# Contents

<table>
<thead>
<tr>
<th>Table of Contents</th>
<th>iii</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>vii</td>
</tr>
</tbody>
</table>

## Calculus I

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Limits</td>
<td>1</td>
</tr>
<tr>
<td>1.1 An Introduction To Limits</td>
<td>7</td>
</tr>
<tr>
<td>1.2 Epsilon-Delta Definition of a Limit</td>
<td>15</td>
</tr>
<tr>
<td>1.3 Finding Limits Analytically</td>
<td>25</td>
</tr>
<tr>
<td>1.4 One Sided Limits</td>
<td>39</td>
</tr>
<tr>
<td>1.5 Limits Involving Infinity</td>
<td>47</td>
</tr>
<tr>
<td>1.6 Continuity</td>
<td>59</td>
</tr>
<tr>
<td>2 Derivatives</td>
<td>77</td>
</tr>
<tr>
<td>2.1 Instantaneous Rates of Change: The Derivative</td>
<td>77</td>
</tr>
<tr>
<td>2.2 Interpretations of the Derivative</td>
<td>92</td>
</tr>
<tr>
<td>2.3 Basic Differentiation Rules</td>
<td>100</td>
</tr>
<tr>
<td>2.4 The Product and Quotient Rules</td>
<td>109</td>
</tr>
<tr>
<td>2.5 The Chain Rule</td>
<td>121</td>
</tr>
<tr>
<td>2.6 Implicit Differentiation</td>
<td>132</td>
</tr>
<tr>
<td>3 The Graphical Behavior of Functions</td>
<td>143</td>
</tr>
<tr>
<td>3.1 Extreme Values</td>
<td>143</td>
</tr>
<tr>
<td>3.2 The Mean Value Theorem</td>
<td>152</td>
</tr>
<tr>
<td>3.3 Increasing and Decreasing Functions</td>
<td>158</td>
</tr>
<tr>
<td>3.4 Concavity and the Second Derivative</td>
<td>169</td>
</tr>
<tr>
<td>3.5 Curve Sketching</td>
<td>178</td>
</tr>
</tbody>
</table>
13.9 Lagrange Multipliers ........................................ 844

14 **Multiple Integration** .......................................... 851
   14.1 Iterated Integrals and Area .................................. 851
   14.2 Double Integration and Volume ............................... 861
   14.3 Double Integration with Polar Coordinates ................. 873
   14.4 Center of Mass ................................................. 881
   14.5 Surface Area ...................................................... 893
   14.6 Volume Between Surfaces and Triple Integration .......... 900
   14.7 Change of Variables in Multiple Integrals ................. 921

15 **Line and Surface Integrals** .................................... 929
   15.1 Line Integrals ................................................... 929
   15.2 Properties of Line Integrals .................................. 938
   15.3 Green’s Theorem ............................................... 947
   15.4 Surface Integrals and the Divergence Theorem .............. 954
   15.5 Stokes’ Theorem ................................................. 965
   15.6 Gradient, Divergence, Curl and Laplacian .................. 979

A **Solutions To Selected Problems** .................................. A.1

Index .................................................................. A.15
A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three-volume series on Calculus. The first part covers material taught in many “Calculus 1” courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6. The second text covers material often taught in “Calculus 2”: integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 7 through 10. The third text covers topics common in “Calculus 3” or “Multivariable Calculus”: parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 11 through 15. All three are available separately for free.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased separately.

A result of this splitting is that sometimes material is referenced that is not contained in the present text. The context should make it clear whether the “missing” material comes before or after the current portion. Downloading the appropriate pdf, or the entire APEX Calculus LT pdf, will give access to these topics.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High-level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower-level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for “why you should care,” and ends with a look ahead to see how the just-learned material helps address future problems. Additionally, each chapter includes a section zero, which provides a basic review and practice problems of pre-calculus skills. Since this content is a pre-requisite for calculus, reviewing and mastering these skills are considered your responsibility. This means that it is your responsibility to seek assistance outside of class from your instructor, a math resource center or other math tutoring available on-campus. A solid understanding of these skills are essential to your success in solving calculus problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth
of theorems, and the application of mathematical techniques. When you encounter a sentence you don’t understand, read it again. If it still doesn’t make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

*You don’t have to read every equation.* The examples generally show “all” the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

*Some proofs have been delayed until later (or omitted completely).* In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the more difficult proofs in the text. The interested reader is highly encouraged to find other proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem means and how to apply it without knowing fully why it is true.

*Work through the examples.* The best way to learn mathematics is to do it. Reading about it (or watching someone else do it) is a poor substitute. For this reason, every page has a place for you to put your notes so that you can work out the examples. That being said, sometimes it is useful to watch someone work through an example. For this reason, this text also provides links to online videos where someone is working through a similar problem. If you want even more videos, these are generally chosen from

- Khan Academy: [https://www.khanacademy.org/](https://www.khanacademy.org/)

Some other sites you may want to consider are

- Larry Green’s Calculus Videos: [http://www.ltcconline.net/greenl/courses/105/videos/VideoIndex.htm](http://www.ltcconline.net/greenl/courses/105/videos/VideoIndex.htm)

All of these sites are completely free (although some will ask you to donate). Here’s a sample one:

![Watch the video: Practical Advice for Those Taking College Calculus at](https://youtu.be/ILNfpJTZLxk)

**Interactive, 3D Graphics**

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.
As of this writing, the only PDF viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC / Mac / Unix / Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the PDF viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX — Affordable Print and Electronic texts

APEX is a consortium of authors who collaborate to produce high-quality, low-cost textbooks. The current textbook-writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text “cost” the authors of this book their time, and that was not enough. APEX Calculus would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution — Noncommercial 4.0 copyright. That means you can give the .pdf to
anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are “missing” or remove sections that your students won’t need. The source files can be found at https://github.com/APEXCalculus.

You can learn more at www.vmi.edu/APEX.

Greg Hartman

Creating APEX LT

Starting with the source at https://github.com/APEXCalculus, faculty at the University of North Dakota made several substantial changes to create APEX Late Transcendentals. The most obvious change was to rearrange the text to delay proving the derivative of transcendental functions until Calculus 2. UND added Sections 7.1 and 7.3, adapted several sections from other resources, created the prerequisite sections, included links to videos and Geogebra, and added several examples and exercises. In the end, every section had some changes (some more substantial than others), resulting in a document that is about 20% longer. The source files can now be found at https://github.com/teepeemm/APEXCalculusLT_Source.

Extra thanks are due to Michael Corral for allowing us to use portions of his Vector Calculus, available at www.mecmath.net/ (specifically, Sections 11.7, 13.9, and 14.7, and Chapter 15) and to Paul Dawkins for allowing us to use portions of his online math notes from tutorial.math.lamar.edu/ (specifically, Sections 8.5 and 9.7, as well as “Area with Parametric Equations” in Section 10.3). The work on Calculus III was partially supported by the NDUS OER Initiative.
Calculus III
11: VECTORS

This chapter introduces a new mathematical object, the vector. Defined in Section 11.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force is applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into “space.” That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

11.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the $x$-$y$ plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of solid objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in “2D,” we live in a “3D” world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point $P$ in space can be represented with an ordered triple, $P = (a, b, c)$, where $a$, $b$ and $c$ represent the relative position of $P$ along the $x$-, $y$- and $z$-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the right hand rule. This rule states that when the fingers of the right hand extend in the direction of the positive $x$-axis and curve toward the positive $y$-axis, then the extended thumb will point in the direction of the positive $z$-axis. (It may take some thought to verify this, but this system is inherently different from the one created by using

Figure 11.1: Illustrating the right hand rule. Figure courtesy of user:Schorsch12 / Wikimedia Commons / Public Domain.
Another way to view the rule is that when the index finger of the right hand extends in the direction of the positive $x$-axis, and the middle finger (bent “inward” so it is perpendicular to the palm) points along the positive $y$-axis, then the extended thumb will point in the direction of the positive $z$-axis.

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 11.2 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the $x$-$y$ plane is drawn in its standard way, with the $z$-axis down to the left. The perspective is that the paper represents the $x$-$y$ plane and the positive $z$-axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the $x$-$y$ plane as being a horizontal plane in, say, a room, where the positive $z$-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 11.3. The same point $P$ is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

### Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

**Definition 51 Distance In Space**

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance $D$ between $P$ and $Q$ is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$  

We refer to the line segment that connects points $P$ and $Q$ in space as $PQ$, and refer to the length of this segment as $\|PQ\|$. The above distance formula allows us to compute the length of this segment.

**Example 1 Length of a line segment**

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $PQ$ and find its length.
11.1 Introduction to Cartesian Coordinates in Space

The points $P$ and $Q$ are plotted in Figure 11.4; no special consideration needs to be made to draw the line segment connecting these two points; simply connect them with a straight line. One cannot actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 51, we have

$$
\|PQ\| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14}.
$$

Spheres

Just as a circle is the set of all points in the plane equidistant from a given point (its center), a sphere is the set of all points in space that are equidistant from a given point. Definition 51 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius $r$. If a point $P = (x, y, z)$ lies on the sphere, then $P$ is $r$ units from $C$; that is,

$$
\|PC\| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.
$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius $r$, as given in the following Key Idea.

**Key Idea 48 Standard Equation of a Sphere in Space**

The standard equation of the sphere with radius $r$, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Watch the video: Example of Equation of a Sphere at https://youtu.be/fE_PWxyohXQ

Example 2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$.

Notes:
To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

\begin{align*}
x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\
(x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) &= 14 \\
(x + 1)^2 + (y - 2)^2 + (z - 3)^2 &= 16
\end{align*}

The sphere is centered at \((-1, 2, 3)\) and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables \(x, y\) and \(z\) are all used. We now consider situations where surfaces are defined where one or two of these variables are absent.

**Introduction to Planes in Space**

The coordinate axes naturally define three planes (shown in Figure 11.5), the **coordinate planes**: the \(x-y\) plane, the \(y-z\) plane and the \(x-z\) plane. The \(x-y\) plane is characterized as the set of all points in space where the \(z\)-value is 0. This, in fact, gives us an equation that describes this plane: \(z = 0\). Likewise, the \(x-z\) plane is all points where the \(y\)-value is 0, characterized by \(y = 0\).

**Example 3**  
**A plane in three dimensions**  
The equation \(x = 2\) describes all points in space where the \(x\)-value is 2. This is a plane, parallel to the \(y-z\) coordinate plane, shown in Figure 11.6.

**Example 4**  
**Regions defined by planes**  
Sketch the region defined by the inequalities \(-1 \leq y \leq 2\).
11.1 Introduction to Cartesian Coordinates in Space

The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 11.7, which are parallel to the $x$-$z$ plane. Thus the region extends infinitely in the $x$ and $z$ directions, and is bounded by planes in the $y$ direction.

Cylinders

The equation $x = 1$ obviously lacks the $y$ and $z$ variables, meaning it defines points where the $y$ and $z$ coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ in space. In the plane, this equation describes a circle of radius 1, centered at the origin. In space, the $z$ coordinate is not specified, meaning it can take on any value. In Figure 11.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant $z$-value of $-1.5$, the middle one has a $z$-value of 0 and the top circle has a $z$-value of 1. By plotting all possible $z$-values, we get the surface shown in Figure 11.8 (b). This surface looks like a “tube,” or a “cylinder”, which leads to our next definition.

Definition 52 Cylinder
Let $C$ be a curve in a plane and let $L$ be a line not parallel to $C$. A cylinder is the set of all lines parallel to $L$ that pass through $C$. The curve $C$ is the directrix of the cylinder, and the lines are the rulings.

In this text, we consider curves $C$ that lie in planes parallel to one of the coordinate planes, and lines $L$ that are perpendicular to these planes, forming right cylinders. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3rd variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the $x$-$y$ plane is the directrix and the rulings are lines parallel to the $z$-axis. (Any circle shown in Figure 11.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 5 Graphing cylinders
Graph the cylinder following cylinders.

1. $z = y^2$
2. $x = \sin z$

Notes:
SOLUTION

1. We can view the equation $z = y^2$ as a parabola in the $y$-$z$ plane, as illustrated in Figure 11.9 (a). As $x$ does not appear in the equation, the rulings are lines through this parabola parallel to the $x$-axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

![Figure 11.9: Sketching the cylinder defined by $z = y^2$.](image)

2. We can view the equation $x = \sin z$ as a sine curve that exists in the $x$-$z$ plane, as shown in Figure 11.10 (a). The rules are parallel to the $y$ axis as the variable $y$ does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

![Figure 11.10: Sketching the cylinder defined by $x = \sin z$.](image)

**Surfaces of Revolution**

One of the applications of integration we learned previously was to find the volume of solids of revolution—solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the $x$-axis. Cross-sections of this surface parallel to the $y$-$z$ plane are circles, as shown in Figure 11.11(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius $r$. 

Notes:
The radius is a function of $x$; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 11.11(b) is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

**Key Idea 49  Surfaces of Revolution, Part 1**

Let $r$ be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the $x$-axis is $y^2 + z^2 = r(x)^2$.

2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the $y$-axis is $x^2 + z^2 = r(y)^2$.

3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the $z$-axis is $x^2 + y^2 = r(z)^2$.

**Example 6  Finding equation of a surface of revolution**

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the $z$-axis.

**Solution** Using Key Idea 49, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 11.12(a) and the surface is drawn in Figure 11.12(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 11.12(a).

This particular method of creating surfaces of revolution is limited. For instance, in Example 6.3.5 of Section 6.3 we found the volume of the solid formed by revolving $y = \sin x$ about the $y$-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the $x$-axis. Trying to rewrite $y = \sin x$ as a function of $y$ is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the $y$-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the $z$-axis. This will give us a more natural way of viewing the surface.

