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Mathematical Modelling of English Coulee: Tanks in Series

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UNDergraduate SHOWCASE
UNIVERSITY OF NORTH DAKOTA

Objectives

- To generate a robust mathematical model for the English Coulee which can be used to help predict and understand the variation of chemical concentrations with time.
- To better understand the characteristics of Tank models and their limitations.

Derivation of Equations

Considering a set of tanks in series, the state of each tank's contents can be described mathematically based upon conservation of mass.
The volume change in the nth tank can be described as :

$$\frac{d}{dt}V_n(t) = Q_{n-1}(t) + I_n(t) - Q_n(t)$$

Where $Q_n(t)$ is the flow leaving the nth tank and $I_n(t)$ represents an outside source or drain.
The mass change in the system of some contaminant in the nth tank can be described as:

$$\frac{d}{dt}M_n(t) = \dot{m}_{n-1}(t) + R_n(t) - \dot{m}_n(t)$$

Where $\dot{m}_n(t)$ is the mass flow of contaminant leaving the nth tank and $R_n(t)$ represents an outside source or drain of the contaminant.

Concentration being: $C_n(t) = \frac{M_n(t)}{V_n(t)}$ allows for the change in concentration to be expressed as:

$$\frac{d}{dt}C_n(t) = \frac{[\dot{m}_{n-1}(t) + R_n(t) - \dot{m}_n(t)] * V_n(t) - M_n(t) * [Q_{n-1}(t) + I_n(t) - Q_n(t)]}{[V_n(t)]^2}$$

Note, if the tanks are assumed to be well mixed, $C_n(t) * Q_n(t) = \dot{m}_n(t)$, then the equation can be written as:

$$\frac{d}{dt}C_n(t) = \frac{1}{V_n(t)} * [C_{n-1}(t) * Q_{n-1}(t) - C_n(t)[Q_{n-1}(t) + I_n(t)] + R_n(t)]$$

Let C $= \begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_n(t) \end{bmatrix}$ and $V^- = \begin{bmatrix} \frac{1}{V_1(t)} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{V_n(t)} \end{bmatrix}$ Then the system can be expressed in matrix form as:

$$\frac{d}{dt}C = V^- * \begin{bmatrix} -(Q_0(t) + I_1(t)) & & & 0 \\ Q_1(t) & -(Q_1(t) + I_2(t)) & & \\ & \ddots & \ddots & \\ 0 & & Q_{n-1}(t) & -(Q_{n-1}(t) + I_n(t)) \end{bmatrix} C + \begin{bmatrix} C_0(t)Q_0(t) + R_1(t) \\ R_2(t) \\ \vdots \\ R_n(t) \end{bmatrix}$$

If the simplifying assumption of constant volume is made, the fow rate leaving each tank can be expressed as $Q_n(t) = Q_0(t) + \sum_{i=1}^{n} I_i(t)$. This yields the following system:

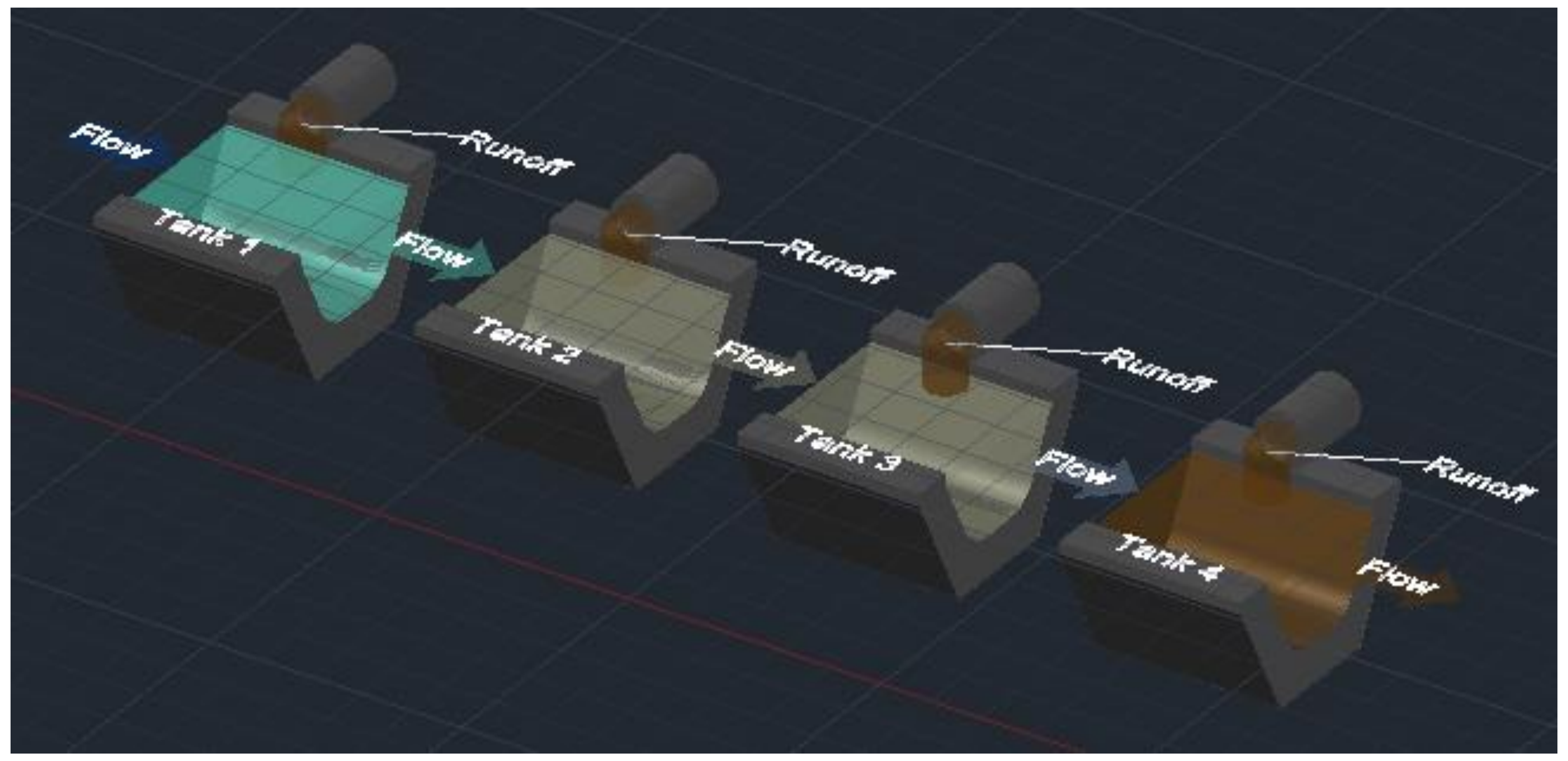
$$\frac{d}{dt}C = V^- * \begin{bmatrix} -(Q_0(t) + I_1(t)) & & & 0 \\ Q_0(t) + I_1(t) & -(Q_0(t) + I_1(t) + I_2(t)) & & \\ & \ddots & \ddots & \\ 0 & & Q_0(t) + \sum_{i=1}^{n-1} I_i(t) & -(Q_0(t) + \sum_{i=1}^{n} I_i(t)) \end{bmatrix} C + \begin{bmatrix} C_0(t)Q_0(t) + R_1(t) \\ R_2(t) \\ \vdots \\ R_n(t) \end{bmatrix}$$

If $Q_0(t)$ is a constant, solutions for such a system can be expressed in terms of the matrix exponential: $C(t) = e^{At} * C_0 + e^{At} * \int_t^{t_0} e^{-At} * G(t)dx$, where G(t) is the matrix representing the non-homogeneous part of the previous system.

Numerical Methods

- RK4 method programmed in Mathematica.
- Provides numerical solutions to the above system.
- Error of method is $O(h^4)$.
- Allows for quick simulation while varying model parameters.
- Allows for easy visualization and manipulation of results.
- Allows for the addition of probabilistic elements:
  - Selection of input parameters from normal distributions of likely values.
  - Model ran a great number of times with run information saved.
  - Histograms and cumulative frequency distributions plotted for specific times, allowing for an understanding of likely concentration behavior.
  - Assuming the validity of chosen distributions, can allow for the determination of Confidence Intervals.

