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Constraints And Equivalence In Cvar Portfolio Optimization

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Constraints and Equivalence in CVaR Portfolio Optimization

by

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This thesis, submitted by Casey Rozowski in partial fulfillment of the requirements for the Degree of Master of Science from the University of North Dakota, has been read by the Faculty Advisory Committee under whom the work has been done, and is hereby approved.

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This thesis is being submitted by the appointment advisory committee as having met all of the requirements of the Graduate School at the University of North Dakota and is hereby approved.

Dean of the Graduate School

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Contents

List of Figures

List of Tables

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2 ABSTRACT

The implementation of constraints are standard practice for a portfolio manager. The effect, mathematical and practical, is a study still in development. This thesis offers a standardized approach to finding a potential explanation of these constraints. It considers this approach in both the MVO and CVaR settings both theoretically by providing a mathematical proof, and practically by simulating portfolio optimizations with both generated and actual market data. In the MVO setting an MVO with constraints is equivalent to an unconstrained MVO with perturbed covariance matrix. With regards to CVaR, the CVaR optimization with constraints is equivalent to an unconstrained CVaR optimization with a perturbed asset returns matrix. While this study clarifies how this process can be applied to other portfolio optimizations under constraint it leaves room for a deeper study of how the adjustment of particular individual constraints can effect portfolio optimization.

3 CHAPTERS

3.1 INTRODUCTION

3.1.1 Review of Literature

Economist Harry Markowitz introduced modern portfolio theory in 1952 (Markowitz, 1952). Modern portfolio theory involves weighting the assets of a portfolio in a way that maximizes return and minimizes portfolio variance. This practice is often referred to as Mean-Variance Optimization (MVO). The portfolio variance is calculated using the sample covariance matrix of the securities. Mean-variance optimization was the standard portfolio optimization method from its introduction until nearly 50 years later. More recently the limitations of the MVO framework have challenged its widespread use. The mathematical framework for MVO is based upon an assumption of a normal distribution of returns. In application this is a problem since returns tend toward a log-normal distribution. MVO also fails to take significant losses into consideration since it relies only on the first two moments of the return distribution. The sample covariance matrix, which is central to the MVO process, is estimated with significant error (Ledoit, 2003). Consequently, alternatives have emerged which attempt to remedy or replace MVO. Ledoit and Wolf introduced a method of shrinkage that will decrease the estimation error of the sample covariance matrix (Ledoit, 2003). Attempts were also made to complement the MVO with the use of Value-at-Risk in order to incorporate another moment.

Value-at-Risk (VaR) is defined at a given beta value (typically .90, .95, or, .99) as the portfolio loss associated with the given beta value. Including a VaR calculation with an MVO optimization gave managers a more robust understanding of the risk associated with a portfolio and allowed managers to compare differing portfolios in a new way. However, this has been short lived. Soon after the implementation of VaR as a risk measure the limitations of this measure were recognized. VaR has undesirable mathematical characteristics like subadditivity and convexity. In the case of subadditivity the VaR of a portfolio can be larger than the sum of the risks of its components. It is also only coherent when based on the assumption of a normal distribution of returns (Rockafellar, 1999). In practice, the use of VaR models has recently been shown to have masked a 2 billion dollar loss at JP Morgan (Whittall, 2012).

Meanwhile, Conditional Value-at-Risk (CVaR) improves upon these limitations by giving a more coherent representation of risk. CVaR is often referred to as expected shortfall or tail risk. While VaR only represents the loss at a particular beta value, CVaR is the expected loss beyond the beta value. This measure takes the shape of the tail of the loss distribution into consideration. CVaR also has better mathematical properties than VaR (Artzner, 1997). CVaR, unlike MVO and VaR is not dependent on the assumption of a normal distribution of returns (Acerbi, 2002). The use of CVaR does not preclude the use of VaR or portfolio variance as risk measures but simply provides a more coherent measure of risk.

Mean-CVaR optimization is performed separate from MVO and without the use of a sample covariance matrix, reducing estimation error concerns. This new portfolio optimization technique was first introduced by Rockafellar and Uryasev in 1999 (Rockafellar, 1999). By minimizing CVaR given a minimum expected return, the optimization necessarily produces a low VaR as well. Consequently Mean-CVaR optimization provides portfolio managers an opportunity to limit and evaluate numerous portfolio allocation risk metrics.

Portfolio theory has undergone major challenges and changes in the last decade. The recognition of the limitations of MVO and subsequently VaR has led to increased interest in the properties and utilization of CVaR in portfolio optimization. In fact, the Basel report signified the shift from the use of VaR to CVaR in May of this year.

> ”Moving from value-at-risk to expected shortfall a number of weaknesses have been identified with using value-at-risk (VaR) for determining regulatory capital requirements, including its inability to capture tail risk. For this reason, the Committee has considered alternative risk metrics, in particular expected shortfall (ES)....The Committee recognizes that moving to ES could entail certain operational challenges; nonetheless it believes that these are outweighed by the benefits of replacing VaR with a measure that better captures tail risk. Accordingly, the Committee is proposing the use of ES for the internal models-based approach and also intends to determine risk weights for the standardised approach using an ES methodology” (Haug, 2012).

However, the use of CVaR as a risk metric is not yet standard in the finance industry and there are properties of Mean-CVaR optimization that have yet to be investigated. This includes the effects of constraints on portfolio characteristics.

3.1.2 Hypothesis

A common occurrence in modern portfolio theory is the use of constraints. Theoretically, an unconstrained portfolio allocation method should yield superior performance results. In MVO, this is not true; constrained portfolios typically outperform their unconstrained counterparts by decreasing estimation error (Ledoit, 2003). Constraints are important because not all allocations are possible for all managers. There may be restrictions on short selling, maximum weight allocations, and beta values. The effects of constraints in the MVO setting was first investigated by Jagannathan and Ma (2002). Their research centered on understanding why imposing constraints improved portfolio performance in practice. They found that constraints effectively shrank the covariance matrix, reduced sampling error, and consequently improved performance. However, with the emergence of new portfolio optimization techniques a more thorough and universal understanding of the effect of constraints is needed. In this paper, I examine how constraints affect two types of portfolio optimization problems. First, I examine the use of constraints in a mean-variance optimization (MVO) setting, and then I move to the Conditional Value-at-Risk (CVaR) setting.

I will first generalize the results of Jagannathan and Ma in the MVO setting. I will provide a proof of how numerous constraints affect the sample covariance matrix. We will see that the no-shortselling constraint decreases covariance among individual assets. Meanwhile, maximum allocation constraints increases covariance among returns and maximum beta exposure leads to an increase in covariance which is proportional to the betas of the individual assets. Finally the minimum expected return condition negatively affects the covariance of assets whose expected returns exceed the minimum return while it positively affects the covariance of assets with a lower expected returns than the minimum return. Then I will provide numerical examples highlighting this effect. First by performing MVO using generated normally distributed market returns and secondly by preforming MVO using actual market returns.

Next I will extend this work to the CVaR setting. In the CVaR setting there is the possibility for numerous perturbations that will satisfy the Karush-Kuhn-Tucker (KKT) equivalence conditions. One particular perturbation is considered. From this result, the no-shortselling and maximum allocation constraints adjust returns of individual assets up and down by the Lagrange multiplier. Similarly to the MVO scenario, the minimum expected return and maximum beta exposure conditions affect the unconstrained problem by adjusting returns up and down proportionally to the Lagrange multipliers. Again, I will then perform CVaR portfolio optimization using generated normally distributed market returns. Lastly I will highlight the effects with actual market data.

3.2 METHOD

3.2.1 Mean-Variance Optimization

The goal in the MVO setting is to minimize the expected portfolio variance given a minimum expected return for the portfolio. A typical constrained MVO problem with expected portfolio variance $\omega^T \Sigma \omega$, and expected return α, is

$$
\begin{aligned}
\text{minimize: } & \omega^T \Sigma \omega \\
\text{subject to: } & w^T \mu \geq \alpha \\
& \sum_{i=1}^{n} \omega_i = 1 \\
& 0 \leq \omega_i \leq \bar{\omega}
\end{aligned}
$$

where ω represents the weights on the assets in the portfolio, Σ represents the covariance matrix, and μ represents the expected returns of the assets. Next, we show how to find an equivalent problem in the unconstrained case.

