\to$ **sSet** is given by defining the following simplicial set as a composition:

$$\begin{array}{ccc} \mathbf{\Delta}^{\mathsf{op}} & \xrightarrow{\;i\;} & \mathbf{Quiv}^{\mathsf{op}} \\ & {\scriptstyle N_i(Q)} \searrow & \downarrow {\scriptstyle \hom_{\mathbf{Quiv}}(-,Q)} \\ & & \mathbf{Set} \end{array}$$

That is, we consider $\hom_{\mathbf{Quiv}}(-, Q) \circ i$ to get a simplicial set $N_i(Q) : \mathbf{\Delta}^{\mathsf{op}} \to$ **Set**. $N_i(Q)$ is called the **nerve** of Q with respect to i.

The idea of geometric realization of a simplicial set involves more topology than should be included here, but suffice it to say there exists a functor $|-|$: **sSet** $\to$ **Top** called geometric realization. Thus given a functor $i : \mathbf{\Delta} \to$ **Quiv** we may *geometrically realize* any quiver as a topological space by taking $|N_i(Q)|$. Moreover, we have an adjunction $|-| \dashv N_i$ where the left adjoint $|-|$ freely adds topological structure and the right adjoint N_i faithfully forgets structure. The *singular nerve*

Top $\to$ **sSet** is a special nerve functor which preserves higher homotopies on hom-sets, thus allowing one to "do homotopy theory" in **sSet** (this being their primary motivation for study).

Let us return to what it means to geometrically realize a quiver Q. Realization is done with respect to some functor $i : \mathbf{\Delta} \to$ **Quiv**. What choice of i would potentially be meaningful? Here it helps to change views and consider $i : \mathbf{\Delta} \to \mathbf{Q} \to$ **Set**, which we can rewrite yet again as $i : \mathbf{\Delta} \times \mathbf{Q} \to$ **Set**. Now recall the product category $\mathbf{\Delta} \times \mathbf{Q}$ consists of objects being ordered pairs $([n], X)$, where $X = E, V$, and morphisms ordered pairs applied component-wise. We end this section with two ideas for further study:

1. Considering the three (equivalent) formulations of i above, does one naturally admit a "meaningful" choice of definition of i so that quivers may be used to study homotopy, or vice-versa? The study of the meaning of the nerve under various choices of i could pose some interesting questions.

2. The nerve construction above can be done for any category, so another option relevant to our topics would be to study various properties of representation categories $\mathbf{Rep}_k(Q)$. With some additional thought, it may be clear how to define the functor i in a natural way corresponding to dimensions of the spaces in a representation. These are very preliminary ideas, but could lead to interesting questions.

We have considered homotopy theory in this section, but we mention briefly a relationship to homology in the conclusion. In the next section we expand our definition of quiver and representation to get a very general category which may admit even more ways of capturing algebraic and topological geometric ideas.

Fraïssé Limits and the Random Quiver

In the previous section we saw how to send every directed graph to a simplicial set, then on to a topological space. Ultimately we would like to use a single category of the form $\mathbf{Rep}_k(Q)$ to study as much geometry as possible. We will show that a great candidate for Q is a countable quiver[3] which contains every finite quiver as a subquiver. This quiver (unique up to isomorphism) is known as the *random quiver*. Note this is infinite, which is an exception to the idea that our quivers would be finite. We have defined *subrepresentation* of a quiver previously, but not subquiver.

Definition III.6. Let Q be a quiver. A *subquiver $Q' \subset Q$ of Q* is a pair (Q'_0, Q'_1) with $Q'_0 \subset Q_0$ and $Q'_1 \subset Q_1$ such that for every $a \in Q'_1$, we have $sa, ta \in Q'_0$.

In other words, we obtain a (proper) subquiver by deleting one or more of the vertices of Q, then throwing out all of the arrows which went to or from those vertices. We can then opt to delete additional arrows between the remaining vertices.

(6.1)
A quiver Q

(6.2)
A subquiver of Q

Figure 6: An example of a subquiver

In Figure 6, the subquiver was obtained, in part, by deleting x_2. This forced the deletion of all arrows going to or from x_2. Additionally, one of the arrows at top was deleted (just for fun!).

[3]At this point we are using quiver in the sense of our original definition: a directed graph. In this section we will be careful to indicate when we are talking about quiver representations.

In order to construct the random quiver, we need a result from model theory that guarantees the existence of what is know as a Fraïssé limit. The full story on this *Fraïssé construction* can be found in [7]. First, a preliminary definition which has a more general formulation.

Definition III.7. The *age* of a quiver Q is the class $\mathcal{A}(Q)$ of all finite quivers embeddable in Q.

At this point we have mentioned quiver isomorphisms and embeddings with the assumption that the meanings are intuitive based on experience with other topics. To be completely precise we will define quiver morphism and its special flavors. In order to avoid confusion with the morphisms of quivers defined previously (which were really morphisms of quiver *representations*) we will use the term morphism of bare quivers.

Definition III.8. Let Q, R be quivers. A map $f : Q_0 \amalg Q_1 \to R_0 \amalg R_1$ (or simply $Q \to R$) is a *morphism of bare quivers* if $f(x) \in R_0$ for each $x \in Q_0$ and $f(a) \in R_1$ for each $a \in Q_1$, such that $s_R f(a) = f(s_Q a)$ and $t_R f(a) = f(t_Q a)$. We say f is an *embedding* if f is injective and an isomorphism if it is a surjective embedding. If there is an isomorphism $Q \to R$, we write $Q \cong R$. In particular, if $f : Q \to R$ is an isomorphism we may write $f : Q \cong R$.

Note that we take the domain to be the union of the two underlying sets of Q so as to avoid such distinctions as “injective on vertices” and “injective on arrows.” The distinction may be valuable in certain settings, but will not be necessary here. It is harmless to assume that for each quiver, its underlying sets are disjoint. Treating these morphisms more like functors would give access to a wealth of category-theoretic terminology, but, again, that is overkill for our purposes.

Before precisely defining the random quiver, we need one more concept. This is the concept that makes the random quiver particularly special (unique).

Definition III.9. A quiver Q is *homogeneous* if any isomorphism between two finite subquivers of Q extends to an automorphism of Q.

We will have more to say about homogeneity when we investigate the random quiver.

Definition III.10. *The random quiver*, denoted Q^*, is the unique (up to isomorphism) homogeneous countable quiver such that every finite quiver embeds in Q^*.

How can we be guaranteed such a Q^* exists? This follows from a result of Roland Fraïssé, known as the Fraïssé construction. The result of a Fraïssé construction is sometimes called a Fraïssé limit, and we will see Q^* is one example of such an animal.