A value of $x$ is a measurement of distance from the $z$-axis. At the distance $r$, we plot a $z$-height of $f(r)$. When rotating $f(x)$ about the $z$-axis, we want all points a distance of $r$ from the $z$-axis in the $x$-$y$ plane to have a $z$-height of $f(r)$. All such...
points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing $r$ with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

**Key Idea 50 Surfaces of Revolution, Part 2**
Let $z = f(x)$, $x \geq 0$, be a curve in the $x$-$z$ plane. The surface formed by revolving this curve about the $z$-axis has equation $z = f(\sqrt{x^2 + y^2})$.

### Example 7 Finding equation of surface of revolution
Find the equation of the surface found by revolving $z = \sin x$ about the $z$-axis.

**Solution** Using Key Idea 50, the surface has equation $z = \sin(\sqrt{x^2 + y^2})$.

The curve and surface are graphed in Figure 11.13.

### Quadric Surfaces
Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

**Definition 53 Quadric Surface**
A **quadric surface** is the graph of the general second-degree equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$ 

When the coefficients $D$, $E$ or $F$ are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 11.14. If we intersect this shape with the plane $z = d$ (i.e., replace $z$ with $d$), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$
Divide both sides by \( d \):

\[
1 = \frac{x^2}{4d} + \frac{y^2}{d}.
\]

This describes an ellipse – so cross sections parallel to the \( x-y \) coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the \( x-z \) plane. For instance, letting \( y = 0 \) gives the equation \( z = \frac{x^2}{4} \), clearly a parabola. Intersecting with the plane \( x = 0 \) gives a cross section defined by \( z = y^2 \), another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.
Elliptic Paraboloid, \[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = d )</td>
<td>Parabola</td>
</tr>
<tr>
<td>( y = d )</td>
<td>Parabola</td>
</tr>
<tr>
<td>( z = d )</td>
<td>Ellipse</td>
</tr>
</tbody>
</table>

One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the \( z \) variable. The paraboloid will “open” in the direction of this variable’s axis. Thus \( x = y^2/a^2 + z^2/b^2 \) is an elliptic paraboloid that opens along the \( x \)-axis.

Multiplying the right hand side by \((-1)\) defines an elliptic paraboloid that “opens” in the opposite direction.

Elliptic Cone, \[ z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>Crossed Lines</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>Crossed Lines</td>
</tr>
<tr>
<td>( x = d )</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>( y = d )</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>( z = d )</td>
<td>Ellipse</td>
</tr>
</tbody>
</table>

One can rewrite the equation as \( z^2 - x^2/a^2 - y^2/b^2 = 0 \). The one variable with a positive coefficient corresponds to the axis that the cones “open” along.
Ellipsoid, \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = d)</td>
<td>Ellipse</td>
</tr>
<tr>
<td>(y = d)</td>
<td>Ellipse</td>
</tr>
<tr>
<td>(z = d)</td>
<td>Ellipse</td>
</tr>
</tbody>
</table>

If \(a = b = c \neq 0\), the ellipsoid is a sphere with radius \(a\); compare to Key Idea 48.

Hyperboloid of One Sheet, \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = d)</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>(y = d)</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>(z = d)</td>
<td>Ellipse</td>
</tr>
</tbody>
</table>

The one variable with a negative coefficient corresponds to the axis along which the hyperboloid “opens”.

643
Hyperboloid of Two Sheets, \[ \frac{x^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = d)</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>(y = d)</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>(z = d)</td>
<td>Ellipse</td>
</tr>
</tbody>
</table>

The one variable with a positive coefficient corresponds to the axis along which the hyperboloid “opens”. In the case illustrated, when \(|d| < |c|\), there is no trace in the plane \(z = d\).

Hyperbolic Paraboloid, \[ z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = d)</td>
<td>Parabola</td>
</tr>
<tr>
<td>(y = d)</td>
<td>Parabola</td>
</tr>
<tr>
<td>(z = d)</td>
<td>Hyperbola</td>
</tr>
</tbody>
</table>

The parabolic traces will open along the axis of the one variable that is raised to the first power.
Example 8  

**Sketching quadric surfaces**

Sketch the quadric surface defined by the given equation.

1. \[ y = \frac{x^2}{4} + \frac{z^2}{16} \]
2. \[ x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \]
3. \[ z = y^2 - x^2. \]

**SOLUTION**

1. \[ y = \frac{x^2}{4} + \frac{z^2}{16} : \]
   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is \( y \), we note the paraboloid opens along the \( y \)-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces \( x = 0 \) and \( z = 0 \) form parabolas that outline the shape.
   \[ x = 0: \] The trace is the parabola \( y = \frac{z^2}{16} \)
   \[ z = 0: \] The trace is the parabola \( y = \frac{x^2}{4} \).

   Graphing each trace in the respective plane creates a sketch as shown in Figure 11.15(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. \[ x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 : \]
   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.
   \[ x = 0: \] The trace is the ellipse \( \frac{y^2}{9} + \frac{z^2}{4} = 1 \). The major axis is along the \( y \)-axis with length 6 (as \( b = 3 \), the length of the axis is 6); the minor axis is along the \( z \)-axis with length 4.
   \[ y = 0: \] The trace is the ellipse \( x^2 + \frac{z^2}{4} = 1 \). The major axis is along the \( z \)-axis, and the minor axis has length 2 along the \( x \)-axis.
Chapter 11  Vectors

11.12 Vectors

1. \( z = 0 \): The trace is the ellipse \( x^2 + \frac{y^2}{9} = 1 \), with major axis along the \( y \)-axis.

Graphing each trace in the respective plane creates a sketch as shown in Figure 11.16(a). Filling in the surface gives Figure 11.16(b).

3. \( z = y^2 - x^2 \):

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the \( y-z \) and \( x-z \) planes:

\( x = 0 \): The trace is \( z = y^2 \), a parabola opening up in the \( y-z \) plane.

\( y = 0 \): The trace is \( z = -x^2 \), a parabola opening down in the \( x-z \) plane.

Sketching these two parabolas gives a sketch like that in Figure 11.17(a), and filling in the surface gives a sketch like (b).

**Example 9  Identifying quadric surfaces**

Consider the quadric surface shown in Figure 11.18. Which of the following equations best fits this surface?

\[
\begin{align*}
(a) \quad & x^2 - y^2 - \frac{z^2}{9} = 0 \\
(b) \quad & x^2 - y^2 - z^2 = 1 \\
(c) \quad & z^2 - x^2 - y^2 = 1 \\
(d) \quad & 4x^2 - y^2 - \frac{z^2}{9} = 1
\end{align*}
\]

**Solution**  The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to \( \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \).

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid “opens” along the \( x \)-axis, meaning \( x \) must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the \( z \)-direction than in the \( y \)-direction, so we need an equation where \( c > b \). This eliminates (b), leaving us with (d). We should verify that the equation given in (d), \( 4x^2 - y^2 - \frac{z^2}{9} = 1 \), fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the \( x \)-direction and is wider in the \( z \)-direction than in the \( y \). Now note the coefficient of the \( x \)-term. Rewriting \( 4x^2 \) in standard form, we have: \( 4x^2 = \frac{x^2}{(1/2)^2} \). Thus when \( y = 0 \) and \( z = 0 \), \( x \) must be \( 1/2 \); i.e., each hyperboloid “starts” at \( x = 1/2 \). This matches our figure.

We conclude that \( 4x^2 - y^2 - \frac{z^2}{9} = 1 \) best fits the graph.
This section has introduced points in space and shown how equations can describe surfaces. The next sections explore vectors, an important mathematical object that we'll use to explore curves in space.
Exercises 11.1

Terms and Concepts

1. Axes drawn in space must conform to the ________ __________ rule.
2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.
3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.
4. Which quadric surface looks like a Pringles® chip?
5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the $x$-axis, what quadric surface is formed?
6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the $y$-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2), B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.
8. The points $A = (1, 1, 3), B = (3, 2, 7), C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?
9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.
10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

In Exercises 11–14, describe the region in space defined by the inequalities.

11. $x^2 + y^2 + z^2 < 1$
12. $0 \leq x \leq 3$
13. $x \geq 0, y \geq 0, z \geq 0$
14. $y \geq 3$

In Exercises 15–18, sketch the cylinder in space.

15. $z = x^3$
16. $y = \cos z$
17. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
18. $y = \frac{1}{x}$

In Exercises 19–22, give the equation of the surface of revolution described.

19. Revolve $z = \frac{1}{1 + y^2}$ about the $y$-axis.
20. Revolve $y = x^2$ about the $x$-axis.
21. Revolve $z = x^2$ about the $z$-axis.
22. Revolve $z = 1/x$ about the $z$-axis.

In Exercises 23–26, a quadric surface is sketched. Determine which of the given equations best fits the graph.

23. (a) $x = y^2 + \frac{z^2}{9}$ (b) $x = y^2 + \frac{z^2}{3}$
24. (a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$
25. (a) $x^2 + \frac{y^2}{3} + \frac{z^2}{2} = 1$ (b) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
26. (a) $y^2 - x^2 - z^2 = 1$ (b) $y^2 + x^2 - z^2 = 1$

In Exercises 27–32, sketch the quadric surface.

27. $z - y^2 + x^2 = 0$
28. $z^2 = x^2 + \frac{y^2}{4}$
29. $x = -y^2 - z^2$
30. $16x^2 - 16y^2 - 16z^2 = 1$
31. $\frac{x^2}{9} - y^2 + \frac{z^2}{25} = 1$
32. $4x^2 + 2y^2 + z^2 = 4$
11.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction (“... with winds from the southeast gusting up to 30 mph ...”). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, direction is important. Because of this, we study vectors, mathematical objects that convey both magnitude and direction information.

One “bare-bones” definition of a vector is based on what we wrote above: “a vector is a mathematical object with magnitude and direction parameters.” This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

**Definition 54 Vector**

A vector is a directed line segment.

Given points \( P \) and \( Q \) (either in the plane or in space), we denote with \( \overrightarrow{PQ} \) the vector from \( P \) to \( Q \). The point \( P \) is said to be the initial point of the vector, and the point \( Q \) is the terminal point.

The magnitude, length or norm of \( \overrightarrow{PQ} \) is the length of the line segment \( PQ \): \( \| \overrightarrow{PQ} \| = \| PQ \| \).

Two vectors are equal if they have the same magnitude and direction.

Figure 11.19 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use \( \mathbb{R}^2 \) (pronounced “r two”) to represent all the vectors in the plane, and use \( \mathbb{R}^3 \) (pronounced “r three”) to represent all the vectors in space.

Consider the vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{RS} \) as shown in Figure 11.20. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point.

---

**Notes:**
to reach the terminal point. One can analyze this movement to measure the magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through \(P\) and \(Q\) or \(R\) and \(S\)). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is displacement; that is, how far in the \(x\), \(y\) and possibly \(z\) directions the terminal point is from the initial point. Both the vectors \(\overrightarrow{PQ}\) and \(\overrightarrow{RS}\) in Figure 11.20 have an \(x\)-displacement of 2 and a \(y\)-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose \(x\)-displacement is \(a\) and whose \(y\)-displacement is \(b\) will have terminal point \((a, b)\) when the initial point is the origin, \((0, 0)\). This leads us to a definition of a standard and concise way of referring to vectors.

### Definition 55

**Component Form of a Vector**

1. The component form of a vector \(\vec{v}\) in \(\mathbb{R}^2\), whose terminal point is \((a, b)\) when its initial point is \((0, 0)\), is \(\langle a, b \rangle\).

2. The component form of a vector \(\vec{v}\) in \(\mathbb{R}^3\), whose terminal point is \((a, b, c)\) when its initial point is \((0, 0, 0)\), is \(\langle a, b, c \rangle\).

The numbers \(a\), \(b\) (and \(c\), respectively) are the components of \(\vec{v}\).

**Note:** Instead of \(\vec{v}\), some texts use boldface: \(\mathbf{v}\). The advantage of \(\mathbf{v}\) is that it tends to be easier to read. The advantage of \(\vec{v}\) is that it’s easier to write.

It follows from the definition that the component form of the vector \(\overrightarrow{PQ}\), where \(P = (x_1, y_1)\) and \(Q = (x_2, y_2)\) is

\[
\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle
\]

in space, where \(P = (x_1, y_1, z_1)\) and \(Q = (x_2, y_2, z_2)\), the component form of \(\overrightarrow{PQ}\) is

\[
\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.
\]

We practice using this notation in the following example.
Example 1 Using component form notation for vectors

1. Sketch the vector \( \vec{v} = \langle 2, -1 \rangle \) starting at \( P = (3, 2) \) and find its magnitude.

2. Find the component form of the vector \( \vec{w} \) whose initial point is \( R = (-3, -2) \) and whose terminal point is \( S = (-1, 2) \).

3. Sketch the vector \( \vec{u} = \langle 2, -1, 3 \rangle \) starting at the point \( Q = (1, 1, 1) \) and find its magnitude.

Solution

1. Using \( P \) as the initial point, we move 2 units in the positive \( x \)-direction and \(-1 \) units in the positive \( y \)-direction to arrive at the terminal point \( P' = (5, 1) \), as drawn in Figure 11.21(a).

The magnitude of \( \vec{v} \) is determined directly from the component form:

\[
\| \vec{v} \| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.
\]

2. Using the paragraph following Definition 55, we have

\[
\overrightarrow{RS} = (1 - (-3), 2 - (-2)) = (2, 4).
\]

One can readily see from Figure 11.21(a) that the \( x \)- and \( y \)-displacement of \( \overrightarrow{RS} \) is 2 and 4, respectively, as the component form suggests.