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Constrained MVO}\\[1em] \min\limits_{\omega\in\mathcal{W}} \quad \omega^T\Sigma\omega \\ \text{s.t. } \ 0\le\omega\le\bar{\omega}\end{array}} \arrow[d, "\text{J \& M}"'] & \boxed{\begin{array}{c}\text{Constrained CVaR}\\[1em] \min\limits_{\omega\in\mathcal{W}} \mathrm{CVaR}_\alpha(\omega) \\ \text{s.t. } \ 0\le\omega\le\bar{\omega}\end{array}} \\
\boxed{\begin{array}{c}\text{Unconstrained MVO}\\[1em] \min\limits_{\omega\in\mathcal{W}} \quad \omega^T\tilde{\Sigma}\omega\end{array}} & \boxed{\begin{array}{c}\text{Unconstrained CVaR}\\[1em] \min\limits_{\omega\in\mathcal{W}} \mathrm{CVaR}_\alpha(\omega)\end{array}}
\end{tikzcd}
$$

MVO Constrained-to-Unconstrained Jagannathan and Ma (2002) converted constrained MVO problems into unconstrained MVO problems via Lagrange multipliers (or, more precisely, KKT conditions) and recognized this as a problem with perturbed $\tilde{\Sigma}$.

$$
\begin{array}{ccc}
\text{Constrained} & & \text{Unconstrained} \\[1em]
\begin{aligned} \min \ & \omega^T\Sigma\omega \\ \text{s.t. } & \sum_{i=1}^{n}\omega_i = 1 \\ & 0\le\omega_i\le\bar{\omega} \end{aligned}
& \xRightarrow[]{\substack{\text{KKT conditions}\\ \text{(Lagrange multipliers)}}} &
\begin{aligned} \min \ & \omega^T\tilde{\Sigma}\omega \\ \text{s.t. } & \sum_{i=1}^{n}\omega_i = 1 \end{aligned}
\end{array}
$$

KKT first-order conditions

Constrained MVO	Unconstrained MVO
$\nabla_\omega \mathcal{L} = \Sigma\omega - \lambda_0 \mathbf{1} - \lambda + \delta = 0$	$\nabla_\omega \tilde{\mathcal{L}} = \tilde{\Sigma}\omega - \tilde{\lambda}_0 \mathbf{1} = 0$
$\omega^T \mathbf{1} = 1$	$\omega^T \mathbf{1} = 1$
$\lambda_i \omega_i = 0$	
$\delta_i(\omega_i - \bar{\omega}) = 0$	
$\lambda_i \geq 0, \delta_i \geq 0$	

In order to solve the unconstrained MVO using the solution of the constrained MVO we first match the problems by finding $\tilde{\Sigma}, \tilde{\lambda}$ in terms of $\Sigma, \lambda_0, \lambda, \delta$ that satisfy $\nabla_\omega \tilde{\mathcal{L}} = 0$. We then make sure that $\tilde{\Sigma}$ is a symmetric and positive semi-definite matrix.

$$\implies \quad \tilde{\Sigma} = \Sigma + (\delta \mathbf{1}^T + \mathbf{1}\delta^T) - (\lambda \mathbf{1}^T + \mathbf{1}\lambda^T) \tag{1}$$

Interpretation

The introduction of minimum/maximum-weight constraints effectively changes covariance between securities. Meanwhile, the no short-selling restriction decreases covariance of individual assets uniformly with all securities. Conversely, the maximum holding restriction increases covariance of individual assets uniformly with all securities.

$$\tilde{\Sigma}_{i,j} = \Sigma_{i,j} + \underbrace{(\delta_i + \delta_j)}_{\text{Max holding}} - \underbrace{(\lambda_i + \lambda_j)}_{\text{No short sells}} \tag{2}$$

Additional Constraints

Generalizing the approach of Jagannathan and Ma to other constraints I found that limiting maximum portfolio β increases covariance of individual assets uniformly with all securities. The increase is proportional to β_i of the securities. The minimum-expected-return condition negatively affects covariance of assets whose expected returns exceed the minimum return while it positively affects covariance of assets with lower returns than the minimum return.

$$\tilde{\Sigma} = \Sigma + (\boldsymbol{\delta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\delta}^T) - (\boldsymbol{\lambda}\mathbf{1}^T + \mathbf{1}\boldsymbol{\lambda}^T) + \eta_1(\boldsymbol{\beta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\beta}^T) - \eta_2((\boldsymbol{\mu} - \bar{\mu}\mathbf{1})\mathbf{1}^T + \mathbf{1}(\boldsymbol{\mu} - \bar{\mu}\mathbf{1})^T) \quad (3)$$

See Appendix for proof.

Application Using Generated Data Market data were generated using the one-factor model. Forty assets with 500 months of trailing data were generated. Each asset has normally distributed monthly market returns (r_m) using expected return ($E(r_m)$)=.01, and variance ($V(r_m)$) equal to the square root of 0.2337. The beta value (β_i) for each individual stock is generated normally with mean 1 and standard deviation 0.4. The error term $\epsilon_i \sim N(0, \sigma_i^2)$ was generated with a σ_i^2 for each i based on the imported VIX data. The one-factor model was then used to generate the month returns (r_i) for 40 securities via

$$r_i = \beta_i \cdot r_m + \epsilon_i \quad (4)$$

Calculate Estimated Expected Return and Covariance Matrix

In order to solve for the optimal allocation of the constrained problem we first calculated μ and Σ,

μ: Calculated as the monthly return for each security for the 500 month period.

$$\mu_i = \frac{1}{500}\sum_{k=1}^{500} r_i(k)$$

Σ: Calculated as the covariance matrix of the 500×10 matrix of monthly returns.

$$\Sigma_{i,j} = \frac{1}{500}(r_i(k) - \mu_i)(r_j(k) - \mu_j)$$

Then, using the barrier method implemented by the *quadprog* function in Matlab I calculated the optimal allocation. This solver also provided me with the Lagrange multipliers. I used these Lagrange multipliers to calculate $\tilde{\Sigma}$ according to equation (3). This allowed me to investigate the validity of the proof and the effects of the constraints more closely.

Application Using Market Data In the case of market data, data was provided to me by a Sr Portfolio Manager - Quantitative Equity at Whitebox Advisors. This included three sets of data each containing over 1100 assets with only 126 days of trailing data. The expected returns and covariance matrix was calculated in Matlab. Subsequently, the barrier method was once again implemented in Matlab to calculate the optimal allocation and provide the Lagrange multipliers. These multipliers were used to calculate $\tilde{\Sigma}$ according to equation (3). Once again I was able to investigate the affects of the Lagrange multipliers and how these affects compared to the case of generated data.

3.2.2 Mean-CVaR Optimization

CVaR optimization can be performed to reduce the risk of high losses. CVaR is closely related to value at risk (VaR). With respect to a specified probability level β, the β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α, whereas the β-CVaR is the conditional expectation of losses

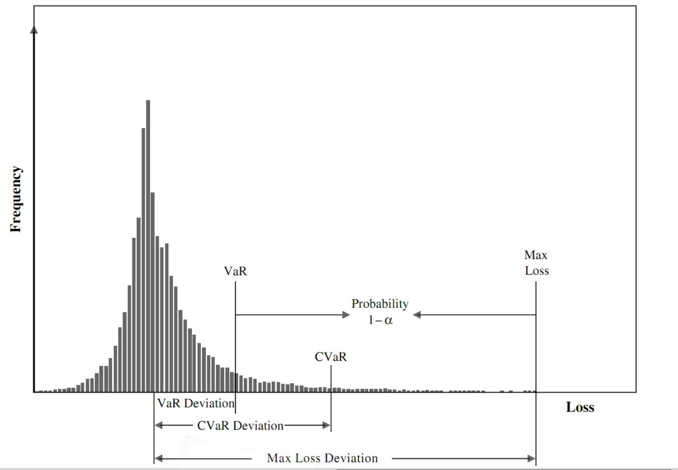


Figure 1: CVaR Example

above that amount α. Both VaR and CVaR are risk measures used to assess the probability of high losses, but we are only concerned with optimizing CVaR. VaR has some undesirable math characteristics, such as a lack of subadditivity. This is important when dealing with portfolio diversifications because if subadditivity fails, we would be better off by splitting our portfolio in order to decrease risks. CVaR does not lack subadditivity. VaR is only coherent when based on standard deviation of normal distributions, where CVaR does not rely on any specific distribution. VaR does not give any information on the degree of the losses, just the percent of extreme loss scenarios whereas CVaR gives the expected loss in those extreme loss scenarios. CVaR also provides a more direct measure of potential loss.

I will first establish a constrained CVaR problem. Rockafellar and Uryasev introduce a performance function and auxiliary variables to model the original problem as a linear programming problem. By discretizing, CVaR is minimized with samples generated from a distribution of scenarios $\mathbf{y}$. Let $f(\mathbf{x}, \mathbf{y})$ denote the loss function, where $\mathbf{x} = [x_1, ..., x_N]^T$ denotes a vector of weights (those of assets in the portfolio) and $\mathbf{y} = [y_1, ..., y_N]^T$ denotes a vector of returns of assets. Let p($\mathbf{y}$) describe the probability density function of $\mathbf{y}$. The probability of $f(\mathbf{x}, \mathbf{y})$ not exceeding a threshold is given by

$$\psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y}) d\mathbf{y}$$

Given a certainty level β, VaR and CVaR are defined as

$$\mathrm{VaR}_\beta = \mathrm{VaR}_\beta(\mathbf{x}) = \min\{\alpha \in \mathbb{R} | \psi(\mathbf{x}, \alpha) \geq \beta\}$$

and

$$\mathrm{CVaR}_\beta = \mathrm{CVaR}_\beta(\mathbf{x}) = (1-\beta)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \geq VaR_\beta(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y}$$

It can be proved that β-CVaR associated with any x can be achieved by minimizing the performance function with respect to α

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1-\beta)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \geq VaR_\beta(\mathbf{x})} [f(\mathbf{x}, \mathbf{y}) - \alpha]_+ p(\mathbf{y}) d\mathbf{y}$$

