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Four Derivations of Schrödinger’s Time Dependent Equation

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Objectives

Obtain Schrödinger’s time-dependent equation using four methods:

- Classical Simple Harmonic Motion (SHM)
- Classical electromagnetic wave equation
- Schrödinger’s time independent equation
- Hamilton-Jacobi equation

Explain the applications of Schrödinger’s equation, and failures of Newtonian physics where Quantum physics takes its place.

Introduction

Laws and theorems of classical physics apply nicely to macroscopic objects, but at the subatomic level they begin to break down, hence the advent of Quantum Mechanics. Just as Newton’s laws describe large scale motion, the Schrödinger Equation attempts to describe the motion and properties of particles, such as electrons about a nucleus, and photons. It did not appear out of thin air, there a multitudes of ways to obtain the equation using classical starting points and a few logical assumptions.

Prerequisite Knowledge

- Algebra - How to manipulate an equation for a specific variable or result
- Calculus I - Second order derivatives
- Physics:
  - Energy: \( \frac{1}{2} m v^2 \) (Classical) and \( E = \frac{1}{2} m v^2 \)
  - Frequency: \( \nu = \frac{1}{\lambda} \)
  - Maxwell’s equations of electrodynamics
  - \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)
  - \( \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \)
  - \( \nabla \cdot \mathbf{E} = 0 \)
  - \( \nabla \cdot \mathbf{B} = 0 \)
  - Complex numbers: \( e^{-i\theta} = \cos(\theta) - i\sin(\theta) \)
  - Hamilton’s equations for Canonical transformations
  - \( \frac{dx}{dt} = \frac{\partial H}{\partial p} \)
  - \( \frac{dp}{dt} = -\frac{\partial H}{\partial x} \)

Why Do We Need To Know Quantum Physics?

There should be no limit for energy in the ultraviolet range, but that is impossible

Method I: Simple Harmonic Motion

Begin with equation of a wave propagating through a medium:

\[
x(t) = A \cdot \cos(\omega t - \phi) - i \sin(\omega t - \phi)
\]

\[
\Rightarrow \psi(t) = A \cdot e^{i(\omega t - \phi)}
\]

(1)

Take the spatial derivative of \( \psi \) twice and the time derivative once to find

\[
\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \psi
\]

\[
\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} E \psi
\]

(2)

(3)

Apply eq. 2 and eq. 3 to classical total energy to obtain the TDSE.

Method II: Time-Independent Schrödinger Equation

\[
\nabla^2 \psi + \frac{2m}{\hbar^2} (E - V) \psi = 0
\]

(4)

Assume a solution

\[
\psi = A \cdot e^{-iEt}
\]

(5)

Differentiate with respect to time, substitute, and manipulate eq. 4 to obtain the time-dependent equation.

The difference between the time-dependent and independent equations is analogous to Statics and Dynamics in Classical Physics.

Method III: Classical Wave Equation

Assume the wave equation for a particle in free space

\[
\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0,
\]

(7)

whose solution is

\[
E(x, t) = E_0 e^{i(kx - \omega t)}
\]

(8)

Assume non-zero rest mass, and approximate relativistic energy using \( \gamma c^2 = m c^2 \approx \sqrt{\gamma^2 c^4 + \frac{p^2 c^2}{m^2}} \). Discard the small (insignificant) terms. This method is along the lines of how Erwin Schrödinger originally developed his equations.

Method IV: Hamilton-Jacobi Equation

Assume a solution of \( \psi(X, t) = e^{iS(X, t)} \) and solve for \( S(X, t) \). Take the spatial and time derivatives of \( S \) and apply them to the Hamilton-Jacobi equation for a particle with mass \( m \) and potential \( V \).

\[
\frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{\partial S}{\partial y} + \frac{\partial S}{\partial z} + V - \frac{\partial S}{\partial t} = 0
\]

(9)

This method makes the fewest assumptions and approximations.

Applications and Technology

Research: QM provides a means for understanding properties of electrons and then allows scientists to utilize those properties to understand the structures of materials, from the skeletons of rockets, to the solid state hard drive in your computer.

References


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