Theorem III.11 (Fraïssé)**.** *Suppose $\mathcal{A}$ is a non-empty class of finite quivers such that:*

- ***(i)*** *(Hereditary Property) if $Q \in \mathcal{A}$ and $R \subset Q$ is a subquiver, then there is $R' \in \mathcal{A}$ with $R \cong R'$,*
- ***(ii)*** *(Joint Embedding Property) if $Q, R \in \mathcal{A}$ then there is $S \in \mathcal{A}$ such that both Q and R embed in S, and*
- ***(iii)*** *(Amalgamation Property) if $Q, R, S \in \mathcal{A}$ such that there are embeddings $f_1 : Q \to R$ and $f_2 : Q \to S$, then there is $T \in \mathcal{A}$ such that there are embeddings $g_1 : R \to T$ and $g_2 : S \to T$ satisfying $g_1(f_1(Q)) = g_2(f_2(Q))$.*

In this case, there is a unique homogeneous countable quiver with age $\mathcal{A}$.

Proof. This is a special instance of a more general proof in [7]. □

We will prove that the class of all finite quivers satisfies the assumptions of the above theorem. This will establish, once and for all, that the random quiver Q^* exists. We will be able to use a subset of its representations in the category $\textbf{Rep}_k(Q^*)$ to describe topological structures.

Theorem III.12. *Let $\mathcal{A}$ be the class of all finite quivers. Then $\mathcal{A}$ satisfies the hereditary property, the joint embedding property, and the amalgamation property.*

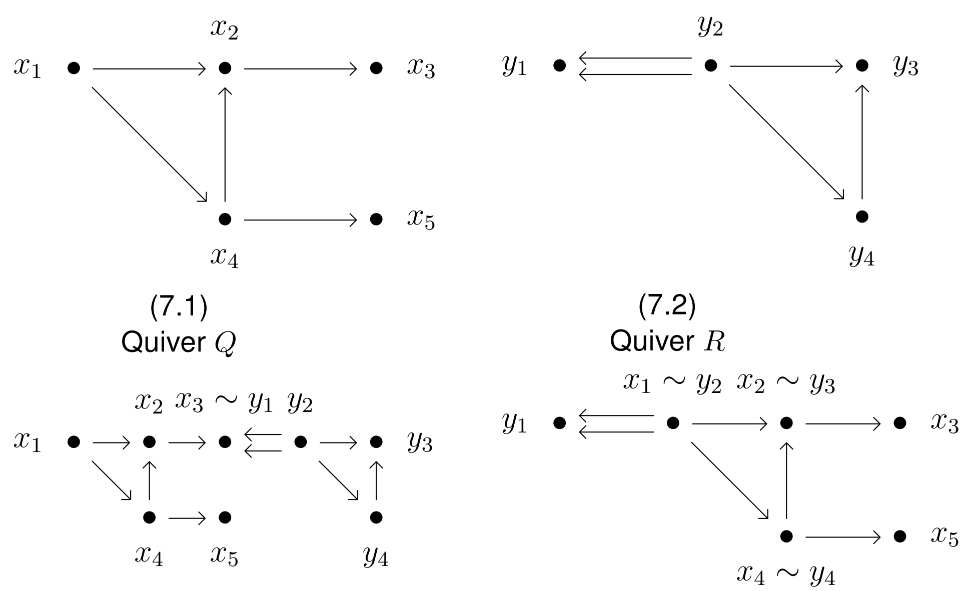
Proof. (Hereditary) Let $Q \in \mathcal{A}$ with $R \subset Q$ a subquiver. Since Q is finite, so is R. Since $\mathcal{A}$ contains all finite quivers, $R \in \mathcal{A}$. Trivially, $R \cong R$.

(Joint Embedding) There are several intuitive ways to show this. Let $Q, R \in \mathcal{A}$. Clearly there exists R' such that $Q_0 \cap R'_0 = \emptyset$ and $Q_1 \cap R'_1 = \emptyset$ with $R \cong R'$ (simply relabel the elements of R_0 and R_1 so they have nothing in common with elements of Q_0 and Q_1, and adjust the functions s_R and t_R accordingly). Now define $Q \amalg R' := (Q_0 \amalg R'_0, Q_1 \amalg R'_1)$. Observe $Q \amalg R'$ is finite so $Q \amalg R' \in \mathcal{A}$. Moreover, clearly Q and R embed in $Q \amalg R'$. (Intuitively, $Q \amalg R'$ looks like a copy of Q and a copy of R side-by-side with no arrows between them.)

(Amalgamation) Let $Q, R, S \in \mathcal{A}$ and suppose $f_1 : Q \to R$ and $f_2 : Q \to S$ are embeddings (R and S each contain a copy of Q as a subquiver). The goal is to construct a finite quiver T such that R and S are subquivers of T which overlap on their internal copies of Q. Begin by identifying the vertices and arrows $f_1(Q) = f_2(Q)$ and define $T_0^0 = \{f_1(R_0) \cup f_2(S_0)\}$. Now well-order the elements of $R_0 \setminus f_1(Q_0) = \{x_1, x_2, \ldots, x_m\}$ and the elements of $S_0 \setminus f_2(Q_0) = \{x_{m+1}, \ldots, x_n\}$. Choose $z_1 \notin T_0$ and define $T_0^1 = T_0^0 \cup \{z_1\}$ and in our embedding take $x_1 \mapsto z_1$. Inductively choose a point $z_i \notin T_0^{i-1}$ and take $T_0^i = T_0^{i-1} \cup \{z_i\}$, $1 \leq i \leq n$, with $x_i \mapsto z_i$. Define $T_0 = T_0^n$. Now that T_0 is defined, the process of adding arrows and extending the embedding

onto T_1 is done similarly. Finally, observe $|T_0| \leq |R_0| + |S_0|$ (resp. $|T_1| \leq |R_1| + |S_1|$). In particular, $T \in \mathcal{A}$. $\square$

Below we see examples of what the quivers guaranteed by joint embedding and amalgamation might look like. Since the "disjoint" embedding used in the proof above is not particularly aesthetically appealing, Figure 7 shows how two quivers can be jointly embedded by gluing them at a point. This is equivalent to an amalgamation where the common subquiver is simply taken to be a single point.



(7.1) Quiver Q

(7.2) Quiver R

(7.3) Admits a Joint Embedding

(7.4) An Amalgamation

Figure 7: Joint Embedding and Amalgamation Properties

Since the class of "all finite quivers" is not particularly restrictive, there are many trivial games that can be played to find different ways to satisfy the joint embedding and amalgamation properties. It is not worth explicitly expounding on any of these alternatives. It may, however, be worth silently considering how some of these constructions might work. For an amalgamation of two quivers, we simply need to choose a common subquiver and paste them together such that they agree on the specified subquiver. In the example of Figure 7, Q and R are made to agree on their triangular subquiver. Joint embeddings and amalgamations differ primarily in the fact that two quivers need not agree on any vertices or arrows of a joint embedding. In fact, a joint embedding object can be quite large with a large "distance" between the two original quivers. Even an amalgamation quiver can be much larger than necessary. When building an amalgamation, if the common subquiver is maximal, then the proof above constructs the smallest possible amalgamation. The fact that the class of finite quivers satisfies our three conditions is nearly trivial, but the proof of existence of the random quiver (the proof that Fraïssé limits exist) is where most of the work is done.