By discretizing, $\mathrm{F}_\beta(\mathbf{x}, \alpha)$ can be approximated with samples generated from the distribution of $\mathbf{y}$, i.e. $\mathbf{y}_k$, with k = 1, 2,..., q by

$$\hat{F}_\beta(\mathbf{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [f(\mathbf{x}, \mathbf{y}_k) - \alpha]_+$$

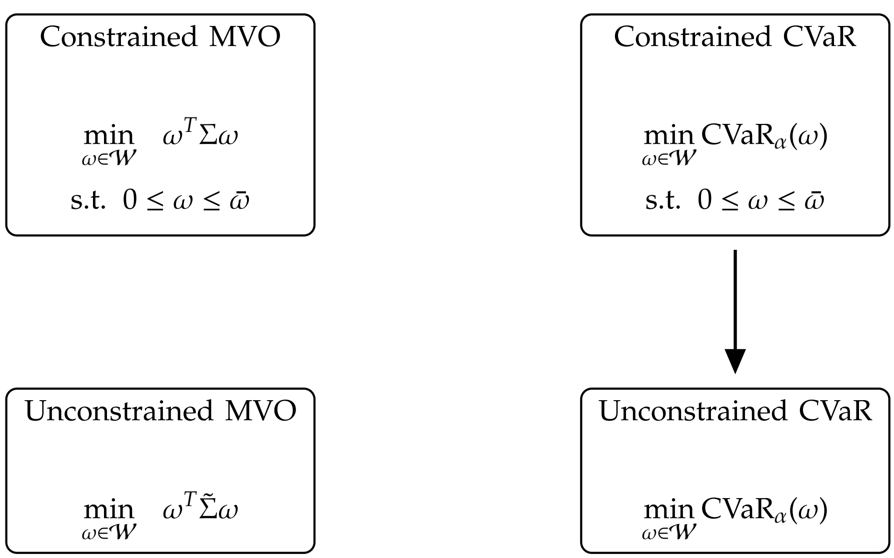
I use the loss function $f(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T\mathbf{y}$. With the introduction of auxiliary variables u_k, k = 1, 2,..., q, we may rewrite the objective function as

$$\hat{F}_\beta(\mathbf{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k$$

with constraints $u_k \geq 0$ and $\mathbf{x}^T\mathbf{y} + \alpha + u_k \geq 0$, k = 1,2,..., q. Therefore the original problem is converted into a linear programming problem and can be presented as

$$\begin{aligned}
\text{minimize: } \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k \\
\text{subject to: } \mathbf{x}^T\mathbf{y} + \alpha + u_k \geq 0 \\
u_k \geq 0 \\
\mathbf{x} \geq 0 \\
\mathbf{1}^T\mathbf{x} = 1 \\
-\mu^T\mathbf{x} \leq -R
\end{aligned}$$

Hence, I minimize the CVaR value given the above constraints. Just as I did in the MVO setting, I would like to find an equivalent portfolio expected return value.



CVaR Constrained-to-Unconstrained By solving a constrained Mean-CVaR optimization problem, extracting the Lagrange multipliers, and finding an appropriate perturbation I expect to find the same solution with the perturbed unconstrained Mean-CVaR optimization. This is done by using KKT conditions (Lagrange multipliers) and matching solutions of a perturbed unconstrained problem to the constrained problem in the following manner.

$$
\begin{array}{ccc}
\text{Constrained} & & \text{Unconstrained} \\
\begin{aligned}
&\min_{\mathbf{x}} \ \mathrm{CVaR}_\alpha(\mathbf{x}) \\
&\text{s.t.} \ \sum_{i=1}^{n} x_i = 1 \\
&\quad\ \ 0 \le x_i \le \bar{x} \\
&\quad\ \ \mathbf{x}^T \boldsymbol{\beta} \le \bar{\beta} \\
&\quad\ \ \mathbf{x}^T \boldsymbol{\mu} \ge \underline{\mu}
\end{aligned}
& \xrightarrow[\text{(Lagrange multipliers)}]{\text{KKT conditions}} &
\begin{aligned}
&\min_{\mathbf{x}} \ \widetilde{\mathrm{CVaR}}_\alpha(\mathbf{x}) \\
&\text{s.t.} \ \sum_{i=1}^{n} x_i = 1
\end{aligned}
\end{array}
$$

$$
\text{where} \quad \widetilde{\mathrm{CVaR}}_\alpha(\mathbf{x}) = \tfrac{1}{1-\alpha} \int_{\tilde{f}(\mathbf{x},\mathbf{y}) \ge \tilde{\mathrm{VaR}}_\alpha(\mathbf{x})} \tilde{f}(\mathbf{x},\mathbf{y}) p(\mathbf{y}) d\mathbf{y}
$$

Conversion to Unconstrained CVaR

Now it is necessary to equate the first-order KKT conditions at solution $\mathbf{x}^*$ of the contrained problem so that first-order KKT conditions are the same.

$$
\begin{aligned}
\nabla_{\mathbf{x}} \widetilde{\mathcal{L}}(\mathbf{x}^*) &= \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*) \\
\nabla_{\mathbf{x}} \widetilde{\text{CVaR}}_\alpha(\mathbf{x}^*) - \tilde{\eta}_0 \mathbf{1} &= \nabla_{\mathbf{x}} \text{CVaR}_\alpha(\mathbf{x}^*) - \eta_0 \mathbf{1} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1 \boldsymbol{\beta} - \eta_2 \boldsymbol{\mu}
\end{aligned}
$$

One possible perturbation is

$$
\tilde{f}(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T(\mathbf{y} + \boldsymbol{\lambda} - \boldsymbol{\delta} - \eta_1 \boldsymbol{\beta} + \eta_2 \boldsymbol{\mu}) \tag{5}
$$

See Appendix for proof.

Interpretation

From this result, the no-shortselling constraint and the maximum-allocation constraint affect the unconstrained problem by adjusting returns of individual assets up and down by the Lagrange multiplier. Similarly, the introduction of a minimum expected-return and maximum β-exposure conditions affect the unconstrained problem by adjusting the returns up and down proportionally to μ and β, respectively.

If one prefers, the full-allocation constraint can be absorbed into individual returns in the unconstrained problem. In general, any inequality constraints can be treated in the same way as I have in this thesis.

Application Using Generated Data Data for this CVaR application was generated via the same program that was used in the corresponding MVO application.

Then, using the barrier method implemented by the *linprog* function in Matlab

I calculated the optimal allocation. Similar to *quadprog*, *linprog* also supplied me with the Lagrange multipliers. I used these multipliers to shift the matrix of returns according to equation (5). The validity of the proof and the effects of the constraints were then considered.

Application Using Market Data This CVaR application utilitized the same sets of market data that were used in the corresponding MVO application.

The same method of implementation was used as in the generated data CVaR application. Again, the proof and constraint effects were considered. Both simulations will be performed with a β of .95.

3.3 RESULTS

3.3.1 MVO

The minimum expected return was increased and decrease over an interval of 0.1 in increments of 0.00005. By letting my minimum expected return fluctuate in both the original and perturbed MVO I was able to graphically compare the estimated return and estimated variance of both constrained MVO and particularly perturbed unconstrained MVO using $\tilde{\Sigma}$. This same approach was taken in order to consider how the constrained problem and the perturbed unconstrained problem compared over different minimum expected returns. Lastly, the effect of the constraints was highlighted by comparing the perturbed unconstrained MVO to the traditional unconstrained MVO. Consider the following diagrams comparing these effects using both generated market data and actual market data.

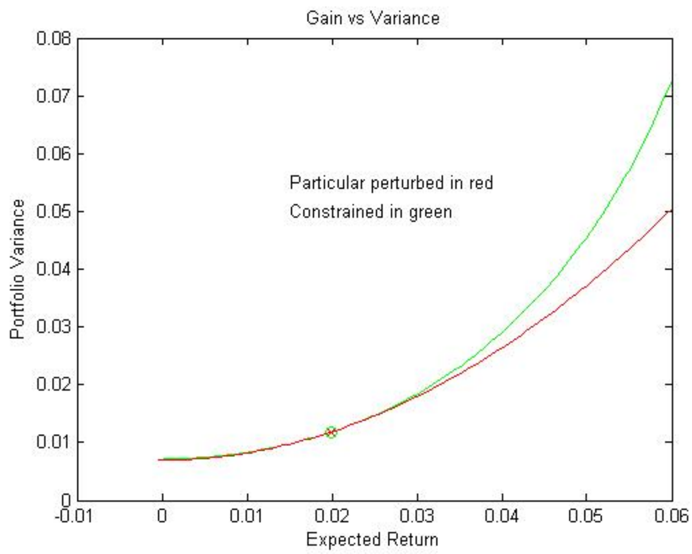


Figure 2: Generated MVO 1

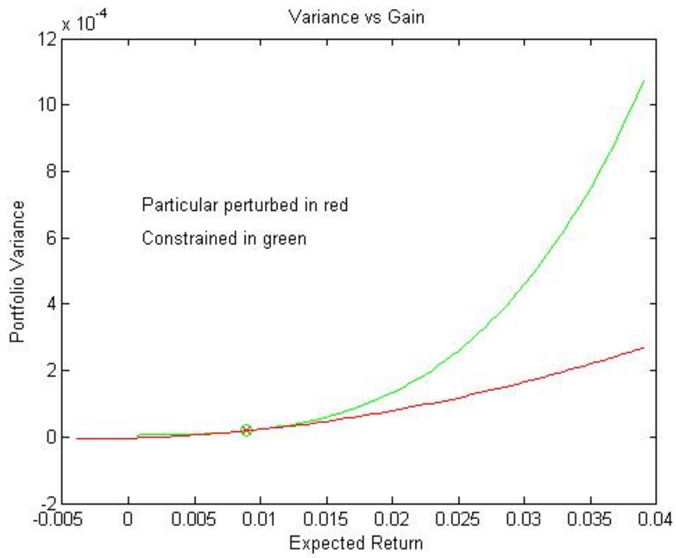


Figure 3: Actual MVO 1

The previous figures illustrate that equivalence only occurs at the initial value. This supports the uniqueness of the Lagrangians.

