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Equations of Motion in a Rotating Noninertial Reference Frame

The Coriolis Force
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Objectives
To demonstrate how "fictitious" forces arise from a frame of reference that isn’t in a state of inertia, a familiar model is constructed in the form of a rotating planet.
- Describe the motion of a sphere rotating about a stationary axis
- Determine the equations of motion of an object moving in the frame of the planet’s surface
- Test the solutions with expectations under different parameters

Introduction
Newton’s first law of mechanics states that a body remains at rest or in uniform motion unless acted upon by a force. Though not explicitly stated, this law defines an inertial reference frame. If a reference frame is subject to acceleration intrinsic to its motion, like the surface of a rotating sphere, it is a noninertial frame of reference. Seemingly measurable forces that manifest from this frame are termed fictitious forces and are artificial corrections required due to attempts to extend Newton’s equations to a noninertial system [1].

Homogeneous Solution
When solving differential equations, a general solution for the equation $\ddot{x} = A\dot{x}$ is:

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} V^{-1},$$

where matrix $V$ is a composite of the eigenvectors of $A$, $V^{-1}$ is its inverse, and $\lambda_i$ are the eigenvalues of $A$.

Particular Solution
The particular solution can be represented with a linear operator such as $L_{op}\vec{p}_p = \vec{g}(t)$, and solved using a Green’s function.

$$L_{op}\vec{g}(t, t_0) = \delta(t - t_0)$$

A function can be rewritten with a Dirac delta so that,

$$L_{op}\vec{p}_p = \int_{-\infty}^{\infty} \vec{g}(t, t_0) \delta(t - t_0) \, dt_o$$

Complete Solution
$$v_x = g_s \sin^2 \omega t + v_{yo} \cos 2\omega t - v_{yo} \sin 2\omega t - v_{xo} \sin 2\omega t$$
$$v_y = g_s (\sin 2\omega t - 2\omega t) - v_{yo} \sin 2\omega t + v_{yo} \omega t (\cos 2\omega t + \alpha) + v_{xo} \sin 2\omega t + \omega t (\alpha - \sin 2\omega t)$$
$$v_x = -v_y (\sin^2 \phi + \sin^2 \omega t) + v_{yo} \sin 2\omega t + v_{yo} \omega t (\cos 2\omega t + \alpha) + v_{xo} \sin 2\omega t + \omega t (\alpha - \sin 2\omega t)$$

Change of Frame Transformation
If the rotating sphere is embedded in a “fixed” [1] frame the equation that relates measurements from an observer rotating on the surface to that of a celestial, “fixed”, observer is as follows:

$$\frac{dv}{dt} = \frac{dv}{dt}_{\text{rotating}} + \vec{\omega} \times \vec{v}$$

The velocity, $\vec{v}$, as measured by the fixed observer is dependent on the angular velocity of the sphere. To derive the fictitious forces, the same process can be carried out to determine acceleration $\frac{dv}{dt}_{\text{rotating}}$ corrections between the frames.

$$F_{\text{fictitious}} = F - m\ddot{x} \times \vec{v} - m\omega^2 \vec{v} \times (\vec{\omega} \times \vec{v}) - 2m\omega \times \vec{v} \times \vec{w}. \tag{5}$$

Each term in equation 5 can be interpreted physically as:

- $F$: sum of the forces acting on the object as measured in the fixed system
- $-m\ddot{x} \times \vec{v}$: result of rotational acceleration
- $-m\omega^2 \vec{v} \times (\vec{\omega} \times \vec{v})$: centrifugal force
- $-2m\omega \times \vec{v} \times \vec{w}$: Coriolis force.

Results
Explicit values for the aforementioned equations are as follows:

- $\ddot{z} = 0 \quad \ddot{\phi} = -\frac{g_s}{\omega} \sin 2\omega t$,
- $\ddot{\omega} = -\frac{g_s}{\omega} \cos 2\omega t + \frac{2}{\omega} \sin 2\omega t - \frac{2\omega^2}{\omega} \cos 2\omega t$,
- $L_{op} = \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$.

To test whether these solutions agree with expectations $\omega$ and $\alpha$ can be altered. If an object is dropped over a pole ($\alpha = 0$) it should only be affected by gravity. If dropped at the equator ($\alpha = \frac{\pi}{2}$) an additional easterly velocity should occur. If there is no rotation, only the gravity term should survive. Additionally, in the Northern Hemisphere a particle projected in a horizontal plane will be directed towards the right of the particle’s motion [1]. All deflections in the Southern Hemisphere are opposite to the Northern. For the velocity vector function, $\vec{v}(\phi, \alpha, t)$, these constraints result in:

- $\vec{v}(0, 0, \omega) = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$
- $\vec{v}(\frac{\pi}{2}, \alpha, 0) = \left[ \begin{array}{c} \frac{g_s}{\omega} \sin 2\omega t \\ \frac{g_s}{\omega} \cos 2\omega t \end{array} \right]$
- $\vec{v}(\alpha, \pi, 0) = \left[ \begin{array}{c} 0 \\ 0 \\ -\frac{g_s}{\omega} \sin 2\omega t \end{array} \right]$
- $\vec{v}(\alpha, \frac{\pi}{2}, 0) = \left[ \begin{array}{c} \frac{g_s}{\omega} \cos 2\omega t \\ \frac{g_s}{\omega} \sin 2\omega t \end{array} \right]$

Figure 1: The path an object traces when experiencing “fictitious” forces induced by a noninertial frame of reference. A target due south on a globe (a) is deflected from a straight path (b) by the rotational motion of the globe. © Encyclopaedia Britannica

Figure 2: Easterly deflection demonstrated by the change in velocity, $v_y$, over one minute. Motion in the Northern Hemisphere deflects to the right, while motion in the southern hemisphere deflects left.

Additional Information
The southerly deflection is on the order of a million times smaller than the easterly deflection. Despite many attempts, no credible evidence that the southerly deflection has been detected has been correctly measured [2].

References

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