Unfortunately, proving many of the nice properties that follow from homogeneity requires quite a bit more model theory than can be included here. However, there are a few ways we may think about homogeneity. Perhaps the easiest example to visualize is the linear order $(\mathbb{Q}, \leq)$, which is the Fraïssé limit of the class of all finite linear orders. In this case we have a notion of "locality," and the homogeneity of $(\mathbb{Q}, \leq)$ can be rephrased loosely as saying that the order looks the same locally, no matter where you look (any local symmetry is in fact global). The idea of being local can be adapted to digraphs, but requires thinking about "in-neighbors" and "out-neighbors," and unlike $\mathbb{Q}$, two vertices of Q^* may not be related by edges at all.

Homogeneity can also be thought of as saying Q^* is "robust" against deletion of finitely many edges and vertices. For our purposes we are not interested in literally removing components, but rather in knowing that we can find any finite number of disjoint copies of some finite quiver. For example, we can find any finite number of disjoint Reineke (k, n)-quivers within Q^*.

Model Categories

Model categories (due to Quillen) are categories which allow one to "do homotopy theory." That is, they have distinguished classes of maps that behave like those of interest in homotopy theory. The category-theoretic terminology used below may be found in Appendix A.

Throughout the remainder of this chapter we must make yet another concession. In addition to dealing with an infinite quiver, we must allow the vertex representations to be infinite-dimensional k-vector spaces. The reason for this is that **Vect** is bicomplete, while **FinVect** is not. It will be apparent where this matters.

Definition III.13. A **Model Category C** is one in which

(i) **C** is complete and cocomplete (bicomplete);

(ii) there are three distinguished classes of morphisms: **weak equivalences**, **fibrations**, and **cofibrations**. These need not be disjoint. In particular, a (co)fibration which is also a weak equivalence is called an **acyclic (co)fibration**; and

(iii) all of the above obey the following axioms:

  a. (two-out-of-three condition) Given composable morphisms f, g, if any two of f, g, or gf is a weak-equivalence, then so is the third, and

  b. The pair of cofibrations and acyclic fibrations is a weak factorization system for **C** (resp. fibrations and acyclic cofibrations).

The goal of this section is to show that $\mathbf{Rep}_k(Q^*)$ is a model category. The main challenge is in deciding how to select the weak equivalences and (co)fibrations. We will first show **Rep**$_k(Q^*)$ is complete and cocomplete. The following, which may be found in [1], gives a nice condition for completeness.

Theorem III.14 (Existence of Limits Theorem)**.** *In a category **C**, the following are equivalent:*

(i) ***C*** *is complete.*

(ii) ***C*** *has equalizers and products.*

Lemma III.15. *The category **Rep**$_k(Q^*)$ is complete.*

Proof. We will show the existence of equalizers first. Let $\Phi, \Psi : V \to W$ be morphisms of representations of the random quiver. We will need a representation E, so to start, for each $x \in Q_0^*$ let $\dim E_x = \sup\{\operatorname{rank} X^a \mid sa = x \text{ or } ta = x, X = V, W\}$. Note this supremum exists because the set of arrows is countable and $\dim V_a$ and $\dim W_a$ provide bounds. The idea here is we are ensuring the universal property of equalizers holds by providing enough room to account for all arrows, but no more. Similarly, for every $a \in Q_0^*$, let E^a be the trivial map. This ensures the universal property by forcing the uniqueness of quiver morphisms into E: each component must be trivial[4].

For each $x \in Q_0^*$, let $\{b_\alpha \mid \alpha \in \kappa\}$ be a basis for E_x and treat V_x with its standard basis $\{\epsilon_\alpha \mid \alpha \in \kappa_x\}$. Now for each x define $e_x(b_\alpha) = \epsilon_\alpha$ if $\varphi_x(\epsilon_\alpha) = \psi_x(\epsilon_\alpha)$. Otherwise, define $e_x(b_\alpha) = 0$. It is easy to see E with the collection $\{e_x \mid x \in Q_0^*\}$ equalizes Φ, Ψ and is universal among such objects.

Now to see we have products, let $\{V_i\}$ be a (possibly infinite) family of representations of Q^*. Define P_x to be the product $\prod_i V_{i,x}$ in **Vect**. For any arrow $a \in Q_0^*$,

[4]This is due to homogeneity of the random quiver: every vertex must be the source of some arrow and the target of some arrow.

define P^a component-wise on the spaces V_{sa}. Now clearly P is a product of $\{V_i\}$. By the existence of limits theorem, **Rep**$_k(Q^*)$ is complete. □

As should be expected, a category **C** being cocomplete is equivalent to the existence of coequalizers and coproducts.

Lemma III.16. *The category **Rep**$_k(Q^*)$ is cocomplete.*

Proof. This mimics the proof of completeness almost exactly. Let $\Phi, \Psi : W \to V$ be morphisms of quivers. Instead of using rank to determine $\dim E_x$, we use $\dim E_x = \dim V_x/(\operatorname{Im}\varphi_x \cap \operatorname{Im}\psi_x)$ and instead of using trivial maps, we use maps of maximal rank (since, up to bases, rank is the only distinguishing feature of linear maps).

Similarly, the direct sum of representations discussed in the introduction gives the coproduct under the condition that all but finitely many components are nonzero. □

Since **Rep**$_k(Q^*)$ is bicomplete, we are on our way to having a model category. The trick now is to cleverly selected the classes of quiver morphisms which will satisfy the axioms of the definition of model category. We close this section with remarks on future work.

1. As just stated, the first goal is to find *some* collections of quiver morphisms satisfying the axioms two-of-three axiom and the weak factorization system requirement.

2. Equipped with a model structure, **Rep**$_k(Q^*)$ could be used to study homotopy theory. The next step would be to determine if this sheds light on old ideas, or opens doors to new ones.

Homotopy is currently of importance not just geometrically (topologically), but also foundationally in the the study of Homotopy Type Theory. Model categories were used somewhat recently by Steve Awodey and Michael Warren to construct a model of Intensional Martin-Löf Type Theory. See [15] for more on the topic.

Abelian Categories

In this section we introduce the notion of an abelian category. Again, we will neglect to mention the specific applications of abelian categories, but will just note that they are of interest in algebraic geometry and homological algebra. They are so-named because they capture some important properties of the category **Ab** of abelian groups. An excellent introduction to the pure theory is [6].