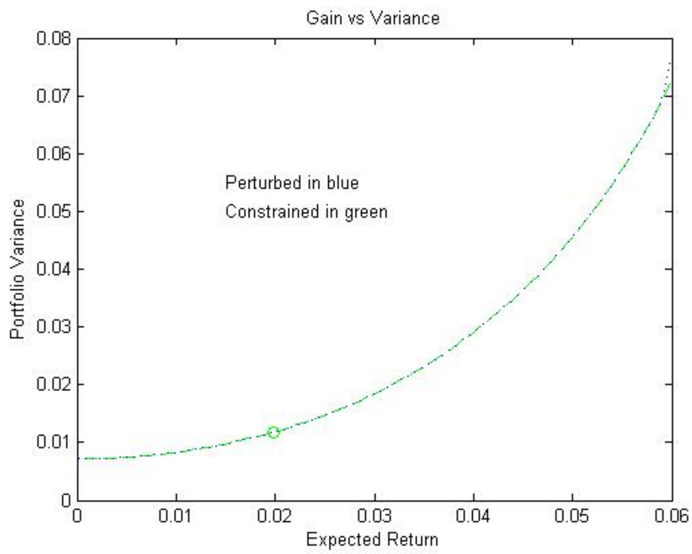


Figure 4: Generated MVO 2

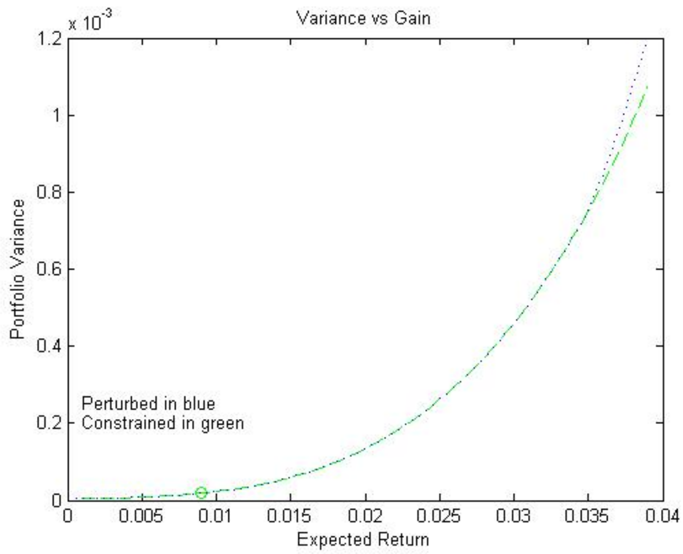


Figure 5: Actual MVO 2

If $\tilde{\Sigma}$ is, in fact, capturing the effect of the constraints then the optimal allocation variance should be the same as we allow the minimum expected return to fluctuate. This is illustrated in the previous figures.

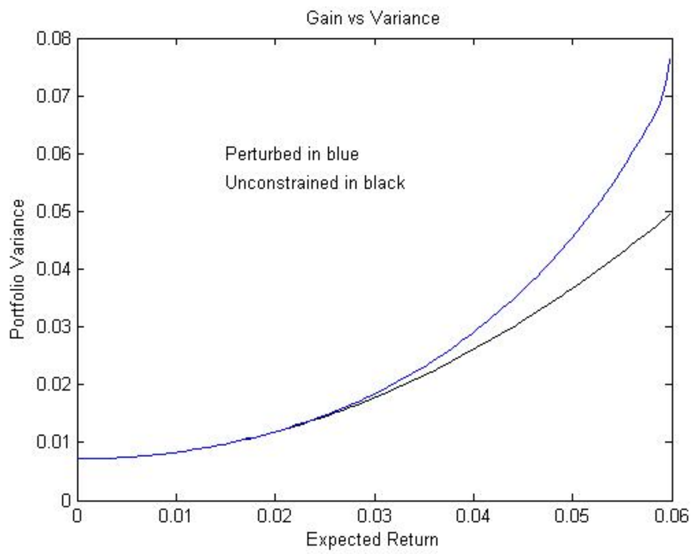


Figure 6: Generated MVO 3

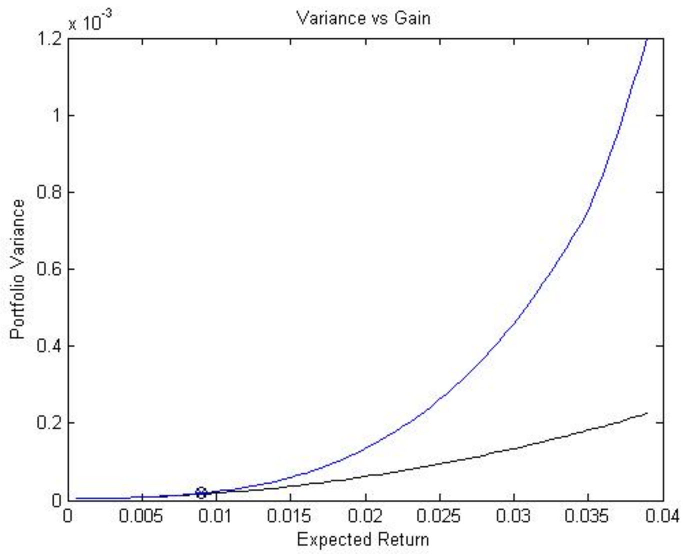


Figure 7: Actual MVO 3

Recalculating $\tilde{\Sigma}$ and the optimal allocation for each minimum expected return allowed me to illustrate how a change in expected return alters the significance of the Lagrange multipliers.

As expected, the MVO allocations were identical. That is, the expected portfolio return, the portfolio variance, and the individual asset weight allocations matched identically.

Table 1: Generated MVO Equivalence

Generated MVO	Constrained	Perturbed Unconstrained
Portfolio Variance	0.011813	0.011813
Expected Return	0.019843	0.019843

Table 2: Actual MVO Equivalence

Actual MVO	Constrained	Perturbed Unconstrained
Portfolio Variance	0.000019	0.000019
Expected Return	0.008972	0.008972

3.3.2 CVaR

In a similar fashion the minimum expected return was increased and decreased over an interval of 0.05 in increments of 0.00005. This allowed me to create plots of the data in order to consider whether the perturbation of the individual returns effectively explained the effect of the constraints. Consider the following figures (Constrained is blue, Perturbed Unconstrained is red);