3. Using \( Q \) as the initial point, we move 2 units in the positive \( x \)-direction, \(-1 \) unit in the positive \( y \)-direction, and 3 units in the positive \( z \)-direction to arrive at the terminal point \( Q' = (3, 0, 4) \), illustrated in Figure 11.21(b).

The magnitude of \( \vec{u} \) is:

\[
\| \vec{u} \| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.
\]

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an algebra on vectors.

Notes:
Chapter 11  Vectors

Definition 56  Vector Algebra

1. Let \( \mathbf{u} = \langle u_1, u_2 \rangle \) and \( \mathbf{v} = \langle v_1, v_2 \rangle \) be vectors in \( \mathbb{R}^2 \), and let \( c \) be a scalar.
   
   (a) The addition, or sum, of the vectors \( \mathbf{u} \) and \( \mathbf{v} \) is the vector
   
   \[ \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle . \]
   
   (b) The scalar product of \( c \) and \( \mathbf{v} \) is the vector
   
   \[ c\mathbf{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle . \]

2. Let \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) be vectors in \( \mathbb{R}^3 \), and let \( c \) be a scalar.
   
   (a) The addition, or sum, of the vectors \( \mathbf{u} \) and \( \mathbf{v} \) is the vector
   
   \[ \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle . \]
   
   (b) The scalar product of \( c \) and \( \mathbf{v} \) is the vector
   
   \[ c\mathbf{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle . \]

In short, we say addition and scalar multiplication are computed “component-wise.”

Example 2  Adding vectors

Sketch the vectors \( \mathbf{u} = \langle 1, 3 \rangle \), \( \mathbf{v} = \langle 2, 1 \rangle \) and \( \mathbf{u} + \mathbf{v} \) all with initial point at the origin.

**SOLUTION**  We first compute \( \mathbf{u} + \mathbf{v} \).

\[ \mathbf{u} + \mathbf{v} = \langle 1, 3 \rangle + \langle 2, 1 \rangle \]
\[ = \langle 3, 4 \rangle . \]

These are all sketched in Figure 11.22.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding \( \mathbf{u} + \mathbf{v} \) suggests the following idea:

“Starting at an initial point, go out \( \mathbf{u} \), then go out \( \mathbf{v} \).”

---

Notes:
This idea is sketched in Figure 11.23, where the initial point of \( \vec{v} \) is the terminal point of \( \vec{u} \). This is known as the “Head to Tail Rule” of adding vectors. Vector addition is very important. For instance, if the vectors \( \vec{u} \) and \( \vec{v} \) represent forces acting on a body, the sum \( \vec{u} + \vec{v} \) gives the resulting force. Because of various physical applications of vector addition, the sum \( \vec{u} + \vec{v} \) is often referred to as the resultant vector, or just the “resultant.”

Analytically, it is easy to see that \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \). Figure 11.23 also gives a graphical representation of this, using gray vectors. Note that the vectors \( \vec{u} \) and \( \vec{v} \), when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector \( \vec{u} + \vec{v} \) is defined by forming the parallelogram defined by the vectors \( \vec{u} \) and \( \vec{v} \); the initial point of \( \vec{u} + \vec{v} \) is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in \( \mathbb{R}^3 \) as well.

It follows from the properties of the real numbers and Definition 56 that

\[
\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}.
\]

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

**Example 3 Vector Subtraction**

Let \( \vec{u} = \langle 3, 1 \rangle \) and \( \vec{v} = \langle 1, 2 \rangle \). Compute and sketch \( \vec{u} - \vec{v} \).

**Solution** The computation of \( \vec{u} - \vec{v} \) is straightforward, and we show all steps below. Usually the formal step of multiplying by \((-1)\) is omitted and we “just subtract.”

\[
\begin{align*}
\vec{u} - \vec{v} &= \vec{u} + (-1)\vec{v} \\
&= \langle 3, 1 \rangle + \langle -1, -2 \rangle \\
&= \langle 2, -1 \rangle .
\end{align*}
\]

Figure 11.24 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum \( \vec{u} + (-\vec{v}) \). The figure also illustrates how \( \vec{u} - \vec{v} \) can be obtained by looking only at the terminal points of \( \vec{u} \) and \( \vec{v} \) (when their initial points are the same).

**Example 4 Scaling vectors**

1. Sketch the vectors \( \vec{v} = \langle 2, 1 \rangle \) and \( 2\vec{v} \) with initial point at the origin.
2. Compute the magnitudes of \( \vec{v} \) and \( 2\vec{v} \).
Chapter 11  Vectors

**Chapter 11 Vectors**

1. We compute $2\mathbf{v}$:

$$2\mathbf{v} = 2 \langle 2, 1 \rangle = \langle 4, 2 \rangle .$$

Both $\mathbf{v}$ and $2\mathbf{v}$ are sketched in Figure 11.25. Make note that $2\mathbf{v}$ does not start at the terminal point of $\mathbf{v}$; rather, its initial point is also the origin.

2. The figure suggests that $2\mathbf{v}$ is twice as long as $\mathbf{v}$. We compute their magnitudes to confirm this.

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}.$$  
$$\|2\mathbf{v}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}.$$

As we suspected, $2\mathbf{v}$ is twice as long as $\mathbf{v}$.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\mathbf{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0, 0 \rangle$; in $\mathbb{R}^3$, it is $\langle 0, 0, 0 \rangle$. Usually the context makes it clear whether $\mathbf{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

**SOLUTION**

Notes:
Theorem 86  Properties of Vector Operations
The following are true for all scalars \( c \) and \( d \), and for all vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \), where \( \vec{u}, \vec{v} \) and \( \vec{w} \) are all in \( \mathbb{R}^2 \) or where \( \vec{u}, \vec{v} \) and \( \vec{w} \) are all in \( \mathbb{R}^3 \):

1. \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \)  Commutative Property
2. \( (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \)  Associative Property
3. \( \vec{v} + \vec{0} = \vec{v} \)  Additive Identity
4. \( (cd)\vec{v} = c(d\vec{v}) \)
5. \( c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \)  Distributive Property
6. \( (c + d)\vec{v} = c\vec{v} + d\vec{v} \)  Distributive Property
7. \( 0\vec{v} = \vec{0} \)
8. \( \|c\vec{v}\| = |c| \cdot \|\vec{v}\| \)
9. \( \|\vec{u}\| = 0 \) if, and only if, \( \vec{u} = \vec{0} \).

As stated before, each vector \( \vec{v} \) conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as \( \|\vec{v}\| \).

Unit vectors are a way of extracting just the direction information from a vector.

Definition 57  Unit Vector
A unit vector \( \vec{v} \) is a vector \( \vec{v} \) with a magnitude of 1; that is,

\[ \|\vec{v}\| = 1. \]

Consider this scenario: you are given a vector \( \vec{v} \) and are told to create a vector of length 10 in the direction of \( \vec{v} \). How does one do that? If we knew that \( \vec{u} \) was the unit vector in the direction of \( \vec{v} \), the answer would be easy: \( 10\vec{u} \). So how do we find \( \vec{u} \)?

Property 8 of Theorem 86 holds the key. If we divide \( \vec{v} \) by its magnitude, it becomes a vector of length 1. Consider:

\[
\frac{1}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|} \|\vec{v}\| \quad \text{(we can pull out } \frac{1}{\|\vec{v}\|} \text{ as it is a scalar)}
\]

\[ = 1. \]

Notes:
Chapter 11  Vectors

So the vector of length 10 in the direction of \( \vec{v} \) is \( 10 \frac{\vec{v}}{||\vec{v}||} \). An example will make this more clear.

**Example 5  Using Unit Vectors**

Let \( \vec{v} = \langle 3, 1 \rangle \) and let \( \vec{w} = \langle 1, 2, 2 \rangle \).

1. Find the unit vector in the direction of \( \vec{v} \).
2. Find the unit vector in the direction of \( \vec{w} \).
3. Find the vector in the direction of \( \vec{v} \) with magnitude 5.

**Solution**

1. We find \( ||\vec{v}|| = \sqrt{10} \). So the unit vector \( \vec{u} \) in the direction of \( \vec{v} \) is
   \[
   \vec{u} = \frac{1}{\sqrt{10}} \vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.
   \]

2. We find \( ||\vec{w}|| = 3 \), so the unit vector \( \vec{u} \) in the direction of \( \vec{w} \) is
   \[
   \vec{u} = \frac{1}{3} \vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.
   \]

3. To create a vector with magnitude 5 in the direction of \( \vec{v} \), we multiply the unit vector \( \vec{u} \) by 5. Thus \( 5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10} \rangle \) is the vector we seek. This is sketched in Figure 11.26.

The basic formation of the unit vector \( \vec{u} \) in the direction of a vector \( \vec{v} \) leads to an interesting equation. It is:

\[
\vec{v} = ||\vec{v}|| \frac{1}{||\vec{v}||} \vec{v}.
\]

We rewrite the equation with parentheses to make a point:

\[
\vec{v} = \left( \frac{1}{||\vec{v}||} \right) \vec{v}.
\]

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying only direction information. Identifying unit vectors with direction allows us to define parallel vectors.

Notes:
11.2 An Introduction to Vectors

Definition 58 Parallel Vectors

1. Unit vectors \( \vec{u}_1 \) and \( \vec{u}_2 \) are parallel if \( \vec{u}_1 = \pm \vec{u}_2 \).

2. Nonzero vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) are parallel if their respective unit vectors are parallel.

It is equivalent to say that vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) are parallel if there is a scalar \( c \neq 0 \) such that \( \vec{v}_1 = c\vec{v}_2 \) (see marginal note).

If one graphed all unit vectors in \( \mathbb{R}^2 \) with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in \( \mathbb{R}^2 \) is \( \langle \cos \theta, \sin \theta \rangle \) for some angle \( \theta \).

A similar construction in \( \mathbb{R}^3 \) shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in \( \mathbb{R}^2 \). Important concepts about unit vectors are given in Key Idea 51 below.

Key Idea 51 Unit Vectors

1. The unit vector in the direction of \( \vec{v} \) is

\[
\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}.
\]

2. A vector \( \vec{u} \) in \( \mathbb{R}^2 \) is a unit vector if, and only if, its component form is \( \langle \cos \theta, \sin \theta \rangle \) for some angle \( \theta \).

3. A vector \( \vec{u} \) in \( \mathbb{R}^3 \) is a unit vector if, and only if, its component form is \( \langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle \) for some angles \( \theta \) and \( \varphi \).

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 6 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 11.27. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

Notes: \( \vec{0} \) is directionless; because \( \|\vec{0}\| = 0 \), there is no unit vector in the “direction” of \( \vec{0} \). Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, \( \vec{0} \) is parallel to all vectors as \( \vec{0} = c\vec{v} \) for all \( \vec{v} \).

We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that \( \vec{0} \) is parallel to all vectors if they desire. (See also the marginal note on page 680.)

Figure 11.27: A diagram of a weight hanging from 2 chains in Example 6.
Knowing that gravity is pulling the 50lb weight straight down, we can create a vector \( \vec{F} \) to represent this force.

\[
\vec{F} = 50 \langle 0, -1 \rangle = \langle 0, -50 \rangle.
\]

We can view each chain as “pulling” the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let \( \vec{F}_1 \) represent the force from the chain making an angle of 30° with the vertical, and let \( \vec{F}_2 \) represent the force from the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 11.28), and apply Key Idea 51. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use \( m_1 \) and \( m_2 \) to represent them.

\[
\vec{F}_1 = m_1 \langle \cos 120°, \sin 120° \rangle \\
\vec{F}_2 = m_2 \langle \cos 45°, \sin 45° \rangle
\]

As the weight is not moving, we know the sum of the forces is \( \vec{0} \). This gives:

\[
\langle 0, -50 \rangle + m_1 \langle \cos 120°, \sin 120° \rangle + m_2 \langle \cos 45°, \sin 45° \rangle = \vec{0}
\]

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

\[
m_1 \cos 120° + m_2 \cos 45° = 0 \\
m_1 \sin 120° + m_2 \sin 45° = 50
\]

This is a simple 2-equation, 2-unknown system of linear equations. We leave it to the reader to verify that the solution is

\[
\begin{align*}
m_1 &= 50(\sqrt{3} - 1) \text{lb} \\
m_2 &= \frac{50\sqrt{2}}{1 + \sqrt{3}} \text{lb}
\end{align*}
\]

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the vertical components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an “additional” horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component-wise. Because such calculations are often necessary, the standard unit vectors can be useful.
**Definition 59 Standard Unit Vectors**

1. In $\mathbb{R}^2$, the standard unit vectors are
   \[ \vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle. \]

2. In $\mathbb{R}^3$, the standard unit vectors are
   \[ \vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle. \]

**Example 7 Using standard unit vectors**

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.

   **Solution**
   \[
   \vec{v} = \langle 2, -3 \rangle \\
   = \langle 2, 0 \rangle + \langle 0, -3 \rangle \\
   = 2 \langle 1, 0 \rangle - 3 \langle 0, 1 \rangle \\
   = 2\vec{i} - 3\vec{j}
   \]

2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

   \[
   \vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k} \\
   = \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\
   = \langle 4, -5, 2 \rangle
   \]

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering texts use that notation.

**Example 8 Finding Component Force**

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 11.29. What angle will the chain make with the vertical as a result of the wind’s pushing? How much higher will the weight be?

---

**Notes:**

Figure 11.29: A figure of a weight being pushed by the wind in Example 8.
The force of the wind is represented by the vector \( \vec{F}_w = 5\hat{i} \).