Definition III.17. Let **A** be a category. We say **A** is **abelian** if

(i) **A** has a zero object;

(ii) **A** has finite products and coproducts;

(iii) every morphism has both a kernel and a cokernel; and

(iv) every monomorphism is a kernel and every epimorphism is a cokernel.

The definition of any unfamiliar category-theoretic term may be found in Appendix A. The goal of this section is simply the following:

Theorem III.18. *The category* $\mathbf{Rep}_k(Q^*)$ *is an abelian category.*

In fact, we can drop the condition from the last section that we allow infinite-dimensional vector spaces. It is easy to see the category $\mathbf{Rep}_k(Q^*)$ has finite products and coproducts even when restricted to finite-dimensional vector spaces in the representations.

Proof. We have already noted (ii) holds in $\mathbf{Rep}_k(Q^*)$ and it is easy to see the trivial representation is a zero object. Denote the trivial representation **0**. To prove (iii), let $\Phi : V \to W$ be a morphism of quivers. Since kernels exist in **FinVect**, each component φ_x has a kernel $\ker \varphi_x$ in **FinVect**. To define $K = \ker \Phi$, take $K_x = \ker \varphi_x$ for each $x \in Q_0^*$. To ensure universality, let K^a be the trivial map for all $a \in Q_1^*$. Let $\Psi : K \to V$ be a map such that each component is a map $\psi_x : \ker \varphi_x \to V_x$. Now we have

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{0} \\ {\scriptstyle \Psi}\downarrow & & \downarrow \\ V & \underset{\Phi}{\longrightarrow} & W \end{array}$$

commuting, so $K = \ker \Phi$. The existence of cokernels is similar.

It remains to be shown that every monomorphism (resp. epimorphism) is a kernel (resp. cokernel), so let $\Phi : V \to W$ be a monomorphism. We defined kernels as objects, so we mean V is a kernel and Φ is the map k asserted to exist in Definition A.14. Clearly

$$\begin{array}{ccc} V & \longrightarrow & \mathbf{0} \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ W & \longrightarrow & \mathbf{0} \end{array}$$

commutes. Universality follows from the fact that every monomorphism must be an isomorphism since each component is an isomorphism in **FinVect**. We must also assume we are free to identify every space with affine space and free to choose our bases. The proof for cokernels is painfully similar. □

There are a number of important theorems relating abelian categories to other category-theoretic ideas. One of the most famous is the following.

Theorem III.19 (Freyd-Mitchell Embedding Theorem)**.** *Every abelian category is a full subcategory of* R-***mod*** *for some ring* R*.*

Having an abelian category opens up a wealth of possibilities, as the topic has been well-studied.

CHAPTER IV

QUIVER VARIETIES AND GEOMETRIC REPRESENTATION THEORY

In this chapter we discuss the development of what are known as quiver varieties, the original flavor being Nakajima quiver varieties. Quiver varieties have led to the development of what is known as geometric representation theory. We will not discuss their applications, but one highlight is their role in the representation theory of finite-dimensional algebras.

Rather than build up to Nakajima quiver varieties–their construction is in fact far too involved for this paper–we will illustrate the construction of *Lagrangian subvarieties*. This construction is considerably shorter, though still decidedly involved. Portions do overlap so an interested reader would already be well on their way to consulting a paper on Nakajima quiver varieties. Other sorts of quiver varieties do exist (e.g. Lusztig and Demazure), but we will use Demazure quiver varieties only as a stepping stone to our goal. The presentation to follow is an amalgamation of the presentations given in [18], [14], and [3].

Unfortunately, we must omit much of the motivation so as to keep our material focused and because of the lofty prerequisites for full appreciation of their applications and relationships to other structures. The highlight might be their role in the representation of finite-dimensional algebras. When one hears the phrase “geometry of quivers,” this is probably the topic that comes to mind.

Orientations and Path Algebras

To start we must establish our playground, as well as some terminology and notation. Throughout we will assume, without loss of generality, that the set of vertices is $\{1, 2, \cdots, n\}$ for some n. Additionally, we will limit our attention to finite quivers without loops. That is, if we have a quiver $Q = (Q_0, Q_1)$, then for all $a \in Q_1$ we have $sa \neq ta$. It turns out that for each arrow in Q_1, we would like to be an able to talk about an opposite arrow.

Definition IV.1. For any arrow $a \in Q_1$, denote the **opposite arrow** by $\bar{a}$, where $sa = t\bar{a}$ and $ta = s\bar{a}$. Define the set $\overline{Q_1} := Q_1 \amalg \{\bar{a} \mid a \in Q_1\}$. Now we define the **double quiver** by $\overline{Q} := (Q_0, \overline{Q_1})$.

We insist that the map $\bar{\cdot}$ be an involution in the double quiver, meaning $\bar{\bar{a}} = a$ for all $a \in \overline{Q_1}$.

Double quivers will be the objects of interest. From now on, when we will simply say "let Q be a double quiver" instead of starting with a quiver Q then constructing its double. Moreover, we will simply use the notation Q_1 rather than $\overline{Q_1}$, with the understanding that every arrow has an opposite. When we want to indicate we are concerned only with the starting arrows–"half" of Q_1–we will use the notation Q_1'. The defining property of having an involution on all arrows makes the following definition possible.

Definition IV.2. Let Q be a double quiver and $\Omega \subset Q_1$. If $\Omega \cup \overline{\Omega} = Q_1$ and $\Omega \cap \overline{\Omega} = \emptyset$, then Ω is called an **orientation** of Q.

We break momentarily from the discussion of oriented and double quivers, as the discussion of the next few ideas does not require those notions.

Definition IV.3. Let Q be a quiver. A **path** in Q is a finite sequence of arrows $P = a_n a_{n-1} \cdots a_1$ such that $ta_i = sa_{i+1}$ for $1 \leq i < n$. The number n is called the **length** of the path P. We also have an empty path at x for each $x \in Q_0$, denoted e_x. Given a path P, we define $sP := sa_1$ and $tP := ta_n$.

Now we are prepared to define the central notion of this section.

Definition IV.4. Given a quiver Q, the **path algebra**, denoted $\mathbb{C}Q$, is the $\mathbb{C}$-algebra with underlying vector space having as a basis all paths in Q. The bilinear multiplication operation on vectors (paths) is defined by concatenation: $P = a_n a_{n-1} \cdots a_1$ and $P' = b_m b_{m-1} \cdots b_1$ means $PP' = a_n \cdots a_1 b_m \cdots b_1$ if $sa_1 = tb_m$, and $PP' = 0$ otherwise.

It is easy to see this multiplication is associative. One very important concept in the sequel is that of a grading on an algebra.

Definition IV.5. A ring R is a **graded ring** if R decomposes as $R = \bigoplus_{n \in \mathbb{N}} R_n$ such that $R_i R_j \subset R_{i+j}$. A **graded (left) module** A over a graded ring R is one that decomposes as $A = \bigoplus_{n \in \mathbb{N}} A_n$ such that $R_i A_j \subset A_{i+j}$. An algebra A is graded if it is graded as a ring.