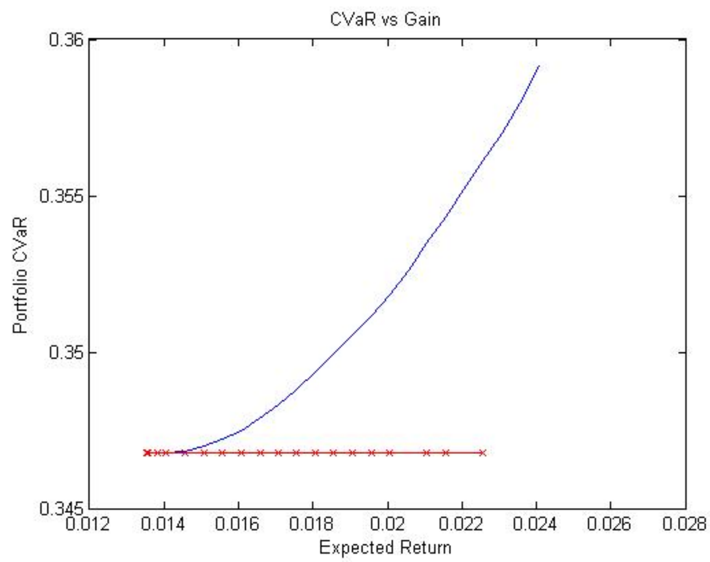


Figure 8: Generated CVaR 1

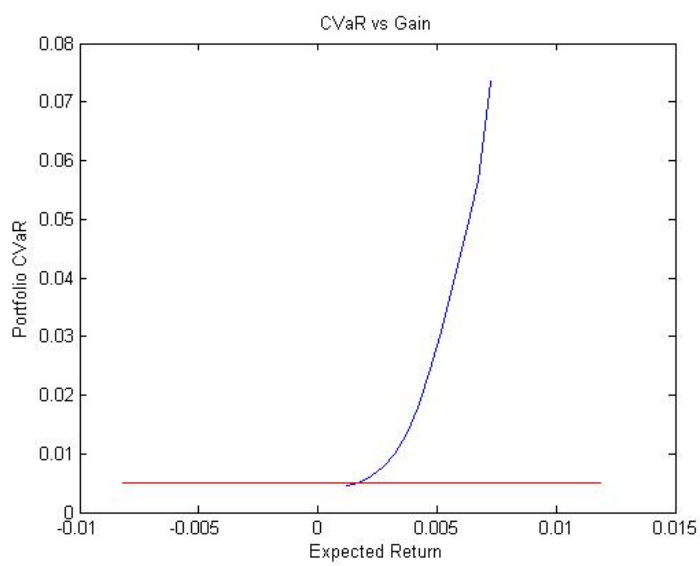


Figure 9: Actual CVaR 1

These figures illustrate that the CVaR values for the portfolio allocations are not identical but converge at a single value. This interssection represents the point at which there is no effect from the Lagrange multipliers. That is to say the interssection point is the point at which the constraints have no effect on portfolio performance. However, the shrinkage effect of the Lagrange multipliers is seen by the consistency of the CVaR values in the perturbed portfolio. These Lagrange multipliers pull extreme values in and shrink the variance of the asset returns matrix. Meanwhile, the effect of the constraints on the CVaR values of the constrained portfolio are evident. In order to achieve higher minimum returns the allocation must become less diversified leading to higher CVaR values.

This is exactly what would be expected considering equation (5). The perturbed portfolio's CVaR value is itself perturbed by the Lagrange multipliers. As the minimum expected return increases the program seeks higher expected returns on individual assets and the effect of the constraint is larger.

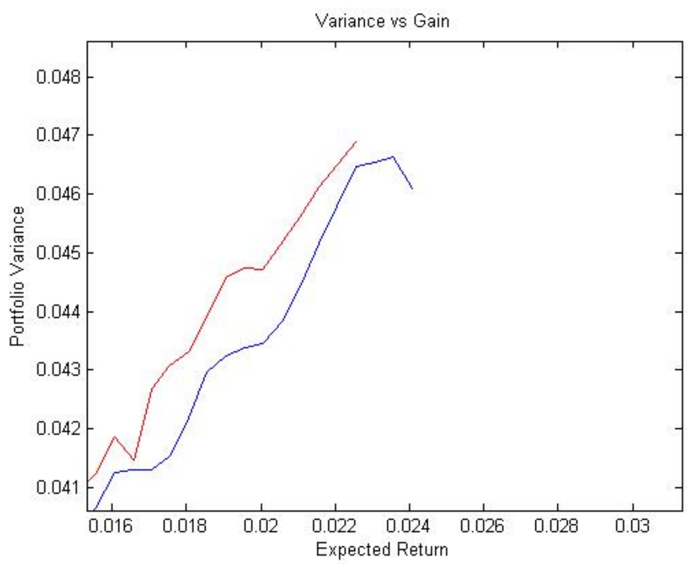


Figure 10: Generated CVaR 2

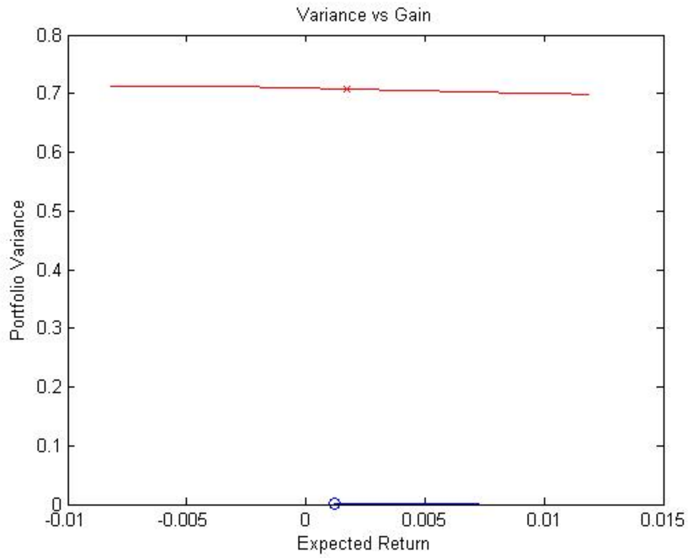


Figure 11: Actual CVaR 2

The previous figures plot the variance of the portfolios as the minimum expected return was allowed to fluctuate to different values. In the case of generated market data the variance of the perturbed portfolio was always greater then that of the original constrained portfolio. However, these values remained within 0.002 of each other and increase as the minimum expected return increased. Using actual market data the results were more difficult to understand. The figure shows almost constant portfolio variance values with the perturbed portfolio and original constrained portfolio. As in the generated data case, the perturbed portfolio's variance was greater then that of the original constrained portfolio. A closer look at the individual curves shows more detail.

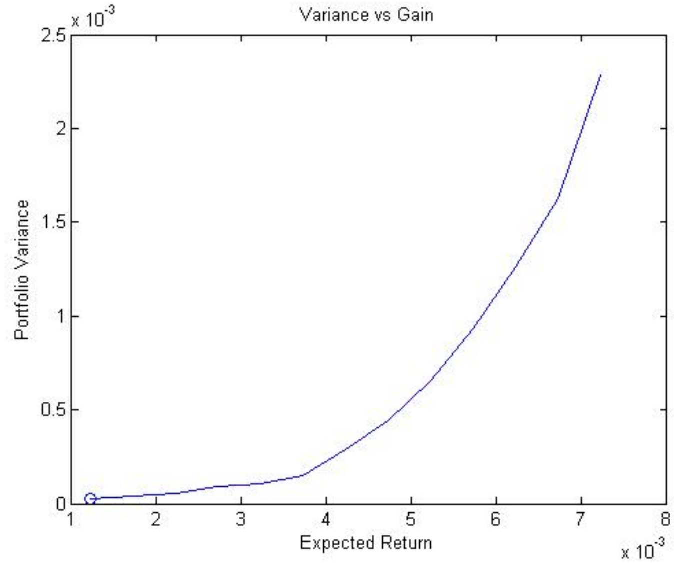


Figure 12: Actual CVaR Constrained

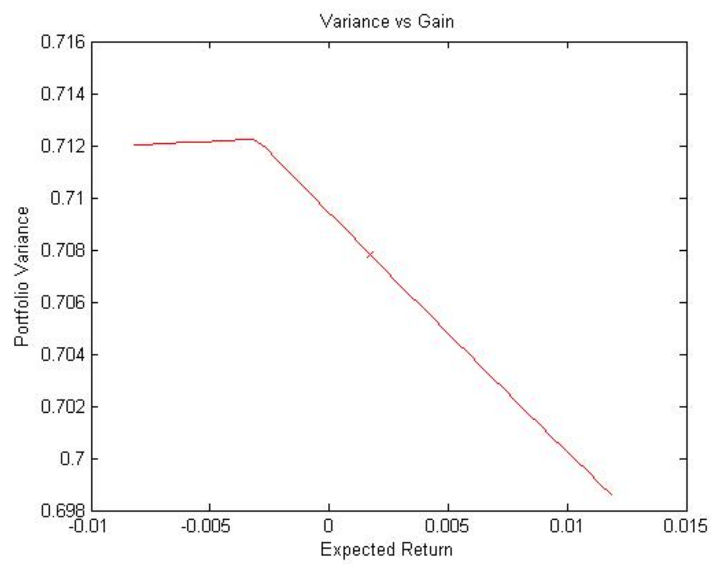


Figure 13: Actual CVaR Perturbed

When I plot the previous curves separately I am able to see how they are performing over differing minimum expected returns. The constrained portfolio recognizes growth in portfolio variance and the perturbed unconstrained portfolio sees a decline. However, these incremental changes are very small in comparison to the overall figure.

Recognizing that the generated market figure and actual market figure are scaled differently also offers some explanation. In each portfolio optimization simulation the number of data points and computed values is based on the ability of Matlab to find a Real valued solution within a specified number of recursions. If Matlab is unable to do so on numerous occasions a break has been built into my code. Consequently, the number of values provided in each simulation may differ.

In both cases the portfolio variance of the perturbed unconstrained optimization is higher then its constrained counterpart. Once again this is representative of the fact that the individual asset returns of the perturbed portfolio were effected by the Lagrange multipliers.

It is important to note that the effect of the constraints, and in effect the Lagrange multipliers, depends on how stringent the constraints are given the distribution of returns of the individual assets. When the effect is minimal it is possible that the constrained portfolio will recognize the same allocation and expected return while maintaining lower risk metrics. However, when the minimum expected return is increased, the perturbation of returns may lead to lower values in any of the risk metrics dependent upon whether it is the lower bound or upper bound that is producing the larger Lagrangian.

Regardless, the CVaR allocations should be identical. Just as in the MVO setting, the expected return matched identically. The asset allocation was similar but contained estimation errors. This occurred because the CVaR calculation is a discretization of the actual CVaR value. Consequently, small differences would be seen in the CVaR, VaR, and asset allocations for both the constrained and perturbed unconstrained portfolios. Numerous simulations produced errors in the expected returns as well. The ones presented here are some of the more accurate calculations.