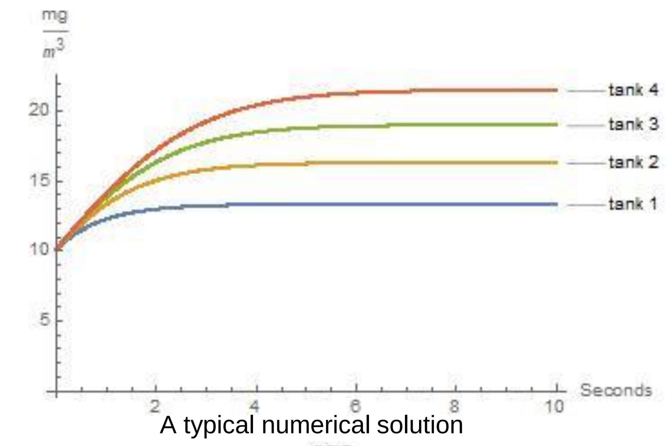
Conceptual Model



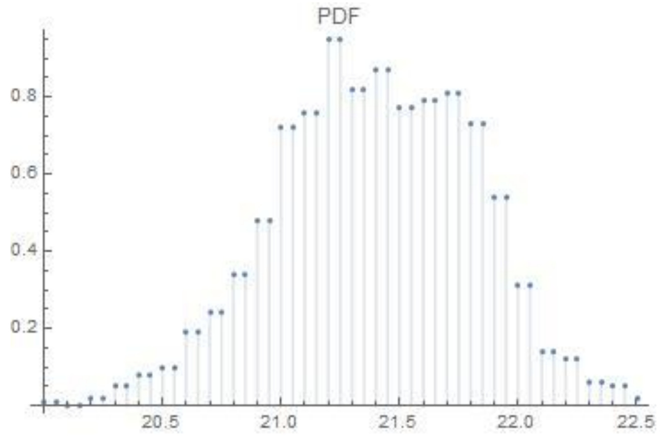
Preliminary Modelling

- Consider a 4 tank system in which each tank start at an initial concentration of 10 mg/m^3, with volume 1 m^3.
- Flow rates and volumes are constant within the system.
- The inline flow into the first tank is 1 m^3/s, and the out of line flow into each tank is .1 m^3/s.
- The concentration of the out of series contribution to each tank is selected for a normal distribution with Mean 5 mg/m^3 and Standard Deviation 0.3 mg/m^3

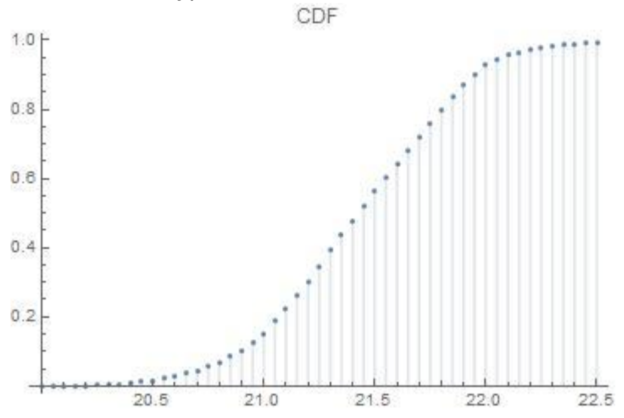
- To the right is shown is a single numerical solution to the parameters described above run from time 0 to 10 sec. The concentration for the fourth tank at time equal to 10
- Below is shown a histogram and cumulative frequency distribution for the fourth tank at time equal to 10 seconds, generated by solving the system 1000 times and recording the end state.