We may associate a grading with $\mathbb{C}Q$ quite easily. Let $\mathrm{span}(\mathbb{C}Q)_n$ denote the space spanned by paths of length n. Then we have $\mathbb{C}Q = \bigoplus_{n \in \mathbb{N}} \mathrm{span}(\mathbb{C}Q)_n$.

Lusztig and Lagrangian Nakajima Quiver Varieties

We now develop some of the more specialized structures that lead up to quiver varieties. The first concept is that of the preprojective algebra on a quiver.

Definition IV.6. Let Q be a double quiver. The **preprojective algebra** of Q is

$$\mathcal{P}(Q) = \mathbb{C}Q / \left(\sum_{a \in Q_1'} a\bar{a} - \bar{a}a \right)$$

When the quiver is clear from context we may simply write $\mathcal{P}$.

It is a simple exercise to see that $\mathcal{P}(Q)$ inherits the grading from $\mathbb{C}Q$. A bit of notation: by P_i we mean the ith component of the grading of $\mathcal{P}$. Most important will be $\mathcal{P}_0$. Conveniently, we can think of $\mathcal{P}_0$-**mod** as the category of finite-dimensional Q_0-graded (recall $Q_0 \subset \mathbb{N}$) vector spaces with morphisms linear maps which also preserve grading. Also note, then, that elements of $\mathcal{P}_0$-**mod** fall into isomorphism classes determined by Q_0-graded dimension.

Definition IV.7. For every Q_0-graded $\mathcal{P}_0$-module V, define the Q_0-grading $\dim V := \sum i \cdot \dim V_i \in \mathbb{N}Q_0$.

Given Q_0-graded $\mathcal{P}_0$-modules V and W, we associate with $\hom_{\mathcal{P}_0}(V, W)$ the sum $\bigoplus_{i \in Q_0} \hom_{\mathbb{C}}(V_i, W_i)$. As usual, $\operatorname{End}_{\mathcal{P}_0} V = \hom_{\mathcal{P}_0}(V, V)$ and $\operatorname{Aut}_{\mathcal{P}_0} V = (\operatorname{End}_{\mathcal{P}_0} V)^{\times}$.

Definition IV.8. Let A be a graded algebra and M an A-module. Then M is **nilpotent** if there exists $n \in \mathbb{N}$ such that $A_n \cdot M = 0$.

Nilpotent modules come together with the following definition and observations to define Lusztig quiver varieties.

Definition IV.9. Let Q be a double quiver. For every $V \in \mathcal{P}_0$-**mod**, define

$$\operatorname{End}_Q V := \bigoplus_{a \in Q_1} \hom_{\mathbb{C}}(V_{sa}, V_{ta}).$$

Let $f \in \operatorname{End}_Q V$. For a path $P = a_n a_{n-1} \cdots a_1$ define $f(P) := f(a_n) \cdots f(a_1)$ Now for any $\sum c_i P_i \in \mathbb{C}Q$, the path algebra, define

$$f\left(\sum c_i P_i\right) := \sum c_i f(P_i).$$

Now notice with this definition for every $f \in \operatorname{End}_Q V$, we get a representation $\mathbb{C}Q \to \operatorname{End}_{\mathbb{C}} V$ where the induced representation of $\operatorname{span}(\mathbb{C}Q)_0$ is completely determined up to isomorphism by $\dim V$.

Definition IV.10. We say $f \in \operatorname{End}_Q V$ is **nilpotent** if there exists a natural $n > 0$ such that for every path $P \in \mathbb{C}Q$ of length $k > n$, $f(P) = 0$.

We now have everything we need to make one of our most important definitions of the chapter. Recall the meaning of the notation Q_1'.

Definition IV.11. For any $V \in \mathcal{P}_0$-**mod**, we get a **Lusztig quiver variety** defined by

$$\Lambda(V) := \left\{ \text{nilpotent } f \in \operatorname{End}_Q V \mid \text{for all } i \in Q_0, \sum_{a \in Q_1', ta = i} f(a) f(\bar{a}) - \sum_{a \in Q_1', sa = i} f(\bar{a}) f(a) = 0 \right\}$$

We can view $\Lambda(V)$ as the set of all nilpotent $\mathcal{P}$-modules with graded dimension $\dim V$ compatible with the given $\mathcal{P}_0$-module structure of V; just as before, we still have a representation $\mathcal{P} \to \operatorname{End}_{\mathbb{C}} V$ for each such f.

Now we would like to define $\Lambda(V, W)$ for two $V, W \in \mathcal{P}_0$-**mod**, so we will take $\Lambda(V, W) = \Lambda(V) \times \hom_{\mathcal{P}_0}(V, W)$. So an element $(f, t) \in \Lambda(V, W)$ is a nilpotent $\mathcal{P}$-module over the $\mathcal{P}_0$-module V, paired with a $\mathcal{P}_0$-module homomorphism $V \to W$.

The following definition rounds out the data we need to define a Langrangian Nakajima quiver variety.

Definition IV.12. Let $V, W \in \mathcal{P}_0$-**mod**. A pair $(f, t) \in \Lambda(V, W)$ is **stable** if for every

f-invariant $\mathcal{P}_0$-submodule of V, call it A, we have $A \subset \ker t$ implies A is trivial. Note this is equivalent to saying that for every $i \in Q_0$, $\ker\big((f|_{V_i}, t|_{V_i})\big) = 0$. The subset of stable elements of $\Lambda(V, W)$ is denoted $\Lambda(V, W)^{\text{stab}}$.

Now the fruits of this long labor.

Definition IV.13. For two $\mathcal{P}_0$-modules V and W, define

$$\mathfrak{L}(V, W) := \Lambda(V, W)^{\text{stab}} / \operatorname{Aut}_{\mathcal{P}_0} V$$

and call $\mathfrak{L}(V, W)$ a **Langrangian Nakajima quiver variety**.

We have remarked a few times on constructs being isomorphic up to Q_0-graded dimension, and in this case $\mathfrak{L}(V, W)$ is determined up to isomorphism by the Q_0-graded dimensions of V and W. This means it may be denoted $\mathfrak{L}(m, n)$ for some $m, n \in \mathbb{N}$.

Geometric Representation Theory

In the last section we built up to the definition of a certain kind of subvarieties of Nakajima quiver varieties. In this section we will talk about Nakajima quiver varieties in general without defining them. We will still be referring to pairs of $\mathcal{P}_0$-modules, and there is not much harm in using the previous section as a guide to thinking about this section. Throughout the last section, the motivation (not to mention the connection to anything geometric) was obscured. In this section we state informally two ways in which Nakajima quiver varieties have been used. The first is a mildly obscure example.