Table 3: Generated CVaR Equivalence

Generated CVaR	Constrained	Perturbed Unconstrained
Portfolio Variance	0.034561	0.027674
Expected Return	0.030693	0.030693
VaR	0.30508	0.33003
CVaR	0.36248	0.37988

Table 4: Actual CVaR Equivalence

Actual CVaR	Constrained	Perturbed Unconstrained
Portfolio Variance	2.64E-05	4.85E-05
Expected Return	0.0010886	0.0010886
VaR	0.0034521	0.004482
CVaR	0.004616	0.0050696

3.4 DISCUSSION

The rapid pace at which the finance industry is attempting to integrate CVaR as a risk metric into their portfolio allocation decisions means that there has not been significant consideration of the characteristics of constrained Mean-CVaR optimized portfolios. In the case of Mean-Variance Optimization, portfolio managers recognized for decades that constrained portfolios performed better than their unconstrained counterparts and that this was a contradiction of the mathematical foundations of MVO. However, it was not until the work of Jagannathan and Ma in 2002 that theorists could explain how constraints effectively increased the accuracy of the sample covariance matrix via shrinkage and how this in effect led to better portfolio results.

MVO has lost favor in finance because of its inability to account for differing distributions of returns and the inaccuracy of finding other risk metric calculations of the original MVO calculations. However, while CVaR has come to the forefront it is not guaranteed to continue to be favored as an effective portfolio optimization tool. Consequently, it is necessary for the finance community to standardize an approach that can be used to understand how constraints effect portfolio allocations. That is what has been attempted here. A solution for equating a particular function's constrained optimization to its correlated unconstrained optimization can be done by the same method that was taken in both the MVO and CVaR settings. The proposed result should be tested, as in this case, under differing constraints and data to verify its validity.

In this particular case the results of the generalized MVO were exactly as had been expected. Using both normally distributed and actual market data, the particular perturbed covariance matrix was in fact a solution only to the original problem. Also, the perturbed solution matched identically with the constrained solution as each portfolio's minimum expected return was allowed to vary. The effect of the constraints was shown to become greater as the minimum expected return increased. The proof states that the optimal allocation, variance, and expected returns should all be identical and in fact they were.

When this concept was extended to CVaR optimization, the proof and numerical results were not as simple. Given the nature of the CVaR calculation it is impossible to avoid rounding errors. The expected return values were typically accurate in that they were identical between the two portfolio optimizations. However, the assets allocation saw estimation error. The computing power and sheer size of the actual market data set made calculations difficult and time consuming. Nonetheless the CVaR equivalence is true analytically and could become more accurate numerically with improved computing resources.

Given the limitations of a student license for Matlab and my personal computer's capabilities I was unable to incorporate substantial amounts of data into this study. I was also unable to run the thousands of simulations that I would have preferred as a single simulation using actual market data took upwards of thirty minutes.

My hope is that this study can provide some insight and direction into the future study of constraints in portfolio optimization. If one prefers, each constraint could be considered individually. Consequently there are many further studies that could be considered. However, I believe time would be better spent considering how particular constraints that could be "loosened" may affect portfolio performance.

4 APPENDICES

Effects of constraints on portfolio variance-optimization problems

This proof is a generalization to the result by Jagannathan and Ma (Jagannathan, 2003). Consider a mean-variance problem with full-utilization, no-shortselling, maximum-allocation, maximum-beta-exposure, and minimum-return constraints:

$$
\begin{aligned}
\min_{\mathbf{x}} \ & \frac{1}{2}\mathbf{x}^T \Sigma \mathbf{x} \\
\text{subject to } & \sum_i x_i = 1 \\
& x_i \geq 0 \\
& x_i \leq \bar{x}_i \\
& \mathbf{x}^T \boldsymbol{\beta} \leq \bar{\beta} \\
& \mathbf{x}^T \boldsymbol{\mu} \geq \bar{\mu}
\end{aligned}
$$

for all i. This problem has a Lagrangian

$$
\mathcal{L}(\mathbf{x}, \eta_0, \boldsymbol{\lambda}, \boldsymbol{\delta}, \eta_1, \eta_2) = \frac{1}{2}\mathbf{x}_T \Sigma \mathbf{x} + \eta_0(1 - \mathbf{x}^T \mathbf{1}) + \boldsymbol{\lambda}^T(-\mathbf{x}) + \boldsymbol{\delta}^T(\mathbf{x} - \bar{\mathbf{x}}) + \eta_1(\mathbf{x}^T \boldsymbol{\beta} - \bar{\beta}) + \eta_2(\bar{\mu} - \mathbf{x}^T \boldsymbol{\mu})
$$

while the KKT conditions read

$$
\begin{aligned}
\nabla_{\mathbf{x}} \mathcal{L} &= \Sigma \mathbf{x} - \eta_0 \mathbf{1} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1 \boldsymbol{\beta} - \eta_2 \boldsymbol{\mu} = 0 \\
\mathbf{x}^T \mathbf{1} &= 1 \\
\lambda_i x_i &= 0 \\
\delta_i (x_i - \bar{x}_i) &= 0 \\
\eta_1(\mathbf{x}^T \boldsymbol{\beta} - \bar{\beta}) &= 0 \\
\eta_2(\bar{\mu} - \mathbf{x}^T \mu) &= 0 \\
\lambda_i \geq 0, \delta_i \geq 0&, \eta_1 \geq 0, \eta_2 \geq 0
\end{aligned}
$$

for all i. We wish to compare this problem with an unconstrained variance-optimization problem subject to full allocation:

$$\begin{aligned}\min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}_T\tilde{\Sigma}\mathbf{x} \\ \text{subject to} \quad & \sum_i x_i = 1\end{aligned}$$

The unconstrained problem has a Lagrangian

$$\tilde{\mathcal{L}} = \frac{1}{2}\mathbf{x}_T\tilde{\Sigma}\mathbf{x} + \tilde{\eta}_0(1 - \mathbf{x}^T\mathbf{1}))$$

with the KKT conditions

$$\begin{aligned}\nabla_{\mathbf{x}}\mathcal{L} &= \tilde{\Sigma}\mathbf{x} - \tilde{\eta}_0\mathbf{1} = 0 \\ \mathbf{x}^T\mathbf{1} &= 1\end{aligned}$$

For suitable $\tilde{\Sigma}$ and $\tilde{\eta}_0$, the solution for the constrained problem also minimizes variance of the unconstrained problem. Suppose $\mathbf{x}^*$ is a solution to the constrained problem with Lagrange multipliers $\eta_0, \boldsymbol{\lambda}, \boldsymbol{\delta}, \eta_1$, and η_2. Let

$$\tilde{\Sigma} = \Sigma + (\boldsymbol{\delta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\delta}^T) - (\boldsymbol{\lambda}\mathbf{1}^T + \mathbf{1}\boldsymbol{\lambda}^T) + \eta_1(\boldsymbol{\beta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\beta}^T) - \eta_2((\boldsymbol{\mu} - \bar{\mu}\mathbf{1})\mathbf{1}^T + \mathbf{1}(\boldsymbol{\mu} - \bar{\mu}\mathbf{1})^T)$$

We show that $\mathbf{x}^*$ is also a solution to the unconstrained problem while $\tilde{\Sigma}$ is a valid covariance matrix. First, it is clear that $\tilde{\Sigma}$ is symmetric. To check that $\tilde{\Sigma}$ is positive-semidefinite, we compute

$$
\begin{aligned}
\mathbf{x}^T\tilde{\Sigma}\mathbf{x} &= \mathbf{x}^T\Sigma\mathbf{x} + \mathbf{x}^T(\boldsymbol{\delta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\delta}^T)\mathbf{x} - \mathbf{x}^T(\boldsymbol{\lambda}\mathbf{1}^T + \mathbf{1}\boldsymbol{\lambda}^T)\mathbf{x} + \\
&\quad \eta_1\mathbf{x}^T(\boldsymbol{\beta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\beta}^T)\mathbf{x} - \eta_2\mathbf{x}^T(\boldsymbol{\mu}\mathbf{1}^T + \mathbf{1}\boldsymbol{\mu}^T - 2\bar{\mu}\mathbf{1}\mathbf{1}^T)\mathbf{x} \\
&= \mathbf{x}^T\Sigma\mathbf{x} + 2(\mathbf{x}^T\mathbf{1})\mathbf{x}^T(\boldsymbol{\delta} - \boldsymbol{\lambda} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu}) + 2\eta_2\bar{\mu}(\mathbf{x}^T\mathbf{1})^2 \\
&= \mathbf{x}^T\Sigma\mathbf{x} + 2(\mathbf{x}^T\mathbf{1})\mathbf{x}^T(-\Sigma\mathbf{x}^* + \eta_0\mathbf{1}) + 2\eta_2\bar{\mu}(\mathbf{x}^T\mathbf{1})^2 \\
&= \mathbf{x}^T\Sigma\mathbf{x} - 2(\mathbf{x}^T\mathbf{1})\mathbf{x}^T\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}\mathbf{x}^* + 2(\eta_0 + \eta_2\bar{\mu})(\mathbf{x}^T\mathbf{1})^2 \\
&\geq \mathbf{x}^T\Sigma\mathbf{x} - 2(\mathbf{x}^T\mathbf{1})(\mathbf{x}^T\Sigma\mathbf{x})^{\frac{1}{2}}(\mathbf{x}^{*T}\Sigma\mathbf{x}^*)^{\frac{1}{2}} + 2(\eta_0 + \eta_2\bar{\mu})(\mathbf{x}^T\mathbf{1})^2 \\
&\geq \mathbf{x}^T\Sigma\mathbf{x} - 2(\mathbf{x}^T\mathbf{1})(\mathbf{x}^T\Sigma\mathbf{x})^{\frac{1}{2}}(\eta_0 + \eta_2\bar{\mu})^{\frac{1}{2}} + 2(\eta_0 + \eta_2\bar{\mu})(\mathbf{x}^T\mathbf{1})^2 \\
&= \left((\mathbf{x}^T\Sigma\mathbf{x})^{\frac{1}{2}} - (\mathbf{x}^T\mathbf{1})(\eta_0 + \eta_2\bar{\mu})^{\frac{1}{2}}\right)^2 + (\eta_0 + \eta_2\bar{\mu})(\mathbf{x}^T\mathbf{1})^2 \\
&\geq 0
\end{aligned}
$$