The force of gravity on the weight is represented by \( \vec{F}_g = -25\hat{j} \). The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

\[
\vec{F}_c = m (\cos \varphi, \sin \varphi) = m \cos \varphi \hat{i} + m \sin \varphi \hat{j}
\]

for some magnitude \( m \) and some angle with the horizontal \( \varphi \). (Note: \( \theta \) is the angle the chain makes with the vertical; \( \varphi \) is the angle with the horizontal.)

As the weight is at equilibrium, the sum of the forces is \( \vec{0} \):

\[
\vec{F}_c + \vec{F}_w + \vec{F}_g = \vec{0}
\]

\[
m \cos \varphi \hat{i} + m \sin \varphi \hat{j} + 5\hat{i} - 25\hat{j} = \vec{0}
\]

Thus the sum of the \( \hat{i} \) and \( \hat{j} \) components are 0, leading us to the following system of equations:

\[
5 + m \cos \varphi = 0 \quad (11.1)
\]

\[
-25 + m \sin \varphi = 0 \quad (11.2)
\]

This is enough to determine \( \vec{F}_c \) already, as we know \( m \cos \varphi = -5 \) and \( m \sin \varphi = 25 \). Thus \( \vec{F}_c = (-5, 25) \). We can use this to find the magnitude \( m \):

\[
m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \text{lb}.
\]

We can then use either equality from Equation (11.2) to solve for \( \varphi \). We choose the first equality as using arccosine will return an angle in the 2\text{nd} quadrant:

\[
5 + 5\sqrt{26} \cos \varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.
\]

Subtracting \( 90^\circ \) from this angle gives us an angle of \( 11.31^\circ \) with the vertical.

We can now use trigonometry to find out how high the weight is lifted. Figure 11.29 shows that a right triangle is formed with the 2\text{ft} chain as the hypotenuse. We have found that the interior angle is \( 11.31^\circ \). The length of the adjacent side (in the diagram, the dashed vertical line) is \( 2 \cos 11.31^\circ \approx 1.96 \text{ft} \). Thus the weight is lifted by about 0.04\text{ft}, almost 1/2\text{in}.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the dot product and the cross product. The next two sections explore each in turn.

Notes:
Exercises 11.2

Terms and Concepts

1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.
2. What is the difference between (1, 2) and (1, 2) ?
3. What is a unit vector?
4. What does it mean for two vectors to be parallel?
5. What effect does multiplying a vector by $-2$ have?

Problems

In Exercises 6–9, points $P$ and $Q$ are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.

6. $P = (2, -1), \quad Q = (3, 5)$
7. $P = (3, 2), \quad Q = (7, -2)$
8. $P = (0, 3, -1), \quad Q = (6, 2, 5)$
9. $P = (2, 1, 2), \quad Q = (4, 3, 2)$
10. Let $\vec{u} = \langle 1, -2 \rangle$ and $\vec{v} = \langle 1, 1 \rangle$.
   a) Find $\vec{u} + \vec{v}, \vec{u} - \vec{v}, 2\vec{u} - 3\vec{v}$.
   b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
   c) Find $\vec{x}$ where $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$.

11. Let $\vec{u} = \langle 1, -1, -1 \rangle$ and $\vec{v} = \langle 2, 1, 2 \rangle$.
   a) Find $\vec{u} + \vec{v}, \vec{u} - \vec{v}, \pi\vec{u} - \sqrt{2}\vec{v}$.
   b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
   c) Find $\vec{x}$ where $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$.

In Exercises 12–15, sketch $\vec{u}, \vec{v}, \vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ on the same axes.

In Exercises 16–19, find $|\vec{u}|, |\vec{v}|, |\vec{u} + \vec{v}|$ and $|\vec{u} - \vec{v}|$.

16. $\vec{u} = \langle 2, 1 \rangle, \quad \vec{v} = \langle 3, -2 \rangle$
17. $\vec{u} = \langle -3, 2, 2 \rangle, \quad \vec{v} = \langle 1, -1, 1 \rangle$
18. $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle -3, -6 \rangle$
19. $\vec{u} = \langle 2, -3, 6 \rangle, \quad \vec{v} = \langle 10, -15, 30 \rangle$
20. Under what conditions is $|\vec{u}| + |\vec{v}| = |\vec{u} + \vec{v}|$?

In Exercises 21–24, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

21. $\vec{v} = \langle 3, 7 \rangle$
22. $\vec{v} = \langle 6, 8 \rangle$
23. $\vec{v} = \langle 1, -2, 2 \rangle$
24. $\vec{v} = \langle 2, -2, 2 \rangle$

25. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.
26. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.
27. Verify, from Key Idea 51, that $\vec{u} = \langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle$ is a unit vector for all angles $\theta$ and $\varphi$.

A weight of 100lb is suspended from two chains, making angles with the vertical of $\theta$ and $\varphi$ as shown in the figure below.

In Exercises 28–31, angles $\theta$ and $\varphi$ are given. Find the force applied to each chain.

28. $\theta = 30^\circ, \quad \varphi = 30^\circ$
29. $\theta = 60^\circ, \quad \varphi = 60^\circ$
30. $\theta = 20^\circ, \quad \varphi = 15^\circ$
31. $\theta = 0^\circ, \quad \varphi = 0^\circ$

A weight of $m$ lb is suspended from a chain of length $\ell$ while a constant force of $F_o$ pushes the weight to the right, making an angle of $\theta$ with the vertical, as shown in the figure below.
In Exercises 32–35, a force $\vec{F}_w$ and length $\ell$ are given. Find the angle $\theta$ and the height the weight is lifted as it moves to the right.

32. $\vec{F}_w = 1\text{lb}$, $\ell = 1\text{ft}$, $p = 1\text{lb}$
33. $\vec{F}_w = 1\text{lb}$, $\ell = 1\text{ft}$, $p = 10\text{lb}$
34. $\vec{F}_w = 1\text{lb}$, $\ell = 10\text{ft}$, $p = 1\text{lb}$
35. $\vec{F}_w = 10\text{lb}$, $\ell = 10\text{ft}$, $p = 1\text{lb}$
### 11.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces a multiplication on vectors called the **dot product**.

**Definition 60 Dot Product**

1. Let \( \vec{u} = \langle u_1, u_2 \rangle \) and \( \vec{v} = \langle v_1, v_2 \rangle \) in \( \mathbb{R}^2 \). The **dot product** of \( \vec{u} \) and \( \vec{v} \), denoted \( \vec{u} \cdot \vec{v} \), is
   \[
   \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.
   \]

2. Let \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) in \( \mathbb{R}^3 \). The **dot product** of \( \vec{u} \) and \( \vec{v} \), denoted \( \vec{u} \cdot \vec{v} \), is
   \[
   \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.
   \]

Note how this product of vectors returns a **scalar**, not another vector.

We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

**Example 1 Evaluating dot products**

1. Let \( \vec{u} = \langle 1, 2 \rangle \) and \( \vec{v} = \langle 3, -1 \rangle \) in \( \mathbb{R}^2 \). Find \( \vec{u} \cdot \vec{v} \).

2. Let \( \vec{x} = \langle 2, -2, 5 \rangle \) and \( \vec{y} = \langle -1, 0, 3 \rangle \) in \( \mathbb{R}^3 \). Find \( \vec{x} \cdot \vec{y} \).

**SOLUTION**

1. Using Definition 60, we have
   \[
   \vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.
   \]

2. Using the definition, we have
   \[
   \vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.
   \]

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

---

Notes:
Chapter 11  Vectors

**Theorem 87  Properties of the Dot Product**
Let \( \vec{u}, \vec{v} \) and \( \vec{w} \) be vectors in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) and let \( c \) be a scalar.

1. \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)  
   Commutative Property
2. \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \)  
   Distributive Property
3. \( c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) \)
4. \( \vec{0} \cdot \vec{v} = 0 \)
5. \( \vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \)

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product mean?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors \( \vec{u} \) and \( \vec{v} \) in the plane, an angle \( \theta \) is clearly formed when \( \vec{u} \) and \( \vec{v} \) are drawn with the same initial point as illustrated in Figure 11.30(a). (We always take \( \theta \) to be the angle in \([0, \pi]\) as two angles are actually created.)

The same is also true of 2 vectors in space: given \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^3 \) with the same initial point, there is a plane that contains both \( \vec{u} \) and \( \vec{v} \). (When \( \vec{u} \) and \( \vec{v} \) are colinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle \( \theta \) between them (and again, \( 0 \leq \theta \leq \pi \)). This is illustrated in Figure 11.30(b).

The following theorem connects this angle \( \theta \) to the dot product of \( \vec{u} \) and \( \vec{v} \).

**Theorem 88  The Dot Product and Angles**
Let \( \vec{u} \) and \( \vec{v} \) be vectors in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Then
\[
\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta,
\]
where \( \theta, 0 \leq \theta \leq \pi \), is the angle between \( \vec{u} \) and \( \vec{v} \).

When \( \theta \) is an acute angle (i.e., \( 0 \leq \theta < \pi/2 \)), \( \cos \theta \) is positive; when \( \theta = \pi/2 \), \( \cos \theta = 0 \); when \( \theta \) is an obtuse angle (\( \pi/2 < \theta \leq \pi \)), \( \cos \theta \) is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 11.31.

Notes:
11.3 The Dot Product

We can use Theorem 88 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem’s equation as

\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \quad \Leftrightarrow \quad \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).
\]

We practice using this theorem in the following example.

**Example 2** Using the dot product to find angles

Let \( \vec{u} = \langle 3, 1 \rangle \), \( \vec{v} = \langle -2, 6 \rangle \) and \( \vec{w} = \langle -4, 3 \rangle \), as shown in Figure 11.32. Find the angles \( \alpha \), \( \beta \) and \( \theta \).

**SOLUTION** We start by computing the magnitude of each vector.

\[
\|\vec{u}\| = \sqrt{10}; \quad \|\vec{v}\| = 2\sqrt{10}; \quad \|\vec{w}\| = 5.
\]

We now apply Theorem 88 to find the angles.

\[
\alpha = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{(\sqrt{10})(2\sqrt{10})} \right) = \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.
\]
\beta = \cos^{-1}\left( \frac{\vec{v} \cdot \vec{w}}{\sqrt{2}(\sqrt{10})(5)} \right) \\
= \cos^{-1}\left( \frac{26}{10\sqrt{10}} \right) \\
\approx 0.6055 \approx 34.7^\circ.

\theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{w}}{\sqrt{10}(5)} \right) \\
= \cos^{-1}\left( \frac{-9}{5\sqrt{10}} \right) \\
\approx 2.1763 \approx 124.7^\circ.

We see from our computation that \( \alpha + \beta = \theta \), as indicated by Figure 11.32. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

Example 3  
Using the dot product to find angles
Let \( \vec{u} = (1, 1, 1) \), \( \vec{v} = (-1, 3, -2) \) and \( \vec{w} = (-5, 1, 4) \), as illustrated in Figure 11.33. Find the angle between each pair of vectors.

**SOLUTION**

1. Between \( \vec{u} \) and \( \vec{v} \):

\[ \theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \]
\[ = \cos^{-1}\left( \frac{0}{\sqrt{3}\sqrt{14}} \right) \]
\[ = \frac{\pi}{2}. \]

2. Between \( \vec{u} \) and \( \vec{w} \):

\[ \theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} \right) \]
\[ = \cos^{-1}\left( \frac{0}{\sqrt{3}\sqrt{42}} \right) \]
\[ = \frac{\pi}{2}. \]

Notes:
3. Between \( \vec{v} \) and \( \vec{w} \):

\[
\theta = \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||} \right) \\
= \cos^{-1} \left( \frac{0}{\sqrt{14} \cdot \sqrt{42}} \right) \\
= \frac{\pi}{2}.
\]

While our work shows that each angle is \( \pi/2 \), i.e., 90\(^\circ\), none of these angles looks to be a right angle in Figure 11.33. Such is the case when drawing three-dimensional objects on the page.

All three angles between these vectors was \( \pi/2 \), or 90\(^\circ\). We know from geometry and everyday life that 90\(^\circ\) angles are “nice” for a variety of reasons, so it should seem significant that these angles are all \( \pi/2 \). Notice the common feature in each calculation (and also the calculation of \( \alpha \) in Example 2): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term orthogonal, which is essentially synonymous to perpendicular.

**Definition 61  Orthogonal**

Vectors \( \vec{u} \) and \( \vec{v} \) are **orthogonal** if their dot product is 0.

**Example 4  Finding orthogonal vectors**

Let \( \vec{u} = \langle 3, 5 \rangle \) and \( \vec{v} = \langle 1, 2, 3 \rangle \).

1. Find two vectors in \( \mathbb{R}^2 \) that are orthogonal to \( \vec{u} \).
2. Find two non–parallel vectors in \( \mathbb{R}^3 \) that are orthogonal to \( \vec{v} \).

**Solution**

1. Recall that a line perpendicular to a line with slope \( m \) has slope \( -1/m \), the “opposite reciprocal slope.” We can think of the slope of \( \vec{u} \) as \( 5/3 \), its “rise over run.” A vector orthogonal to \( \vec{u} \) will have slope \( -3/5 \). There are many such choices, though all parallel:

\( \langle -5, 3 \rangle \) or \( \langle 5, -3 \rangle \) or \( \langle -10, 6 \rangle \) or \( \langle 15, -9 \rangle \), etc.

**Note:** The term perpendicular originally referred to lines. As mathematics progressed, the concept of “being at right angles to” was applied to other objects, such as vectors and planes, and the term orthogonal was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are perpendicular, but common convention gives preference to the word orthogonal.
2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to \( \vec{v} \). Since there are so many, we have great leeway in finding some.