Example IV.14 (Hypertoric Varieties)**.** A *toric variety* is an algebraic variety containing an algebraic torus[5] as an open, dense subset, and such that the action of the torus on itself extends to the entire variety. A *hypertoric variety* is a related structure defined by a certain quotient construction on a torus acting on a $\mathbb{H}$-vector space[6]. It turns out every hypertoric variety can be described by combinatorial data obtained from a Nakajima quiver variety for which the graded dimension $\dim V = 1$.

The next example is one of the primary ideas that gets associated with quiver varieties.

Example IV.15 (Kac-Moody Algebras)**.** A *Kac-Moody Algebra* is given by three pieces of data:

(i) An $n \times n$ generalized Cartan matrix[7] C of rank r;

(ii) a $(2n - r)$-dimensional $\mathbb{C}$-vector space $\mathfrak{h}$; and

(iii) a linearly-independent collection $\{\alpha_i^\vee\}_{i \leq n} \subset \mathfrak{h}$ and another linearly-independent collection $\{\alpha_i\}_{i \leq n} \subset \mathfrak{h}^*$, the dual space of linear functionals, such that $\alpha_i(\alpha_j^\vee) = c_{ji}$.

Generalized Cartan matrices are closely related to the study of Lie groups, and the third condition above is related to simple (co)roots in a semi-simple Lie Algebra. The connection of Nakajima quiver varieties to Cartan matrices is far more apparent following the definition built up in [14] or [13]. Lusztig and Nakajima have played prominent roles in applying quiver varieties to the representation theory of Kac-Moody algebras and quantum groups.

[5]We do not define this, but it is a type of affine abelian algebraic group. We consider such a group acting on itself.

[6]$\mathbb{H}$ denotes the quaternions, a hypercomplex division ring.

[7]Let C be a square matrix over the integers with 2's along the diagonal, non-positive entries elsewhere, such that $c_{ij} = 0$ if and only if $c_{ji} = 0$. Then C is a *generalized Cartan matrix* if $C = DS$ for some diagonal matrix D and symmetric matrix S.

CHAPTER V

CONCLUSION

We have seen several ways in which quivers relate to geometric structures, as well as some directions for future research. The fact that every projective variety is a quiver Grassmannian is likely little more than a novelty. The results on the category **Rep**$_k(Q^*)$ are very preliminary and broad, but offer some interesting prospects for future investigation. Geometric representation theory is the most widely-studied and well-known side of quivers. Below we discuss further ways in which quivers might be applied to geometry.

Future Directions

Homology. On the surface there are obvious similarities between quiver representations and R-module chain complexes if we view them visually, but it seems quivers provide far too much extra freedom for them to be directly applicable to the study of homological algebra. Nevertheless, there are at least some connections between quivers and homology. For instance, the preprojective algebra has connections to (co)homology. Also, consider the following novel example: let Q be a quiver and define a function f on prime-powers by $f(q)$ is the number of absolutely indecomposable representations of Q over $\mathbb{F}_q$. We discussed indecomposable representations in the introduction, but by *absolutely* indecomposable we mean one which remains indecomposable in the algebraic closure of $\mathbb{F}_q$. Now it turns out this function f is a polynomial, and its coefficients come from the ranks of certain homology groups.

A conjecture by Kac stated that the coefficients of such a polynomial were non-negative integers and included a combinatorial formula for the constant term. The complete conjecture was eventually proved, his condition on the coefficients being true because they turned out to be ranks of groups. Connections of (co)homology with quivers and related algebras could be an eventual line of research.

Differential Geometry. A very interesting preprint titled “Dynamics on Networks of Manifolds” (see [4]) demonstrates another way in which quivers can arise, and this viewpoint can provide more evidence for the relevance of quivers to geometry. For some control system (system of ODEs which depends on parameters), we associate a digraph which indicates interactions within the system. The paper considers digraphs satisfying a certain graph lifting property and associates with each vertex a manifold. They then take the categorical product of these manifolds to form a phase space for a system. An arrow between two vertices encodes some information on the dependencies of the state of one system on the state of another. The paper mentioned above goes into great detail, but the punchline is a theorem concerning when synchrony arises within such systems (and why). This is an entirely open line of research, and one place to start may be in investigating what happens when we combine this theory with the usual treatment of quivers: let our manifolds also be topological vector spaces, such as $\mathbb{R}^n$ or Hilbert manifolds (which have local vector space structure that may vary continuously over the manifold).

Group Law on a Plane Cubic. Perhaps one of the most novel prospect is the use of quiver representation categories to emulate the group law on a cubic. Below we outline a few ideas that may eventually lend themselves to this goal. This idea is a bit more fleshed-out than those listed above, so we present now some loose thoughts.

We know that given an elliptic curve $C \subset \mathbb{P}^2$, we may let Q be the Reineke $(1,3)$-quiver

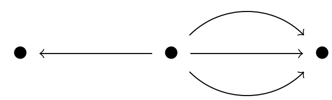


Figure 8: The Reineke $(1,3)$-Quiver

and let V be a representation of Q such that $\mathrm{Gr}_m(V) \cong C$ where $m = (0,1,1)$. This means we can handle any elliptic curve by looking at the Grassmannian functor applied to $\mathbf{Rep}_k(Q)$. One problem is determining how to represent a point of the curve. In other similar endeavors, the strategy is to find a distinguished object $*$ so that arrows $* \to A$ represent points of A. Early intuition also suggests that enriched categories may be very valuable. An enriched category **C** is one in which the $\hom$-sets are actually objects of some monoidal category M, where the monoid structure on M gives the composition rules in **C**. The usual definition of (locally small) categories is that of a category enriched over **Set**. Enriched categories give nice additional structure and interpretations to ordinary categories, and might be used to capture cubic group structure.

Appendices

APPENDIX A

THE LANGUAGE OF CATEGORIES

It is recommended that those completely unfamiliar with category theory refer to a standard text such as [11] or [1]. For convenience, we present here a collection of important definitions and results for those in need of a refresher. To begin at the beginning:

Definition A.1. A **category A** consists of a class of **objects**, denoted $\mathrm{Ob}(\mathbf{A})$, and for each $a, b \in \mathrm{Ob}(\mathbf{A})$ a set[8] $\hom(a, b)$, the elements of which are called **morphisms from** a **to** b. For $f \in \hom(a, b)$ we call a the **domain** of f, denoted $\mathrm{dom}(f)$, and we call b the **codomain** of f, denoted $\mathrm{cod}(f)$. Rather that saying $f \in \hom(a, b)$ we may say $f : a \to b$. All of this is subject to the following requirements:

(i) for all $a, b, c \in \mathrm{Ob}(\mathbf{A})$ and morphisms $f \in \hom(a, b)$ and $g \in \hom(b, c)$, there exists a morphism $g \circ f \in \hom(a, c)$, sometimes denoted gf, called the **composition of** f and g, and this composition is uniquely determined by f and g;

(ii) for any three morphisms f, g, and h of **C**, if $h \circ g$ and $g \circ f$ are defined, then we have $h \circ (g \circ f) = (h \circ g) \circ f$;

(iii) for each $x \in \mathrm{Ob}(\mathbf{A})$ there is $\mathrm{id}_x \in \hom(x, x)$, called the **identity morphism of** x, such that for every $f : a \to b$ we have $f \circ \mathrm{id}_a = \mathrm{id}_b \circ f = f$; and

[8] The condition that the collection of morphisms between two objects be a set means that what we call a category is more precisely a *locally-small* category.