for any $\mathbf{x}$ since

$$
0 \leq \mathbf{x}^T\Sigma\mathbf{x} = -\mathbf{x}^T(-\eta_0\mathbf{1} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu}) = \eta_0 - \bar{\mathbf{x}}^T\boldsymbol{\delta} - \eta_1\bar{\beta} + \eta_2\bar{\mu} \leq \eta_0 + \eta_2\bar{\mu}.
$$

To show that $\mathbf{x}^*$ solves the perturbed unconstrained problem, we first compute

$$
\begin{aligned}
\tilde{\Sigma}\mathbf{x}^* &= \Sigma\mathbf{x}^* + (\boldsymbol{\delta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\delta}^T)\mathbf{x}^* - (\boldsymbol{\lambda}\mathbf{1}^T + \mathbf{1}\boldsymbol{\lambda}^T)\mathbf{x}^* + \eta_1(\boldsymbol{\beta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\beta}^T)\mathbf{x}^* - \eta_2(\boldsymbol{\mu}\mathbf{1}^T + \mathbf{1}\boldsymbol{\mu}^T - 2\bar{\mu}\mathbf{1}\mathbf{1}^T)\mathbf{x}^* \\
&= \Sigma\mathbf{x}^* + \boldsymbol{\delta} - \boldsymbol{\lambda} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu} + \mathbf{1}(\bar{\mathbf{x}}^T\boldsymbol{\delta} + \eta_1\bar{\beta} + \eta_2\bar{\mu}) \\
&= \mathbf{1}(\eta_0 + \bar{\mathbf{x}}^T\boldsymbol{\delta} + \eta_1\bar{\beta} + \eta_2\bar{\mu})
\end{aligned}
$$

So all first-order conditions are satisfied where $\tilde{\eta}_0 = \eta_0 + \bar{\mathbf{x}}^T\boldsymbol{\delta} + \eta_1\bar{\beta} + \eta_2\bar{\mu}$.

Therefore, optimal weights of the constrained problem are also the optimal weights for the unconstrained problem where the covariance matrix is perturbed as follows:

$$
\tilde{\Sigma} = \Sigma + (\boldsymbol{\delta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\delta}^T) - (\boldsymbol{\lambda}\mathbf{1}^T + \mathbf{1}\boldsymbol{\lambda}^T) + \eta_1(\boldsymbol{\beta}\mathbf{1}^T + \mathbf{1}\boldsymbol{\beta}^T) - \eta_2((\boldsymbol{\mu} - \bar{\mu}\mathbf{1})\mathbf{1}^T + \mathbf{1}(\boldsymbol{\mu} - \bar{\mu}\mathbf{1})^T).
$$

This result agrees with the result in Jagannathan and Ma (2000) that no-shortselling and maximum allocation constraints artificially decreases and increases covariance among individual assets in the variance-optimization problem respectively. Moreover, we have shown that the maximum beta-exposure condition leads to an increase in covarinace which is proportional to betas of the individual assets. Finally, we obtained an interesting result that the minimum-expected-return condition negatively affects covariance of assets whose expected returns exceed the minimum return while it positively affects covariance of assets with less returns than the minimum return.

Effects of constraints on Conditional-Value-at-Risk (CVaR) optimization problems

Let the loss function be $f(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T \mathbf{y}$ where $\mathbf{x}$ is current holding and $\mathbf{y}$ is one-time-step return. Then the value-at-risk at probability β (β-VaR) for a given holding x is

$$\alpha_\beta(\mathbf{x}) = \min \left\{ \alpha \in \mathcal{R} : \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha} p(\mathbf{y}) d\mathbf{y} \geq \beta \right\}$$

In addition, the conditional value-at-risk at probability β (β-CVaR) is the expectation of loss given that the loss is greater than β-VaR

$$\phi_\beta(\mathbf{x}) = \frac{1}{1-\beta} \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha_\beta(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y}$$

It is a measure of risk related to the holding which represents expected loss in extreme cases. In some cases, it is beneficial to construct a portfolio which minimizes β-CVaR.

Here, we will examine the effect of holding constraints on the β-CVaR optimization problem. We wish to use KKT conditions to obtain our results so we need $\phi_\beta(\mathbf{x}) \in C^1(\mathbf{x})$. Under certain conditions, Tasche (Tasche, 2000) showed that for

$f(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T\mathbf{y}$, β-CVaR is differentiable. The derivatives are

$$\nabla_{\mathbf{x}}\phi_\beta(\mathbf{x}) = \frac{1}{1-\beta}\int_{f(\mathbf{x},\mathbf{y})\geq\alpha_\beta(\mathbf{x})} -\mathbf{y}p(\mathbf{y})d\mathbf{y}$$

which are continuous in $\mathbf{x}$.

Next, consider a constrained β-CVaR minimization problem

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \phi_\beta(\mathbf{x}) \\
\text{subject to} \quad & \sum_i x_i = 1 \\
& x_i \geq 0 \\
& x_i \leq \bar{x}_i \\
& \sum_i \beta_i x_i \leq \bar{\beta} \\
& \sum_i \mu_i x_i \geq \bar{\mu}
\end{aligned}$$

for all i, where we have full-utilization, no-shortselling, maximum-allocation, maximum-β-exposure, and minimum-expected-return constraints. The Lagrangian for this problem is

$$\mathcal{L}(\mathbf{x}, \eta_0, \boldsymbol{\lambda}, \boldsymbol{\delta}, \eta_1) = \phi_\beta(\mathbf{x}) + \eta_0(1 - \mathbf{1}^T\mathbf{x}) + \boldsymbol{\lambda}^T(-\mathbf{x}) + \boldsymbol{\delta}^T(\mathbf{x} - \bar{\mathbf{x}}) + \eta_1(\mathbf{x}^T\boldsymbol{\beta} - \bar{\beta}) + \eta_2(\bar{\mu} - \mathbf{x}^T\boldsymbol{\mu})$$

while the KKT conditions read

$$
\begin{aligned}
\nabla_{\mathbf{x}}\mathcal{L} &= \nabla_{\mathbf{x}}\phi_\beta(\mathbf{x}) - \eta_0\mathbf{1} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu} = 0 \\
\mathbf{1}^T\mathbf{x} &= 1 \\
\lambda_i x_i &= 0 \\
\delta_i(x_i - \bar{x}_i) &= 0 \\
\eta_1(\mathbf{x}^T\boldsymbol{\beta} - \bar{\beta}) &= 0 \\
\lambda_i &\geq 0, \delta_i \geq 0, \eta_1 \geq 0, \eta_2 \geq 0
\end{aligned}
$$

We wish to compare this to an unconstrained problem (subject to full utilization)

$$
\begin{aligned}
&\min_{\mathbf{x}} \ \tilde{\phi}_\beta(\mathbf{x}) \\
&\text{subject to} \quad \sum_i x_i = 1
\end{aligned}
$$

where $\tilde{\phi}_\beta(\mathbf{x}) = \frac{1}{1-\beta}\int_{\tilde{f}(\mathbf{x},\mathbf{y})\geq\tilde{\alpha}_\beta(\mathbf{x})} \tilde{f}(\mathbf{x},\mathbf{y})\tilde{p}(\mathbf{y})d\mathbf{y}$. The Lagrangian for this problem is

$$
\tilde{\mathcal{L}}(\mathbf{x}, \eta_0, \lambda) = \tilde{\phi}_\beta(\mathbf{x}) + \tilde{\eta}_0(1 - \mathbf{1}^T\mathbf{x})
$$

and the optimization conditions are

$$
\begin{aligned}
\nabla_{\mathbf{x}}\mathcal{L} &= \nabla_{\mathbf{x}}\tilde{\phi}_\beta(\mathbf{x}) - \tilde{\eta}_0\mathbf{1} = 0 \\
\mathbf{1}^T\mathbf{x} &= 1
\end{aligned}
$$