One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let \( \vec{v}_1 = \langle 2, 7, z \rangle \). If \( \vec{v}_1 \) is to be orthogonal to \( \vec{v} \), then \( \vec{v}_1 \cdot \vec{v} = 0 \), so

\[
2 + 14 + 3z = 0 \implies z = \frac{-16}{3}.
\]

So \( \vec{v}_1 = \langle 2, 7, -16/3 \rangle \) is orthogonal to \( \vec{v} \). We can apply a similar technique by leaving the first or second component unknown.

Another method of finding a vector orthogonal to \( \vec{v} \) mirrors what we did in part 1. Let \( \vec{v}_2 = \langle -2, 1, 0 \rangle \). Here we switched the first two components of \( \vec{v} \), changing the sign of one of them (similar to the “opposite reciprocal” concept before). Letting the third component be 0 effectively ignores the third component of \( \vec{v} \), and it is easy to see that

\[
\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0 \rangle \cdot \langle 1, 2, 3 \rangle = 0.
\]

Clearly \( \vec{v}_1 \) and \( \vec{v}_2 \) are not parallel.

An important construction is illustrated in Figure 11.34, where vectors \( \vec{u} \) and \( \vec{v} \) are sketched. In part (a), a dotted line is drawn from the tip of \( \vec{u} \) to the line containing \( \vec{v} \), where the dotted line is orthogonal to \( \vec{v} \). In part (b), the dotted line is replaced with the vector \( \vec{z} \) and \( \vec{w} \) is formed, parallel to \( \vec{v} \). It is clear by the diagram that \( \vec{u} = \vec{w} + \vec{z} \). What is important about this construction is this: \( \vec{u} \) is decomposed as the sum of two vectors, one of which is parallel to \( \vec{v} \) and one that is perpendicular to \( \vec{v} \). It is hard to overstate the importance of this construction (as we’ll see in upcoming examples).

The vectors \( \vec{w}, \vec{z} \) and \( \vec{u} \) as shown in Figure 11.34 (b) form a right triangle, where the angle between \( \vec{v} \) and \( \vec{u} \) is labeled \( \theta \). We can find \( \vec{w} \) in terms of \( \vec{v} \) and \( \vec{u} \).

Using trigonometry, we can state that

\[
\|\vec{w}\| = \|\vec{u}\| \cos \theta. \tag{11.3}
\]

We also know that \( \vec{w} \) is parallel to \( \vec{v} \); that is, the direction of \( \vec{w} \) is the direction of \( \vec{v} \), described by the unit vector \( \frac{\vec{v}}{\|\vec{v}\|} \). The vector \( \vec{w} \) is the vector in the direction \( \frac{1}{\|\vec{v}\|} \vec{v} \) with magnitude \( \|\vec{u}\| \cos \theta \):

\[
\vec{w} = \left( \|\vec{u}\| \cos \theta \right) \frac{1}{\|\vec{v}\|} \vec{v}.
\]

Notes:
11.3 The Dot Product

Replace \( \cos \theta \) using Theorem 88:

\[
\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{1}{\|\vec{v}\|} \vec{v}
\]

Now apply Theorem 87.

Since this construction is so important, it is given a special name.

**Definition 62 Orthogonal Projection**

Let \( \vec{u} \) and \( \vec{v} \) be given. The orthogonal projection of \( \vec{u} \) onto \( \vec{v} \), denoted \( \text{proj}_\vec{v} \vec{u} \), is

\[
\text{proj}_\vec{v} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.
\]

**Example 5 Computing the orthogonal projection**

1. Let \( \vec{u} = \langle -2, 1 \rangle \) and \( \vec{v} = \langle 3, 1 \rangle \). Find \( \text{proj}_\vec{v} \vec{u} \), and sketch all three vectors with initial points at the origin.

2. Let \( \vec{w} = \langle 2, 1, 3 \rangle \) and \( \vec{x} = \langle 1, 1, 1 \rangle \). Find \( \text{proj}_\vec{x} \vec{w} \), and sketch all three vectors with initial points at the origin.

**Solution**

1. Applying Definition 62, we have

\[
\text{proj}_\vec{v} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{-5}{10} \langle 3, 1 \rangle = \left\langle \frac{3}{2}, \frac{1}{2} \right\rangle.
\]

Vectors \( \vec{u} \), \( \vec{v} \) and \( \text{proj}_\vec{v} \vec{u} \) are sketched in Figure 11.35(a). Note how the projection is parallel to \( \vec{v} \); that is, it lies on the same line through the origin as \( \vec{v} \), although it points in the opposite direction. That is because the angle between \( \vec{u} \) and \( \vec{v} \) is obtuse (i.e., greater than 90°).

Notes:

Figure 11.35: Graphing the vectors used in Example 5.
2. Apply the definition:

\[
\text{proj}_\vec{x} \vec{w} = \frac{\vec{w} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} = \frac{6}{3} \langle 1, 1, 1 \rangle = \langle 2, 2, 2 \rangle.
\]

These vectors are sketched in Figure 11.35(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Consider Figure 11.36 where the concept of the orthogonal projection is again illustrated. It is clear that

\[
\vec{u} = \text{proj}_\vec{v} \vec{u} + \vec{z}.
\]

(11.4)

As we know what \(\vec{u}\) and \(\text{proj}_\vec{v} \vec{u}\) are, we can solve for \(\vec{z}\) and state that

\[
\vec{z} = \vec{u} - \text{proj}_\vec{v} \vec{u}.
\]

This leads us to rewrite Equation (11.4) in a seemingly silly way:

\[
\vec{u} = \text{proj}_\vec{v} \vec{u} + (\vec{u} - \text{proj}_\vec{v} \vec{u}).
\]

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression “∥ \(\vec{y}\)” means “is parallel to \(\vec{y}\).” We can use this notation to state “\(\vec{x} \parallel \vec{y}\)” which means “\(\vec{x}\) is parallel to \(\vec{y}\).” The expression “\(\perp \vec{y}\)” means “is orthogonal to \(\vec{y}\),” and is used similarly.)

Key Idea 52 Orthogonal Decomposition of Vectors

Let \(\vec{u}\) and \(\vec{v}\) be given. Then \(\vec{u}\) can be written as the sum of two vectors, one of which is parallel to \(\vec{v}\), and one of which is orthogonal to \(\vec{v}\):

\[
\vec{u} = \text{proj}_\vec{v} \vec{u} + (\vec{u} - \text{proj}_\vec{v} \vec{u}).
\]

We illustrate the use of this equality in the following example.
11.3 The Dot Product

Example 6  Orthogonal decomposition of vectors

1. Let \( \mathbf{u} = \langle 2, 1 \rangle \) and \( \mathbf{v} = \langle 3, 1 \rangle \) as in Example 5. Decompose \( \mathbf{u} \) as the sum of a vector parallel to \( \mathbf{v} \) and a vector orthogonal to \( \mathbf{v} \).

2. Let \( \mathbf{w} = \langle 2, 1, 3 \rangle \) and \( \mathbf{x} = \langle 1, 1, 1 \rangle \) as in Example 5. Decompose \( \mathbf{w} \) as the sum of a vector parallel to \( \mathbf{x} \) and a vector orthogonal to \( \mathbf{x} \).

Solution

1. In Example 5, we found that \( \text{proj}_\mathbf{v} \mathbf{u} = \langle -1.5, -0.5 \rangle \). Let

\[
\mathbf{z} = \mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u} = \langle -2, 1 \rangle - \langle -1.5, -0.5 \rangle = \langle -0.5, 1.5 \rangle.
\]

Is \( \mathbf{z} \) orthogonal to \( \mathbf{v} \)? (i.e., is \( \mathbf{z} \perp \mathbf{v} \)?) We check for orthogonality with the dot product:

\[
\mathbf{z} \cdot \mathbf{v} = \langle -0.5, 1.5 \rangle \cdot \langle 3, 1 \rangle = 0.
\]

Since the dot product is 0, we know \( \mathbf{z} \perp \mathbf{v} \). Thus:

\[
\mathbf{u} = \text{proj}_\mathbf{v} \mathbf{u} + (\mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u}) = \langle 1.5, -0.5 \rangle + \langle -0.5, 1.5 \rangle.
\]

2. We found in Example 5 that \( \text{proj}_\mathbf{x} \mathbf{w} = \langle 2, 2, 2 \rangle \). Applying the Key Idea 52, we have:

\[
\mathbf{z} = \mathbf{w} - \text{proj}_\mathbf{x} \mathbf{w} = \langle 2, 1, 3 \rangle - \langle 2, 2, 2 \rangle = \langle 0, -1, 1 \rangle.
\]

We check to see if \( \mathbf{z} \perp \mathbf{x} \):

\[
\mathbf{z} \cdot \mathbf{x} = \langle 0, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0.
\]

Since the dot product is 0, we know the two vectors are orthogonal. We now write \( \mathbf{w} \) as the sum of two vectors, one parallel and one orthogonal to \( \mathbf{x} \):

\[
\mathbf{w} = \text{proj}_\mathbf{x} \mathbf{w} + (\mathbf{w} - \text{proj}_\mathbf{x} \mathbf{w}) = \langle 2, 2, 2 \rangle + \langle 0, -1, 1 \rangle.
\]

We give an example of where this decomposition is useful.

Notes:
Example 7 Orthogonally decomposing a force vector

Consider Figure 11.37(a), showing a box weighing 50 lb on a ramp that rises 5 ft over a span of 20 ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

SOLUTION As the ramp rises 5 ft over a horizontal distance of 20 ft, we can represent the direction of the ramp with the vector \( \vec{r} = \langle 20, 5 \rangle \). Gravity pulls down with a force of 50 lb, which we represent with \( \vec{g} = \langle 0, -50 \rangle \).

1. To find the force of gravity in the direction of the ramp, we compute \( \text{proj}_r \vec{g} \):

\[
\text{proj}_r \vec{g} = \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}} \vec{r} = \frac{-250}{425} \langle 20, 5 \rangle = \left\langle \frac{-200}{17}, \frac{-50}{17} \right\rangle.
\]

The magnitude of \( \text{proj}_r \vec{g} \) is \( ||\text{proj}_r \vec{g}|| = 50/\sqrt{17} \approx 12.13 \) lb. Though the box weighs 50 lb, a force of about 12 lb is enough to keep the box from sliding down the ramp.

2. To find the component \( \vec{z} \) of gravity orthogonal to the ramp, we use Key Idea 52.

\[
\vec{z} = \vec{g} - \text{proj}_r \vec{g} = \left\langle \frac{200}{17}, \frac{-800}{17} \right\rangle.
\]

The magnitude of this force is \( ||\vec{z}|| = 200/\sqrt{17} \) lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)
11.3 The Dot Product

Application to Work

In physics, the application of a force $F$ to move an object in a straight line a distance $d$ produces work; the amount of work $W$ is $W = Fd$, (where $F$ is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 11.38, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The work done is the amount of force in the direction of $\vec{d}$, $\|\text{proj}_{\vec{d}} \vec{F}\|$, times $\|\vec{d}\|$:

$$\|\text{proj}_{\vec{d}} \vec{F}\| \cdot \|\vec{d}\| = \left| \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|^2} \right| \cdot \|\vec{d}\| \cdot \|\vec{d}\| = \left| \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|^2} \right| \|\vec{d}\|^2 = \vec{F} \cdot \vec{d}.$$

The expression $\vec{F} \cdot \vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F} \cdot \vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in “negative work.” We want to capture this sign, so we drop the absolute value and find that $W = \vec{F} \cdot \vec{d}$.

**Definition 63 Work**

Let $\vec{F}$ be a constant force that moves an object in a straight line from point $P$ to point $Q$. Let $\vec{d} = \overrightarrow{PQ}$. The work $W$ done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F} \cdot \vec{d}$.

**Example 8 Computing work**

A man slides a box along a ramp that rises 3 ft over a distance of 15 ft by applying 50 lb of force as shown in Figure 11.39. Compute the work done.

**Solution** The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50 \langle \cos 30°, \sin 30° \rangle \langle 25\sqrt{3}, 25 \rangle$. The ramp is

Notes:
represented by $\vec{d} = \langle 15, 3 \rangle$. The work done is simply

$$\vec{F} \cdot \vec{d} = \langle 25\sqrt{3}, 25 \rangle \cdot \langle 15, 3 \rangle = 375\sqrt{3} + 75 \text{ ft}\cdot\text{lb}.$$  

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another “product” on vectors, the cross product. Once again, angles play an important role, though in a much different way.
Exercises 11.3

Terms and Concepts
1. The dot product of two vectors is a ______, not a vector.
2. How are the concepts of the dot product and vector magnitude related?
3. How can one quickly tell if the angle between two vectors is acute or obtuse?
4. Give a synonym for “orthogonal.”

Problems
In Exercises 5–10, find the dot product of the given vectors.
5. \( \vec{u} = (2, -4), \vec{v} = (3, 7) \)
6. \( \vec{u} = (5, 3), \vec{v} = (6, 1) \)
7. \( \vec{u} = (1, -1, 2), \vec{v} = (2, 5, 3) \)
8. \( \vec{u} = (3, 5, -1), \vec{v} = (4, -1, 7) \)
9. \( \vec{u} = (1, 1), \vec{v} = (1, 2, 3) \)
10. \( \vec{u} = (1, 2, 3), \vec{v} = (0, 0, 0) \)
11. Create your own vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) in \( \mathbb{R}^2 \) and show that \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \).
12. Create your own vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^3 \) and scalar \( c \) and show that \( c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v}) \).