(iv) the $\hom$-sets are pairwise disjoint.

The final condition is included to insist that every morphism has a *unique* domain and *unique* codomain. Often in practice we think of functions $f : X \to Y$ and $g : X \to f(X)$ (g given by $x \mapsto f(x)$) as being the same function. In a category these are distinct because domain and codomain are an essential piece of data for a morphism.

Given any category we may also consider its *opposite category*.

Definition A.2. Let **A** be a category. Then the **opposite category $\mathbf{A}^{\mathrm{op}}$** is defined by taking $\mathrm{Ob}(\mathbf{A}^{\mathrm{op}}) := \mathrm{Ob}(\mathbf{A})$ and for each $a, b \in \mathrm{Ob}(\mathbf{A})$, $\hom_{\mathbf{A}^{\mathrm{op}}}(a, b) := \hom_{\mathbf{A}}(b, a)$.

Since categories place an emphasis on the morphisms over the objects, once we define categories it becomes interesting to ask what sorts of morphisms we have between categories.

Definition A.3. Given two categories **A** and **B**, a **(covariant) functor** $F : \mathbf{A} \to \mathbf{B}$ is a map such that for $a \in \mathrm{Ob}(\mathbf{A})$ we have a unique object $F(a) \in \mathrm{Ob}(\mathbf{B})$ and for any morphism $f : a \to b$ of **A** we have $F(f) : F(a) \to F(b)$. Moreover, $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition A.4. Given two categories **A** and **B**, a **contravariant functor**[9] $F : \mathbf{A} \to \mathbf{B}$ is a map such that for $a \in \mathrm{Ob}(\mathbf{A})$ we have a unique object $F(a) \in \mathrm{Ob}(\mathbf{B})$ and for any morphism $f : a \to b$ of **A** we have $F(f) : F(b) \to F(a)$. Moreover, $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ and $F(g \circ f) = F(f) \circ F(g)$.

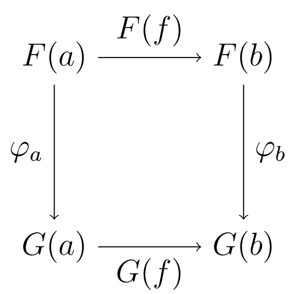
Note that a contravariant functor differs from a covariant functor in that it reverses morphisms. This is precisely what happens when we take the opposite of a category,

[9]When we say "functor" we will always mean a covariant functor. We will only specify the flavor if the functor is contravariant.

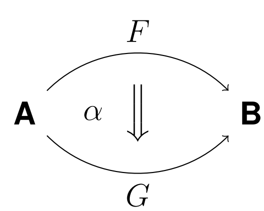
so we can note that a contravariant functor F : **A** $\to$ **B** is simply a covariant functor F' : **A**$^{\mathrm{op}}$ $\to$ **B**.

Now consider for two *sets* A and B the collection of all functions $A \to B$. We denote this set by B^A. We do something similar for functors.

Definition A.5. Let **A** and **B** be categories. The category with objects all functors **A** $\to$ **B**, denoted **B**$^{\mathbf{A}}$, is called a **functor category**. Of course, we must also specify the morphisms. Let F, G : **A** $\to$ **B**. A **natural transformation** $\alpha : F \Rightarrow G$ is a collection of morphisms $\varphi_x : F(x) \to G(x)$, one for each object x of **A**, such that for any $f : a \to b$ in **A** the diagram



commutes. The morphisms φ_x are called the **component of** φ **at** x. Diagrammatically, we denote a natural transformation $\alpha : F \to G$ by



The idea of natural transformation motivates the notion of a *higher morphism*. For us this is particularly relevant in the following context.

Definition A.6. The category **Cat** has as objects all *small* categories (i.e. those with a set of objects rather than a proper class) and as morphisms all functors between them.

As defined above, **Cat** is a category as we have defined them previously, and these are also known as *1-categories* because we only have morphisms between objects (1-dimensional morphisms). We can take our first step into *higher* categories by throwing in natural transformations. This gives us a 2-category, natural transformations being 2-dimensional morphisms since they are morphisms between morphisms. This process can go on and on up to ∞-categories, and higher category theory is of major interest right now to category theorists. Higher categories also arise very naturally in homotopy theory, which has in turn contributed to a potential foundational revolution via homotopy type theory.

To return: there are several important properties functors may satisfy. We give some definitions now.

Definition A.7. Let F : **A** $\to$ **B** be a functor.

- F is **full** if it is surjective on $\hom$-sets. That is, $F|_{\hom(a,b)}$ is surjective for all $a, b \in$ **A**.
- F is **faithful** if it is injective on $\hom$-sets.
- If F is both full and faithful we say it is **fully faithful**.
- F is **essentially surjective** if for every $b \in$ **B** there exists $a \in$ **A** such that $F(a) \cong b$ (as objects of **B**).
- F is an **equivalence** if it is fully faithful and essentially surjective. Two categories are **equivalent** if there is an equivalence between them.

The notion of an isomorphism of categories can also be defined as a bijective functor, but this is a concept far stronger than is necessary. In fact, isomorphic categories rarely occur in practice. It is good enough to consider categories up to equivalence. See [1].

To here the definitions have been more-or-less organized in a logical progression. What follows is a collection of miscellaneous definitions found throughout this paper, arranged in no particular order.

Definition A.8. A category is **complete** if all limits exists, **cocomplete** if all colimits exist, and **bicomplete** if both all limits and all colimits exist.

Note we do not define (co)limits because they are not needed at all. Completeness depends on treating commutative diagrams as functors and is just slightly too involved to include as an aside. It is interesting to note that a product (defined below) is a special case of limits. Our use of (co)completeness is reduced to discussion of the following two definitions.

Definition A.9. Let $f, g : A \to B$ in some category. An **equalizer** is a pair (E, e) of an object E and arrow $e : E \to A$ such that $f \circ e = g \circ e$ and is universal among such pairs. That is, for every C and $h : C \to A$ with $f \circ h = g \circ h$, then there exists a unique $\bar{h}$ making

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ {\scriptstyle \exists ! \bar{h}} \uparrow & \nearrow {\scriptstyle h} & & & \\ C & & & & \end{array}$$

commute. The dual notion is that of a **coequalizer**.