A typical numerical solution



Tank 4, Concentration Vs. Frequency



Tank 4, Concentration Vs. Cumulative frequency

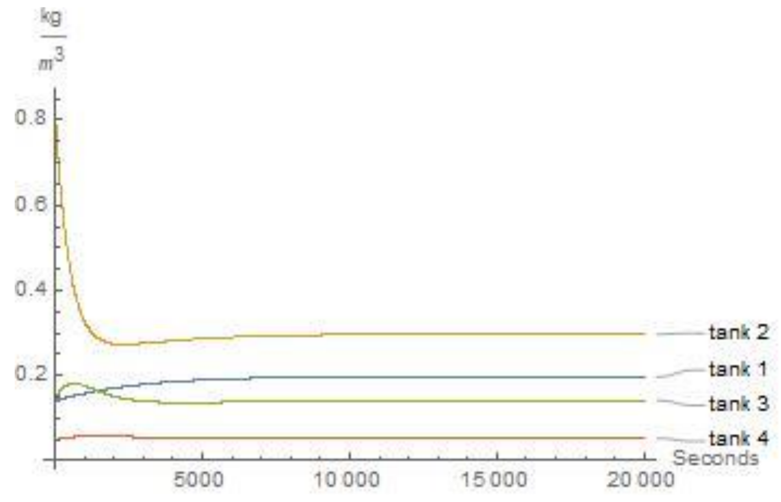
Mar. 2017 Data

Source: 2017 US Masters Project

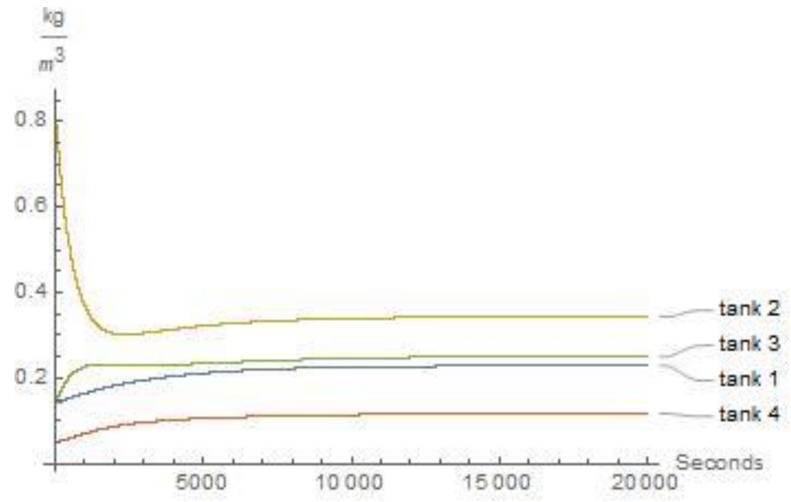
- Data obtained from a 2017 US Masters Project.
- Provided information on cross-sectional areas, flow rates, and TDS concentrations at different sampling sites.
- Flow rates for runoff from storm drains calculated at half and full flow from manning's equation.

Mar. Model Results

- Four tank model chosen because of availability of concentration data.
- Assumes constant volume and flowrates owing to lack of data for variability over time.
- Flow rate contributions from storm drains ran at both half and full flow.
- An arbitrary concentration of 0.1 kg/m^3 selected for runoff flow owing to lack of information.
- Histograms and cumulative frequency distributions not generated due to lack of information



Full flow from runoff



Half-Full flow from runoff

- Equilibrium conditions in both cases reached after approximately 11 hours.

Future Additions

Future information which would be interesting to collect and implement in the model:

- Volume information calculated from gauge station measurements allowing for volume variations with time
- Channel flow rates which vary with time, which could be calculated from gauge information using manning's equation.
- Concentration levels associated with runoff from storm drains and also time varying flow rate measurements for the runoff.
- Greater number of sampling locations

Conclusions

Though relatively simple, using a tank model to represent a natural stream requires a number of unrealistic assumptions:

- Infinitely Dilute: Assumes a species stays aqueous and chemical reactions do not occur; Increases in concentration do not reduce solubility and concentration only changes by advection between tanks.
- Well-Mixed: Assumes instantaneous distribution along a concentration gradient; The contents of any single tank are homogeneous.

For application to the Coulee, lack of sampling data, in addition to lack of channel information prevents a full utilization of the model, which in conjunction with unrealistic assumptions suggests that the model lacks sufficient sophistication and certainly observational support.

Works Cited

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