In Exercises 13–16, find the measure of the angle between the two vectors in both radians and degrees.
13. \( \vec{u} = (1, 1), \vec{v} = (1, 2) \)
14. \( \vec{u} = (-2, 1), \vec{v} = (3, 5) \)
15. \( \vec{u} = (8, 1, -4), \vec{v} = (2, 2, 0) \)
16. \( \vec{u} = (1, 7, 2), \vec{v} = (4, -2, 5) \)

In Exercises 17–20, a vector \( \vec{v} \) is given. Give two vectors that are orthogonal to \( \vec{v} \).
17. \( \vec{v} = (4, 7) \)
18. \( \vec{v} = (-3, 5) \)
19. \( \vec{v} = (1, 1, 1) \)
20. \( \vec{v} = (1, -2, 3) \)

In Exercises 21–26, vectors \( \vec{u} \) and \( \vec{v} \) are given. Find \( \text{proj}_\vec{v} \vec{u} \), the orthogonal projection of \( \vec{u} \) onto \( \vec{v} \), and sketch all three vectors on the same axes.
21. \( \vec{u} = (1, 2), \vec{v} = (-1, 3) \)
22. \( \vec{u} = (5, 5), \vec{v} = (1, 3) \)
23. \( \vec{u} = (-3, 2), \vec{v} = (1, 1) \)
24. \( \vec{u} = (-3, 2), \vec{v} = (2, 3) \)
25. \( \vec{u} = (1, 5, 1), \vec{v} = (1, 2, 3) \)
26. \( \vec{u} = (3, -1, 2), \vec{v} = (2, 2, 1) \)

In Exercises 27–32, vectors \( \vec{u} \) and \( \vec{v} \) are given. Write \( \vec{u} \) as the sum of two vectors, one of which is parallel to \( \vec{v} \) and one of which is perpendicular to \( \vec{v} \). Note: these are the same pairs of vectors as found in Exercises 21–26.
27. \( \vec{u} = (1, 2), \vec{v} = (-1, 3) \)
28. \( \vec{u} = (5, 5), \vec{v} = (1, 3) \)
29. \( \vec{u} = (-3, 2), \vec{v} = (1, 1) \)
30. \( \vec{u} = (-3, 2), \vec{v} = (2, 3) \)
31. \( \vec{u} = (1, 5, 1), \vec{v} = (1, 2, 3) \)
32. \( \vec{u} = (3, -1, 2), \vec{v} = (2, 2, 1) \)

33. A 10lb box sits on a ramp that rises 4fi over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?
34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?
35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?
36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?
37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?
38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?
39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?
11.4 The Cross Product

“Orthogonality” is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors \( \vec{u} \) and \( \vec{v} \) are perpendicular if, and only if, \( \vec{u} \cdot \vec{v} = 0 \).

Given two non–parallel, nonzero vectors \( \vec{u} \) and \( \vec{v} \) in space, it is very useful to find a vector \( \vec{w} \) that is perpendicular to both \( \vec{u} \) and \( \vec{v} \). There is an operation, called the cross product, that creates such a vector. This section defines the cross product, then explores its properties and applications.

**Definition 64 Cross Product**
Let \( \vec{u} = (u_1, u_2, u_3) \) and \( \vec{v} = (v_1, v_2, v_3) \) be vectors in \( \mathbb{R}^3 \). The cross product of \( \vec{u} \) and \( \vec{v} \), denoted \( \vec{u} \times \vec{v} \), is the vector
\[
\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).
\]

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

**Example 1 Computing a cross product**
Let \( \vec{u} = (2, -1, 4) \) and \( \vec{v} = (3, 2, 5) \). Find \( \vec{u} \times \vec{v} \), and verify that it is orthogonal to both \( \vec{u} \) and \( \vec{v} \).

Notes:
11.4 The Cross Product

SOLUTION Using Definition 64, we have
\[ \mathbf{u} \times \mathbf{v} = \langle (-1)5 - (4)2, -(2)5 - (4)3, (2)2 - (-1)3 \rangle = \langle -13, 2, 7 \rangle. \]

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not \( \mathbf{u} \times \mathbf{v} \) is orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \) using the dot product:
\[ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0, \]
\[ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0. \]

Since both dot products are zero, \( \mathbf{u} \times \mathbf{v} \) is indeed orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

A convenient method of computing the cross product starts with forming a particular \( 3 \times 3 \) matrix, or rectangular array. The first row comprises the standard unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \). The second and third rows are the vectors \( \mathbf{u} \) and \( \mathbf{v} \), respectively. Using \( \mathbf{u} \) and \( \mathbf{v} \) from Example 1, we begin with:
\[
\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & -1 & 4 \\
3 & 2 & 5 \\
\end{array}
\]

Now repeat the first two columns after the original three:
\[
\begin{array}{cccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
2 & -1 & 4 & 2 & -1 \\
3 & 2 & 5 & 3 & 2 \\
\end{array}
\]

This gives three full “upper left to lower right” diagonals, and three full “upper right to lower left” diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:
\[
\begin{array}{cccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{j} & \mathbf{k} \\
2 & -1 & 4 & 2 & -1 \\
3 & 2 & 5 & 3 & 2 \\
\end{array}
\]

\[ \mathbf{u} \times \mathbf{v} = ( -5\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} ) - ( -3\mathbf{k} + 8\mathbf{i} + 10\mathbf{j} ) = -13\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} = \langle -13, 2, 7 \rangle. \]

This is equivalent to evaluating the determinant
\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & -1 & 4 \\
3 & 2 & 5 \\
\end{vmatrix}
= \begin{vmatrix}
-1 & 4 \\
2 & 5 \\
3 & 2 \\
\end{vmatrix} \mathbf{i} + \begin{vmatrix}
2 & 4 \\
-1 & 2 \\
3 & 2 \\
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
2 & -1 \\
3 & 2 \\
3 & 2 \\
\end{vmatrix} \mathbf{k}
= (-5 - 8)\mathbf{i} - (10 - 12)\mathbf{j} + (4 - (-3))\mathbf{k} = -13\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}.
\]

We practice using this method.

Notes:
Example 2  Computing a cross product
Let \( \vec{u} = \langle 1, 3, 6 \rangle \) and \( \vec{v} = \langle -1, 2, 1 \rangle \). Compute both \( \vec{u} \times \vec{v} \) and \( \vec{v} \times \vec{u} \).

**SOLUTION**
To compute \( \vec{u} \times \vec{v} \), we form the matrix as prescribed above, complete with repeated first columns:

\[
\begin{array}{ccc|ccc}
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\
1 & 3 & 6 & 1 & 3 \\
-1 & 2 & 1 & -1 & 2
\end{array}
\]

We let the reader compute the products of the diagonals; we give the result:

\[
\vec{u} \times \vec{v} = (3\vec{i} - 6\vec{j} + 2\vec{k}) - (-3\vec{k} + 12\vec{i} + \vec{j}) = \langle -9, -7, 5 \rangle.
\]

To compute \( \vec{v} \times \vec{u} \), we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

\[
\begin{array}{ccc|ccc}
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\
-1 & 2 & 1 & -1 & 2 \\
1 & 3 & 6 & 1 & 3
\end{array}
\]

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice-versa. Thus the result is:

\[
\vec{v} \times \vec{u} = (12\vec{i} + \vec{j} - 3\vec{k}) - (2\vec{k} + 3\vec{i} - 6\vec{j}) = \langle 9, 7, -5 \rangle,
\]

which is the opposite of \( \vec{u} \times \vec{v} \). We leave it to the reader to verify that each of these vectors is orthogonal to \( \vec{u} \) and \( \vec{v} \).

**Properties of the Cross Product**

It is not coincidence that \( \vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) \) in the preceding example; one can show using Definition 64 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Notes:
Theorem 89  Properties of the Cross Product
Let \( \vec{u}, \vec{v} \) and \( \vec{w} \) be vectors in \( \mathbb{R}^3 \) and let \( c \) be a scalar. The following identities hold:

1. \( \vec{u} \times \vec{v} = - (\vec{v} \times \vec{u}) \)  
   Anticommutative Property

2. (a) \( (\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \)  
   Distributive Properties
   (b) \( \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \)

3. \( c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) \)

4. (a) \( (\vec{u} \times \vec{v}) \cdot \vec{u} = 0 \)  
   Orthogonality Properties
   (b) \( (\vec{u} \times \vec{v}) \cdot \vec{v} = 0 \)

5. \( \vec{u} \times \vec{u} = \vec{0} \)

6. \( \vec{u} \times \vec{0} = \vec{0} \)

7. \( \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \)  
   Triple Scalar Product

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 64 satisfies this property. Theorem 89 asserts this property holds; we leave the verification to Exercise 42.

Property 5 from the theorem is also left to the reader to prove in Exercise 43, but it reveals something more interesting than “the cross product of a vector with itself is \( \vec{0} \)” Let \( \vec{u} \) and \( \vec{v} \) be parallel vectors; that is, let there be a scalar \( c \) such that \( \vec{v} = c\vec{u} \). Consider their cross product:

\[
\begin{align*}
\vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\
&= c(\vec{u} \times \vec{u}) \quad \text{(by Property 3 of Theorem 89)} \\
&= \vec{0}, \quad \text{(by Property 5 of Theorem 89)}
\end{align*}
\]

We have just shown that the cross product of parallel vectors is \( \vec{0} \). This hints at something deeper. Theorem 88 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Notes:
Chapter 11  Vectors

Theorem 90  The Cross Product and Angles
Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin \theta,$$

where $\theta$, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note: Definition 61 (through Theorem 88) defines $\vec{u}$ and $\vec{v}$ to be orthogonal if $\vec{u} \cdot \vec{v} = 0$. We could use Theorem 90 to define $\vec{u}$ and $\vec{v}$ to be parallel if $\vec{u} \times \vec{v} = 0$. By such a definition, $\vec{0}$ would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 657.)

Note that this theorem makes a statement about the magnitude of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or $\pi$ (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 86), hence the cross product of parallel vectors is $\vec{0}$.

We demonstrate the truth of this theorem in the following example.

Example 3  The cross product and angles
Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$ as in Example 2. Verify Theorem 90 by finding $\theta$, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

SOLUTION  We use Theorem 88 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||}\right)$$

$$= \cos^{-1}\left(\frac{11}{\sqrt{46} \sqrt{6}}\right)$$

$$\approx 0.8471 = 48.54^\circ.$$

Our work in Example 2 showed that $\vec{u} \times \vec{v} = \langle -9, -7, 5 \rangle$, hence $||\vec{u} \times \vec{v}|| = \sqrt{155}$. Is $||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin \theta$? Using numerical approximations, we find:

$$||\vec{u} \times \vec{v}|| = 12.45.$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin \left(\cos^{-1}\left(\frac{11}{\sqrt{46} \sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46} \sqrt{6}},$$

which allows us to verify the theorem exactly.
11.4 The Cross Product

Right Hand Rule

The anticommutative property of the cross product demonstrates that \( \mathbf{u} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{u} \) differ only by a sign — these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to \( \mathbf{u} \) and \( \mathbf{v} \), we essentially have two directions to choose from, one in the direction of \( \mathbf{u} \times \mathbf{v} \) and one in the direction of \( \mathbf{v} \times \mathbf{u} \). Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another property of the cross product, as defined, is that it follows the right hand rule. Given \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^3 \) with the same initial point, point the index finger of your right hand in the direction of \( \mathbf{u} \) and let your middle finger point in the direction of \( \mathbf{v} \) (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of \( \mathbf{u} \times \mathbf{v} \). One can “practice” this using Figure 11.40. If you switch, and point the index finder in the direction of \( \mathbf{v} \) and the middle finger in the direction of \( \mathbf{u} \), your thumb will now point in the opposite direction, allowing you to “visualize” the anticommutative property of the cross product.

Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond “just” finding a vector perpendicular to two others. We highlight a few here.

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is \( A = bh \), where \( b \) is the length of the base and \( h \) is the height of the parallelogram, as illustrated in Figure 11.41(a). As shown when defining the Parallelogram Law of vector addition, two vectors \( \mathbf{u} \) and \( \mathbf{v} \) define a parallelogram when drawn from the same initial point, as illustrated in Figure 11.41(b). Trigonometry tells us that \( h = \| \mathbf{u} \| \sin \theta \), hence the area of the parallelogram is

\[
A = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = \| \mathbf{u} \times \mathbf{v} \| , \tag{11.5}
\]

where the second equality comes from Theorem 90. We illustrate using Equation (11.5) in the following example.

**Example 4 Finding the area of a parallelogram**

1. Find the area of the parallelogram defined by the vectors \( \mathbf{u} = \langle 2, 1 \rangle \) and \( \mathbf{v} = \langle 1, 3 \rangle \).

Notes:
2. Verify that the points $A = (1, 1, 1)$, $B = (2, 3, 2)$, $C = (4, 5, 3)$ and $D = (3, 3, 2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

**SOLUTION**

1. Figure 11.42(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2, 1, 0 \rangle$ and $\vec{v} = \langle 1, 3, 0 \rangle$. We can now compute the cross product. It is easy to show that $\vec{u} \times \vec{v} = \langle 0, 0, 5 \rangle$; therefore the area of the parallelogram is $A = \|\vec{u} \times \vec{v}\| = 5$.

2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 11.42(b)), we need to show that the opposite sides are parallel. We can quickly show that $\#AB = \#DC = \langle 1, 2, 1 \rangle$ and $\#BC = \#AD = \langle 2, 2, 1 \rangle$. We find the area by computing the magnitude of the cross product of $\#AB$ and $\#BC$:

$$\#AB \times \#BC = \langle 0, 1, -2 \rangle \Rightarrow \|\#AB \times \#BC\| = \sqrt{5}.$$
11.4 The Cross Product

Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the parallelepiped. Each face is parallel to the opposite face, as illustrated in Figure 11.44. By crossing \( \vec{v} \) and \( \vec{w} \), one gets a vector whose magnitude is the area of the base. Dotting this vector with \( \vec{u} \) computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) is

\[
V = |\vec{u} \cdot (\vec{v} \times \vec{w})|.
\]

(11.6)

Note how this is the Triple Scalar Product, first seen in Theorem 89. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any “order” we choose to find the volume. That is,

\[
V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \quad \text{etc.}
\]

As with the cross product, we can also write \( \vec{u} \cdot (\vec{v} \times \vec{w}) \) in terms of a determinant:

\[
\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
\end{vmatrix}.
\]

Because the volume is the absolute value of the determinant, the order of the rows doesn’t matter.