For suitable $\tilde{\phi}_\beta, \tilde{\eta}_0$, a solution for the constrained problem is also a solution for the unconstrained problem if

$$
\nabla_{\mathbf{x}}\tilde{\phi}_\beta(\mathbf{x}^*) - \tilde{\eta}_0\mathbf{1} = \nabla_{\mathbf{x}}\phi_\beta(\mathbf{x}^*) - \eta_0\mathbf{1} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu} \tag{6}
$$

where $\mathbf{x}$ is a solution to the constrained problem. Consider

$$\tilde{f}(\mathbf{x},\mathbf{y}) = -\mathbf{x}^T(\mathbf{y} + \boldsymbol{\lambda} - \boldsymbol{\delta} - \eta_1\boldsymbol{\beta} + \eta_2\boldsymbol{\mu})$$

while other parameters remain the same as in the unconstrained problem. Then

$$\begin{aligned}
\tilde{\alpha}_\beta(\mathbf{x}^*) &= \min\left\{\alpha \in \mathcal{R} : \int_{\tilde{f}(\mathbf{x}^*,\mathbf{y})\geq\alpha} p(\mathbf{y})d\mathbf{y} \geq \beta\right\}\\
&= \min\left\{\alpha \in \mathcal{R} : \int_{-\mathbf{x}^{*T}(\mathbf{y}+\boldsymbol{\lambda}-\boldsymbol{\delta}-\eta_1\boldsymbol{\beta}+\eta_2\boldsymbol{\mu})\geq\alpha} p(\mathbf{y})d\mathbf{y} \geq \beta\right\}\\
&= \min\left\{\alpha \in \mathcal{R} : \int_{-\mathbf{x}^{*T}\mathbf{y}+\bar{\mathbf{x}}^T\boldsymbol{\delta}+\eta_1\bar{\beta}-\eta_2\bar{\mu}\geq\alpha} p(\mathbf{y})d\mathbf{y} \geq \beta\right\}\\
&= \min\left\{\alpha \in \mathcal{R} : \int_{-\mathbf{x}^{*T}\mathbf{y}\geq\alpha-\bar{\mathbf{x}}^T\boldsymbol{\delta}-\eta_1\bar{\beta}+\eta_2\bar{\mu}} p(\mathbf{y})d\mathbf{y} \geq \beta\right\}\\
&= \alpha_\beta(\mathbf{x}^*) + \bar{\mathbf{x}}^T\boldsymbol{\delta} + \eta_1\bar{\beta} - \eta_2\bar{\mu}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla_{\mathbf{x}}\tilde{\phi}_\beta(\mathbf{x}^*) &= \frac{1}{1-\beta}\int_{\tilde{f}(\mathbf{x}^*,\mathbf{y})\geq\tilde{\alpha}_\beta(\mathbf{x}^*)} -(\mathbf{y} + \boldsymbol{\lambda} - \boldsymbol{\delta} - \eta_1\boldsymbol{\beta} + \eta_2\boldsymbol{\mu})p(\mathbf{y})d\mathbf{y}\\
&= \frac{1}{1-\beta}\int_{\tilde{f}(\mathbf{x}^*,\mathbf{y})\geq\tilde{\alpha}_\beta(\mathbf{x}^*)} -\mathbf{y}p(\mathbf{y})d\mathbf{y} + \frac{1}{1-\beta}(-\boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu})\int_{\tilde{f}(\mathbf{x}^*,\mathbf{y})\geq\tilde{\alpha}_\beta(\mathbf{x}^*)} p(\mathbf{y})d\mathbf{y}\\
&= \frac{1}{1-\beta}\int_{-\mathbf{x}^{*T}(\mathbf{y}+\boldsymbol{\lambda}-\boldsymbol{\delta})\geq\alpha_\beta(\mathbf{x}^*)+\bar{\mathbf{x}}^T\boldsymbol{\delta}} -\mathbf{y}p(\mathbf{y})d\mathbf{y} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu}\\
&= \frac{1}{1-\beta}\int_{-\mathbf{x}^{*T}\mathbf{y}\geq\alpha_\beta(\mathbf{x}^*)} -\mathbf{y}p(\mathbf{y})d\mathbf{y} - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu}\\
&= \nabla_{\mathbf{x}}\phi_\beta(\mathbf{x}^*) - \boldsymbol{\lambda} + \boldsymbol{\delta} + \eta_1\boldsymbol{\beta} - \eta_2\boldsymbol{\mu}
\end{aligned}$$

and the condition (1) is satisfied. The solution for the constrained problem is also the solution of an unconstrained problem with adjusted return

$$\tilde{f}(\mathbf{x},\mathbf{y}) = -\mathbf{x}^T(\mathbf{y} + \boldsymbol{\lambda} - \boldsymbol{\delta} - \eta_1\boldsymbol{\beta} + \eta_2\boldsymbol{\mu}).$$

From this result, the no-shortselling constraint and the maximum-allocation constraint affect the unconstrained problem by adjusting returns of individual assets up and down by the Lagrange multiplier. Similarly, introduction of minimum expected-return condition and maximum β-exposure condition affect the unconstrained problem by adjusting the returns up and down proportionally to $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$ respectively.

If one prefers, the full-allocation constraint can be absorbed into individual returns in the unconstrained problem. In general, any equality and inequality constraints can be treated in the same way as we have done in this report.

Note that there might be other perturbations which yield different unconstrained CVaR-optimization problems. Nevertheless, portfolio return adjustment is a plausible way to interpret the effect of constraints on the problem.

MVO Values

Table 5: Generated MVO

Beta	**VaR**	**CVaR**
0.90	0.1591	0.2106
0.95	0.1986	0.2440
0.99	0.2727	0.3095

Table 6: Generated MVO Equivalence

Generated MVO	Constrained	Perturbed Unconstrained
Portfolio Variance	0.011813	0.011813
Expected Return	0.019843	0.019843
Non-Zero Asset Weights (in %)	4.04	4.04
	3.31	3.31
	2.20	2.20
	1.37	1.37
	1.91	1.91
	3.80	3.80
	2.94	2.94
	2.54	2.54
	0.14	0.14
	1.58	1.58
	1.45	1.45
	1.34	1.34
	0.19	0.19
	1.47	1.47
	8.33	8.33
	1.73	1.73
	5.33	5.33
	0.98	0.98
	4.35	4.35
	6.26	6.26
	0.48	0.48
	1.28	1.28
	1.05	1.05

Table 7: Generated MVO Equivalence Continued

Generated MVO	Constrained	Perturbed Unconstrained
	5.96	5.96
	4.90	4.90
	1.13	1.13
	2.70	2.70
	4.01	4.01
	2.28	2.28
	3.98	3.98
	4.75	4.75
	3.21	3.21
	2.11	2.11
	0.03	0.03
	1.93	1.93
	1.81	1.81
	1.37	1.37
	1.78	1.78

Table 8: Actual MVO

Beta	VaR	CVaR
0.90	0.0145	0.0165
0.95	0.0161	0.0179
0.99	0.0190	0.0205

Table 9: Actual MVO Equivalence

Actual MVO	**Constrained**	**Perturbed Unconstrained**
Portfolio Variance	0.000019	0.000019
Expected Return	0.008972	0.008972
Non-Zero Asset Weights (in %)	1.35	1.35
	1.48	1.48
	0.52	0.52
	1.94	1.94
	0.78	0.78
	1.30	1.30
	2.60	2.60
	0.93	0.93
	0.70	0.70
	3.60	3.60
	0.77	0.77
	1.75	1.75
	0.57	0.57
	1.09	1.09
	0.98	0.98
	1.47	1.47
	1.32	1.32
	0.51	0.51
	1.04	1.04
	1.63	1.63
	0.31	0.31
	0.45	0.45
	0.57	0.57
	0.19	0.19

Table 10: Actual MVO Equivalence Continued

Actual MVO	Constrained	Perturbed Unconstrained
	1.66	1.66
	1.43	1.43
	0.54	0.54
	0.93	0.93
	1.49	1.49
	0.42	0.42
	0.50	0.50
	0.20	0.20
	0.92	0.92
	1.26	1.26
	1.65	1.65
	0.14	0.14
	0.22	0.22
	0.52	0.52
	1.38	1.38
	1.72	1.72
	0.08	0.08
	0.37	0.37
	1.58	1.58
	0.17	0.17
	3.85	3.85
	1.00	1.00
	3.14	3.14
	0.40	0.40
	1.88	1.88
	0.99	0.99
	0.30	0.30

Actual MVO	Constrained	Perturbed Unconstrained
	1.62	1.62
	1.31	1.31
	3.01	3.01
	1.60	1.60
	2.29	2.29
	0.93	0.93
	0.86	0.86
	1.09	1.09
	2.14	2.14
	2.18	2.18
	1.30	1.30
	1.05	1.05
	2.16	2.16
	2.59	2.59
	1.44	1.44
	0.48	0.48
	1.35	1.35
	1.83	1.83
	2.44	2.44
	2.28	2.28
	1.94	1.94
	0.23	0.23
	1.08	1.08
	1.21	1.21
	2.81	2.81
	1.78	1.78
	0.43	0.43

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