Definition A.10. Given a collection $\{A_\alpha \mid \alpha \in J\}$ of objects in a category, an object P is a **product** of the A_α if there exists morphisms $\{\pi_\alpha : P \to A_\alpha \mid \alpha \in J\}$ and for every family of morphisms $\{b_\alpha : B \to A_\alpha \mid \alpha \in J\}$ for some object B, there exists a unique $b : B \to P$ so that

We need the following two-for-one definition before we can define weak factorization system:

Definition A.11. Let f, g be morphisms. We say f has the **left lifting property** with respect to g and g has the **right lifting property** with respect to f if for every commutative square

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[v]{} & d \end{array}$$

there is some morphism $h : b \to c$ (not necessarily unique) so that if inserted in the diagram above, both of the resulting triangles commute.

Definition A.12. Let **C** be a category. A pair of classes of morphisms (L, R) is a **weak factorization system** if

- **(i)** Every morphism f of **C** factors as $f = rl$ for some $l \in L$ and $r \in R$;
- **(ii)** L is the class of all morphisms of **C** which have the left-lifting property with respect to R; and
- **(iii)** R is the class of all morphisms of **C** which have the right-lifting property with respect to L.

The following definitions are of interest for Abelian categories.

Definition A.13. An object C is **initial** if for every object A there is a unique arrow $f : C \to A$. Dually, C is **terminal** if for every object A there is a unique arrow $f : A \to C$. We call C a **zero object** if it is both initial and terminal.

In the following definition we avoid using the notion of pullbacks by appealing to the fact that the categories we are interested in have zero objects. Alternatively, we could define them in terms of *zero morphisms*, but either way we avoid making an additional definition.

Definition A.14. Let **C** be a category with zero object **0** (or an initial object if the definition is to be couched in terms of a pullback). Given any morphism $f : A \to B$ in **C**, a **kernel** of f is an object $\ker f$ such that there is a morphism $k : \ker f \to A$ which makes

$$\begin{array}{ccc} \ker f & \longrightarrow & \mathbf{0} \\ {\scriptstyle k}\downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

commute, and is universal among such objects. The dual notion is that of a **cokernel**.

Note in the diagram above fk is the unique morphism $\ker f \to$ **0**.

APPENDIX B

BELMANS'S ALGORITHM

The following four pages present the exact algorithm due to Pieter Belmans, reproduced here with the author's permission. The program, written for SAGE, takes as input a homogeneous polynomial and returns the isomorphic quiver Grassmannian per Reineke's proof. The same code may also be found on its author's blog [2].

The program gives a nice way to verify manual examples, and perhaps make what is going on more intuitive. It is highly recommended that those interested in the construction compare Reineke's proof [16] with Belmans's algorithm.

The ideas driving the algorithm (and hence Reineke's proof) are evident in the example presented in Chapter II.

```
# determine the linear equation of a hypersurface under the d-uple embedding
def getLinearEquation(equation):
  d = equation.degree()
  monomials = getMonomials(equation.parent(), d)
  coefficients = [equation.monomial_coefficient(monomial) for monomial in monomials]
  ring = PolynomialRing(QQ, 'x', len(coefficients))

  return sum(c * x for c, x in zip(coefficients, ring.gens()))

# determine the maximum degree of a list of (homogeneous) equations
def getMaximalDegree(equations):
  return max([equation.degree() for equation in equations])

# get all monomials of a given degree in the polynomial ring
def getMonomials(ring, degree):
  # by sorting and reversing we get the same order as on the bourbaki code
  degrees = reversed(sorted(WeightedIntegerVectors(degree, [1] * ring.ngens())))
  return [prod(x^d for x, d in zip(ring.gens(), exponents)) for exponents in degrees]

def getQuadraticEquations(ring, n, d):
  monomialsMatrix = transpose(matrix(getMonomials(ring, d - 1))) * matrix(ring.gens())
```

```
  monomials = getMonomials(ring, d)
  variablesRing = PolynomialRing(QQ, 'x', binomial(n + d, d) + 1)
  getVariable = lambda i, j: variablesRing.gens()[monomials.index(monomialsMatrix[i, j])]
  variablesMatrix = matrix(variablesRing, monomialsMatrix.nrows(), monomialsMatrix.ncols(), lambda i, j: getVariable(i, j))
  # if one is interested in the quadratic equations
  # print variablesMatrix.minors(2)

  return variablesMatrix.columns()

# add factors of the first variable to make sure the degree of all the equations is the same
def sameDegreeHomogeneous(equations):
  d = getMaximalDegree(equations)
  result = []

  for (i, equation) in enumerate(equations):
      result.append(equation * equation.parent().gen(0)^(d - equation.degree()))

  return result


# determine all the information on the quiver Grassmannian associated to a variety
def printQuiverGrassmannian(variety):
```

```
space = variety.ambient_space()
ring = space.coordinate_ring()

# determining the numbers
n = space.ngens() - 1
d = getMaximalDegree(variety.defining_polynomials())
M = binomial(n + d, d) - 1

print "Considering the projective variety of dimension {} in PG({}, Q) defined by".format(variety.dimension(), n)
for equation in variety.defining_polynomials():
  print "\t{}".format(equation)
print "The same equations, all of the same degree (d = {})".format(d)
equations = sameDegreeHomogeneous(variety.defining_polynomials())
for equation in equations:
  print "\t{}".format(equation)
print "The dimension vector is (1, {}, {})".format(M + 1, binomial(n + d - 1, d - 1))
print "The {} morphism(s) defining the variety (i.e. the maps 2->1) are".format(len(equations))
for equation in variety.defining_polynomials():
  print "\t{}".format(getLinearEquation(equation))
print "The {} morphisms defining the d-uple embedding (i.e. the maps 2->3) are described by".format(n + 1)
for morphism in getQuadraticEquations(ring, n, d):
  print "\t{}".format(morphism)
```

```
P.<x,y,z> = ProjectiveSpace(2, QQ)
E = P.subscheme([x^3-y^2*z+z^3]) # elliptic curve example
printQuiverGrassmannian(E)
print ""
F = P.subscheme([x^3*y+y^3*z+z^3*x]) # Klein quartic
printQuiverGrassmannian(F)
print ""

P.<x,y,z,w> = ProjectiveSpace(3, QQ)
Q = P.subscheme([x^4+y^4+x*y*z*w+z^2*w^2-w^4]) # Fermat quartic
printQuiverGrassmannian(Q)
print ""

P = ProjectiveSpace(4, QQ, 'x')
C = P.subscheme([sum(P.coordinate_ring().gens()), sum([x^3 for x in P.coordinate_ring().gens()])]) # Clebsch surface
printQuiverGrassmannian(C)
```

Bibliography

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