Example 6 Finding the volume of parallelepiped

Find the volume of the parallelepiped defined by the vectors \( \vec{u} = \langle 1, 1, 0 \rangle \), \( \vec{v} = \langle -1, 1, 0 \rangle \) and \( \vec{w} = \langle 0, 1, 1 \rangle \).

**SOLUTION** We apply Equation (11.6). We first find \( \vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle \).

Then

\[
|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle| = 2.
\]

So the volume of the parallelepiped is 2 cubic units. In terms of determinants, we have

\[
\begin{vmatrix}
    1 & 1 & 0 \\
    -1 & 1 & 0 \\
    0 & 1 & 1
\end{vmatrix} = 1 \cdot 1 \cdot 1 - 1 \cdot 0 \cdot 1 - 0 \cdot 1 \cdot 0 = 1 - 0 + 0 = 1.
\]

and the absolute value of this determinant is again 2.

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

---

Notes:

Note: The word “parallelepiped” is pronounced “parallel–eh–pipe–ed.”

Figure 11.44: A parallelepiped is the three dimensional analogue to the parallelogram.

Figure 11.45: A parallelepiped in Example 6.
Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors \( \vec{F} \) and \( \vec{l} \), we see that the bolt moves (because of the threads) in a direction orthogonal to \( \vec{F} \) and \( \vec{l} \). Torque is usually represented by the Greek letter \( \tau \), or tau, and has units of N·m, a Newton–meter, or \( \text{ft-lb} \), a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force \( \vec{F} \) is applied to a lever arm \( \vec{l} \), the resulting torque is

\[
\vec{\tau} = \vec{l} \times \vec{F}.
\]  

(11.7)

Example 7  

Computing torque

A lever of length 2ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10lb is applied to the end of the lever where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 11.46.

Solution

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2ft long, we can state that \( \vec{l} = 2 \langle \cos 45°, \sin 45° \rangle = \langle \sqrt{2}, \sqrt{2} \rangle \).

Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 11.46), we can conclude it is making an angle of \(-45°\) with the horizontal. As it has a magnitude of 10lb, we can state \( \vec{F} = 10 \langle \cos(-45°), \sin(-45°) \rangle = \langle 5\sqrt{2}, -5\sqrt{2} \rangle \).

Using Equation (11.7) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

\[
\vec{\tau} = \vec{l} \times \vec{F} \\
= \langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \langle 5\sqrt{2}, -5\sqrt{2}, 0 \rangle \\
= \langle 0, 0, -20 \rangle
\]

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying “on the page”; our computation of \( \vec{\tau} \) shows that the torque goes “into the page.” This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.
2. Our lever arm can still be represented by \( \vec{\ell} = (\sqrt{2}, \sqrt{2}) \). As our force vector makes a 60° angle with \( \vec{\ell} \), we can see (referencing the right hand side of the figure) that \( \vec{F} \) makes a \(-15^\circ\) angle with the horizontal. Thus

\[
\vec{F} = 10 \left( \cos -15^\circ, \sin -15^\circ \right) = \left( \frac{5(1 + \sqrt{3})}{\sqrt{2}}, \frac{5(1 - \sqrt{3})}{\sqrt{2}} \right).
\]

We again make the third component 0 and take the cross product to find the torque:

\[
\vec{\tau} = \vec{\ell} \times \vec{F} = \left( \sqrt{2}, \sqrt{2}, 0 \right) \times \left( \frac{5(1 + \sqrt{3})}{\sqrt{2}}, \frac{5(1 - \sqrt{3})}{\sqrt{2}}, 0 \right) = \left( 0, 0, -10\sqrt{3} \right).
\]

As one might expect, when the force and lever arm vectors are orthogonal, the magnitude of force is greater than when the vectors are not orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.
Terms and Concepts

1. The cross product of two vectors is a _______, not a scalar.
2. One can visualize the direction of \( \mathbf{u} \times \mathbf{v} \) using the _______ _______ _______.
3. Give a synonym for “orthogonal.”
4. T/F: A fundamental principle of the cross product is that \( \mathbf{u} \times \mathbf{v} \) is orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \).
5. _______ is a measure of the turning force applied to an object.

Problems

6. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
   (a) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)
   (b) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \)
   (c) \( (\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d}) \)
   (d) \( \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}) \)
   (e) \( (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \)

In Exercises 7–15, vectors \( \mathbf{u} \) and \( \mathbf{v} \) are given. Compute \( \mathbf{u} \times \mathbf{v} \) and show this is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).
7. \( \mathbf{u} = (3, 2, -2), \quad \mathbf{v} = (0, 1, 5) \)
8. \( \mathbf{u} = (5, -4, 3), \quad \mathbf{v} = (2, -5, 1) \)
9. \( \mathbf{u} = (4, -5, -5), \quad \mathbf{v} = (3, 3, 4) \)
10. \( \mathbf{u} = (-4, 7, -10), \quad \mathbf{v} = (4, 4, 1) \)
11. \( \mathbf{u} = (1, 0, 1), \quad \mathbf{v} = (5, 0, 7) \)
12. \( \mathbf{u} = (1, 5, -4), \quad \mathbf{v} = (-2, -10, 8) \)
13. \( \mathbf{u} = \mathbf{i}, \quad \mathbf{v} = \mathbf{j} \)
14. \( \mathbf{u} = \mathbf{i}, \quad \mathbf{v} = \mathbf{k} \)
15. \( \mathbf{u} = \mathbf{j}, \quad \mathbf{v} = \mathbf{k} \)
16. Pick any vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^3 \) and show that \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \).
17. Pick any vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^3 \) and show that \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} \).

In Exercises 18–21, the magnitudes of vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^3 \) are given, along with the angle \( \theta \) between them. Use this information to find the magnitude of \( \mathbf{u} \times \mathbf{v} \).
18. \( ||\mathbf{u}|| = 2, \quad ||\mathbf{v}|| = 5, \quad \theta = 30^\circ \)
19. \( ||\mathbf{u}|| = 3, \quad ||\mathbf{v}|| = 7, \quad \theta = \pi/2 \)
20. \( ||\mathbf{u}|| = 3, \quad ||\mathbf{v}|| = 4, \quad \theta = \pi \)
21. \( ||\mathbf{u}|| = 2, \quad ||\mathbf{v}|| = 5, \quad \theta = 5\pi/6 \)

In Exercises 22–25, find the area of the parallelogram defined by the given vectors.
22. \( \mathbf{u} = (1, 1, 2), \quad \mathbf{v} = (2, 0, 3) \)
23. \( \mathbf{u} = (-2, 1, 5), \quad \mathbf{v} = (-1, 3, 1) \)
24. \( \mathbf{u} = (1, 2), \quad \mathbf{v} = (2, 1) \)
25. \( \mathbf{u} = (2, 0), \quad \mathbf{v} = (0, 3) \)

In Exercises 26–29, find the area of the triangle with the given vertices.
26. Vertices: (0, 0, 0), (1, 3, -1) and (2, 1, 1).
27. Vertices: (5, 2, -1), (3, 6, 2) and (1, 0, 4).
28. Vertices: (1, 1), (1, 3) and (2, 2).
29. Vertices: (3, 1), (1, 2) and (4, 3).

In Exercises 30–31, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)
30. Vertices: (0, 0), (1, 2), (3, 0) and (4, 3).
31. Vertices: (0, 0, 0), (2, 1, 1), (-1, 2, -8) and (1, -1, 5).

In Exercises 32–33, find the volume of the parallelepiped defined by the given vectors.
32. \( \mathbf{u} = (1, 1, 1), \quad \mathbf{v} = (1, 2, 3), \quad \mathbf{w} = (1, 0, 1) \)
33. \( \mathbf{u} = (-1, 2, 1), \quad \mathbf{v} = (2, 2, 1), \quad \mathbf{w} = (3, 1, 3) \)

In Exercises 34–37, find a unit vector orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).
34. \( \mathbf{u} = (1, 1, 1), \quad \mathbf{v} = (2, 0, 1) \)
35. \( \mathbf{u} = (1, -2, 1), \quad \mathbf{v} = (3, 2, 1) \)
36. \( \mathbf{u} = (5, 0, 2), \quad \mathbf{v} = (-3, 0, 7) \)
37. \( \mathbf{u} = (1, -2, 1), \quad \mathbf{v} = (-2, 4, -2) \)

38. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.
39. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.
40. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?
41. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?
42. Show, using the definition of the Cross Product, that \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \); that is, that \( \mathbf{u} \) is orthogonal to the cross product of \( \mathbf{u} \) and \( \mathbf{v} \).
43. Show, using the definition of the Cross Product, that \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \).
11.5 Lines

To find the equation of a line in the x-y plane, we need two pieces of information: a point and the slope. The slope conveys direction information. As vertical lines have an undefined slope, the following statement is more accurate:

To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Let \( P \) be a point in space, let \( \vec{p} \) be the vector with initial point at the origin and terminal point at \( P \) (i.e., \( \vec{p} \) “points” to \( P \)), and let \( \vec{d} \) be a vector. Consider the points on the line through \( P \) in the direction of \( \vec{d} \).

Clearly one point on the line is \( P \); we can say that the vector \( \vec{p} \) lies at this point on the line. To find another point on the line, we can start at \( \vec{p} \) and move in a direction parallel to \( \vec{d} \). For instance, starting at \( \vec{p} \) and traveling one length of \( \vec{d} \) places one at another point on the line. Consider Figure 11.47 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with \( \vec{p} \) and moving a certain distance in the direction of \( \vec{d} \). That is, we can define the line as a function of \( t \):

\[
\vec{\ell}(t) = \vec{p} + t \vec{d}. \quad (11.8)
\]

In many ways, this is not a new concept. Compare Equation (11.8) to the familiar “\( y = mx + b \)” equation of a line:

\[
y = b + m x
\]

\[
\vec{\ell}(t) = \vec{p} + t \vec{d}
\]

Figure 11.48: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Figure 11.47: Defining a line in space.

Notes:
Equation (11.8) is an example of a vector–valued function; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

There are other ways to represent a line. Let \( \vec{p} = \langle x_0, y_0, z_0 \rangle \) and let \( \vec{d} = \langle a, b, c \rangle \). Then the equation of the line through \( \vec{p} \) in the direction of \( \vec{d} \) is:

\[
\vec{c}(t) = \vec{p} + t\vec{d} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.
\]

The last line states that the the \( x \) values of the line are given by \( x = x_0 + at \), the \( y \) values are given by \( y = y_0 + bt \), and the \( z \) values are given by \( z = z_0 + ct \). These three equations, taken together, are the parametric equations of the line through \( \vec{p} \) in the direction of \( \vec{d} \).

Finally, each of the equations for \( x, y \) and \( z \) above contain the variable \( t \). We can solve for \( t \) in each equation:

\[
\begin{align*}
x &= x_0 + at & \Rightarrow & & t &= \frac{x - x_0}{a}, \\
y &= y_0 + bt & \Rightarrow & & t &= \frac{y - y_0}{b}, \\
z &= z_0 + ct & \Rightarrow & & t &= \frac{z - z_0}{c},
\end{align*}
\]

assuming \( a, b, c \neq 0 \). Since \( t \) is equal to each expression on the right, we can set these equal to each other, forming the symmetric equations of the line through \( \vec{p} \) in the direction of \( \vec{d} \):

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:
**Definition 65  Equations of Lines in Space**

Consider the line in space that passes through \( \vec{p} = \langle x_0, y_0, z_0 \rangle \) in the direction of \( \vec{d} = \langle a, b, c \rangle \).

1. The **vector equation** of the line is
   \[
   \vec{\ell}(t) = \vec{p} + t\vec{d}.
   \]

2. The **parametric equations** of the line are
   \[
   x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.
   \]

3. The **symmetric equations** of the line are
   \[
   \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
   \]

---

**Example 1  Finding the equation of a line**

Give all three equations, as given in Definition 65, of the line through \( P = (2, 3, 1) \) in the direction of \( \vec{d} = (-1, 1, 2) \). Does the point \( Q = (-1, 6, 6) \) lie on this line?

**SOLUTION**

We identify the point \( P = (2, 3, 1) \) with the vector \( \vec{p} = \langle 2, 3, 1 \rangle \). Following the definition, we have

- the vector equation of the line is \( \vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle \);
- the parametric equations of the line are
  \[
  x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t;
  \]
- the symmetric equations of the line are
  \[
  \frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.
  \]

Does the point \( Q = (-1, 6, 6) \) lie on this line?

---

Notes:
The first two equations of the line are useful when a $t$ value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats.

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 11.49 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of $x$, $y$ and $z$ and see if equality is maintained:

$$\frac{-1 - 2}{-1} = \frac{6 - 3}{1} = \frac{6 - 1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$  

We see that $Q$ does not lie on the line as it did not satisfy the symmetric equations.

**Example 2**  
**Finding the equation of a line through two points**

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

**SOLUTION**  
Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have two points; either one will suffice. The direction of the line can be found by the vector with initial point $P$ and terminal point $Q$: $\overrightarrow{PQ} = (-1, 4, -3)$.

The parametric equations of the line $\ell$ through $P$ in the direction of $\overrightarrow{PQ}$ are:

$$\ell : \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$  

A graph of the points and line are given in Figure 11.50. Note how in the given parametrization of the line, $t = 0$ corresponds to the point $P$, and $t = 1$ corresponds to the point $Q$. This relates to the understanding of the vector equation of a line described in Figure 11.48. The parametric equations "start" at the point $P$, and $t$ determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point $Q$.

**Parallel, Intersecting and Skew Lines**

In the plane, two distinct lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\ell_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\ell_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\ell_1$ and $\ell_2$ are
the same line they share all points;
intersecting lines share only 1 point;
parallel lines \( \vec{d}_1 \parallel \vec{d}_2 \), no points in common; or
skew lines \( \vec{d}_1 \nparallel \vec{d}_2 \), no points in common.

The next two examples investigate these possibilities.

**Example 3 Comparing lines**
Consider lines \( \ell_1 \) and \( \ell_2 \), given in parametric equation form:
\[
\ell_1 : \begin{align*}
x &= 1 + 3t \\
y &= 2 - t \\
z &= t
\end{align*}
\]
\[
\ell_2 : \begin{align*}
x &= -2 + 4s \\
y &= 3 + s \\
z &= 5 + 2s.
\end{align*}
\]

Determine whether \( \ell_1 \) and \( \ell_2 \) are the same line, intersect, are parallel, or skew.

**Solution** We start by looking at the directions of each line. Line \( \ell_1 \) has the direction given by \( \vec{d}_1 = \langle 3, -1, 1 \rangle \) and line \( \ell_2 \) has the direction given by \( \vec{d}_2 = \langle 4, 1, 2 \rangle \). It should be clear that \( \vec{d}_1 \) and \( \vec{d}_2 \) are not parallel, hence \( \ell_1 \) and \( \ell_2 \) are not the same line, nor are they parallel. Figure 11.51 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for \( t \) and \( s \) values such that the respective \( x, y \) and \( z \) values are the same. That is, we want \( s \) and \( t \) such that:
\[
\begin{align*}
1 + 3t &= -2 + 4s \\
2 - t &= 3 + s \\
t &= 5 + 2s.
\end{align*}
\]
This is a relatively simple system of linear equations. Since the last equation is already solved for \( t \), substitute that value of \( t \) into the equation above it:
\[
2 - (5 + 2s) = 3 + s \quad \Rightarrow \quad s = -2, \ t = 1.
\]
A key to remember is that we have three equations; we need to check if \( s = -2, \ t = 1 \) satisfies the first equation as well:
\[
1 + 3(1) \neq -2 + 4(-2).
\]
It does not. Therefore, we conclude that the lines \( \ell_1 \) and \( \ell_2 \) are skew.

**Example 4 Comparing lines**
Consider lines \( \ell_1 \) and \( \ell_2 \), given in parametric equation form:
\[
\ell_1 : \begin{align*}
x &= -0.7 + 1.6t \\
y &= 4.2 + 2.72t \\
z &= 2.3 - 3.36t
\end{align*}
\]
\[
\ell_2 : \begin{align*}
x &= 2.8 - 2.9s \\
y &= 10.15 - 4.93s \\
z &= -5.05 + 6.09s.
\end{align*}
\]

Determine whether \( \ell_1 \) and \( \ell_2 \) are the same line, intersect, are parallel, or skew.
Chapter 11  Vectors

**SOLUTION**  It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the “real world,” most equations that are used do not have nice, integer coefficients. We again start by deciding whether or not each line has the same direction. The direction of $\ell_1$ is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36 \rangle$ and the direction of $\ell_2$ is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09 \rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

\[
\vec{u}_1 = \frac{\vec{d}_1}{\|\vec{d}_1\|} = \langle 0.3471, 0.5901, -0.7289 \rangle
\]

\[
\vec{u}_2 = \frac{\vec{d}_2}{\|\vec{d}_2\|} = \langle -0.3471, -0.5901, 0.7289 \rangle.
\]

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

\[
\vec{d}_1 = \left\langle \frac{16}{10} \cdot \frac{272}{100} - \frac{336}{100} \right\rangle \quad \vec{d}_2 = \left\langle -\frac{29}{10} - \frac{493}{100} \cdot \frac{609}{100} \right\rangle.
\]

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

\[
\vec{u}_1 = \left\langle \frac{10}{83} \cdot \frac{17}{\sqrt{830}} - \frac{21}{\sqrt{830}} \right\rangle \quad \vec{u}_2 = \left\langle -\frac{10}{83} - \frac{17}{\sqrt{830}} \cdot \frac{21}{\sqrt{830}} \right\rangle.
\]

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on $\ell_1$. To determine if this point also lies on $\ell_2$, plug in the $x$, $y$ and $z$ values of $P_1$ into the symmetric equations for $\ell_2$:

\[
\begin{align*}
(-0.7) - 2.8 & \equiv (4.2) - 10.15 \equiv (2.3) - (-5.05) \\
-2.9 & \equiv -4.93 & \equiv 6.09 \\
\Rightarrow & \quad 1.2069 = 1.2069 = 1.2069.
\end{align*}
\]

The point $P_1$ lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 11.52 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

---

Notes:
Distances

Given a point \( Q \) and a line \( \vec{l}(t) = \vec{p} + t\vec{d} \) in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of “distance,” i.e., the length of the shortest line segment from the point to the line.) Identifying \( \vec{p} \) with the point \( P \), Figure 11.53 will help establish a general method of computing this distance \( h \).

From trigonometry, we know \( h = \|\vec{PQ}\| \sin \theta \). We have a similar identity involving the cross product: \( \|\vec{PQ} \times \vec{d}\| = \|\vec{PQ}\| \|\vec{d}\| \sin \theta \). Divide both sides of this latter equation by \( \|\vec{d}\| \) to obtain \( h \):

\[
h = \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|}.
\] (11.9)

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines \( \ell_1(t) = \vec{p}_1 + t\vec{d}_1 \) and \( \ell_2(t) = \vec{p}_2 + t\vec{d}_2 \) be given, as shown in Figure 11.54. To find the direction orthogonal to both \( \vec{d}_1 \) and \( \vec{d}_2 \), we take the cross product: \( \vec{c} = \vec{d}_1 \times \vec{d}_2 \). The magnitude of the orthogonal projection of \( \vec{P}_1\vec{P}_2 \) onto \( \vec{c} \) is the distance \( h \) we seek:

\[
h = \|\text{proj}_{\vec{c}}\vec{P}_1\vec{P}_2\|
= \left\| \frac{\vec{P}_1\vec{P}_2 \cdot \vec{c}}{\|\vec{c}\|} \right\|\vec{c}\|
= \frac{\vec{P}_1\vec{P}_2 \cdot \vec{c}}{\|\vec{c}\|}
= \frac{\|\vec{P}_1\vec{P}_2 \cdot \vec{c}\|}{\|\vec{c}\|}.
\]

Exercise 30 shows that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: \( \vec{P}_1\vec{P}_2 \cdot \vec{c} = \vec{P}_1\vec{P}_2 \cdot (\vec{d}_1 \times \vec{d}_2) \).

The following Key Idea restates these two distance formulas.

Notes:
1. Let $P$ be a point on a line $\ell$ that is parallel to $\vec{d}$. The distance $h$ from a point $Q$ to the line $\ell$ is:

$$h = \frac{\|PQ \times \vec{d}\|}{\|\vec{d}\|}.$$ 

2. Let $P_1$ be a point on line $\ell_1$ that is parallel to $\vec{d}_1$, and let $P_2$ be a point on line $\ell_2$ parallel to $\vec{d}_2$, and let $\vec{c} = \vec{d}_1 \times \vec{d}_2$, where lines $\ell_1$ and $\ell_2$ are not parallel. The distance $h$ between the two lines is:

$$h = \frac{|P_1P_2 \cdot \vec{c}|}{\|\vec{c}\|}.$$ 

Example 5  Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\ell(t) = (1, -1, 1) + t(2, 3, 1)$.

**Solution**  The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $PQ = (0, 2, 2)$. The equation also gives $\vec{d} = (2, 3, 1)$. Following Key Idea 53, we have the distance as:

$$h = \frac{\|PQ \times \vec{d}\|}{\|\vec{d}\|} = \frac{\|(-4, 4, -4)\|}{\sqrt{14}} = \frac{4\sqrt{3}}{\sqrt{14}}.$$

The point $Q$ is approximately 1.852 units from the line $\ell(t)$.

Example 6  Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{align*}
x &= 1 + 3t \\
y &= 2 - t \\
z &= t
dot
\end{align*} \quad \ell_2: \begin{align*}
x &= -2 + 4s \\
y &= 3 + s \\
z &= 5 + 2s.
dot$$

Notes:
SOLUTION These are the same lines as given in Example 3, where we showed them to be skew. The equations allow us to identify the following points and vectors:

\[ P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \vec{P_1P_2} = (-3, 1, 5). \]

\[ \vec{d}_1 = (3, -1, 1) \quad \vec{d}_2 = (4, 1, 2) \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = (-3, -2, 7). \]

From Key Idea 53 we have the distance \( h \) between the two lines is

\[
h = \frac{\|P_1P_2 \cdot \vec{c}\|}{\|\vec{c}\|} = \frac{42}{\sqrt{62}}.
\]

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and are asked in the exercises) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the plane, which we study in detail in the next section. Many complex three-dimensional objects are studied by approximating their surfaces with lines and planes.
Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?
2. Two distinct lines in the plane can intersect or be ________.
3. Two distinct lines in space can intersect, be ________ or be ________.
4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5–14, write the vector, parametric and symmetric equations of the lines described.

5. Passes through \( P = (2, -4, 1) \) parallel to \( \vec{d} = (9, 2, 5) \).
6. Passes through \( P = (6, 1, 7) \), parallel to \( \vec{d} = (-3, 2, 5) \).
7. Passes through \( P = (2, 1, 5) \) and \( Q = (7, -2, 4) \).
8. Passes through \( P = (1, -2, 3) \) and \( Q = (5, 5, 5) \).
9. Passes through \( P = (0, 1, 2) \) and orthogonal to both \( \vec{d}_1 = \langle 2, -1, 7 \rangle \) and \( \vec{d}_2 = \langle 7, 1, 3 \rangle \).
10. Passes through \( P = (5, 1, 9) \) and orthogonal to both \( \vec{d}_1 = \langle 1, 0, 1 \rangle \) and \( \vec{d}_2 = \langle 2, 0, 3 \rangle \).
11. Passes through the point of intersection of \( \vec{L}_1(t) \) and \( \vec{L}_2(t) \) and orthogonal to both lines, where \( \vec{L}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, -2 \rangle \) and \( \vec{L}_2(t) = \langle -2, -1, 2 \rangle + t \langle 3, 1, -1 \rangle \).
12. Passes through the point of intersection of \( \vec{L}_1(t) \) and \( \vec{L}_2(t) \) and orthogonal to both lines, where
\[
\vec{L}_1 = \begin{cases} 
  x = t \\
  y = -2 + 2t \\
  z = 1 + t 
\end{cases} \quad \text{and} \quad \vec{L}_2 = \begin{cases} 
  x = 2 + t \\
  y = 2 - t \\
  z = 3 + 2t 
\end{cases}
\]
13. Passes through \( P = (1, 1) \), parallel to \( \vec{d} = (2, 3) \).
14. Passes through \( P = (-2, 5) \), parallel to \( \vec{d} = (0, 1) \).

In Exercises 15–22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

15. \( \vec{L}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle \), \( \vec{L}_2(t) = \langle 3, 3, 3 \rangle + t \langle -4, 2, -2 \rangle \).
16. \( \vec{L}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, 3 \rangle \), \( \vec{L}_2(t) = \langle 14, 5, 9 \rangle + t \langle 1, 1, 1 \rangle \).
17. \( \vec{L}_1(t) = \langle 3, 4, 1 \rangle + t \langle 2, -3, 4 \rangle \), \( \vec{L}_2(t) = \langle -3, 3, -3 \rangle + t \langle 3, -2, 4 \rangle \).
18. \( \vec{L}_1(t) = \langle 1, 1, 1 \rangle + t \langle 3, 1, 3 \rangle \), \( \vec{L}_2(t) = \langle 7, 3, 7 \rangle + t \langle 6, 2, 6 \rangle \).
19. \( \vec{L}_1 = \begin{cases} 
  x = 1 + 2t \\
  y = 3 - 2t \\
  z = t 
\end{cases} \quad \text{and} \quad \vec{L}_2 = \begin{cases} 
  x = 3 - t \\
  y = 3 + 5t \\
  z = 2 + 7t 
\end{cases}
\]
20. \( \ell_1 = \begin{cases} 
  x = 1.1 + 0.6t \\
  y = 3.77 + 0.9t \\
  z = -2.3 + 1.5t 
\end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} 
  x = 3.11 + 3.4t \\
  y = 2 + 5.1t \\
  z = 2.5 + 8.5t 
\end{cases}
\]
21. \( \ell_1 = \begin{cases} 
  y = 1.33 - 0.45t \\
  z = -4.2 + 1.05t 
\end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} 
  x = 0.86 + 9.2t \\
  y = 0.835 - 6.9t \\
  z = -3.045 + 16.1t 
\end{cases}
\]
22. \( \ell_1 = \begin{cases} 
  y = 2.9 - 1.5t \\
  z = 3.2 + 1.6t 
\end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} 
  x = 4 - 2.1t \\
  y = 1.8 + 7.2t \\
  z = 3.1 + 1.1t 
\end{cases}
\)

In Exercises 23–26, find the distance from the point to the line.

23. \( P = (1, 1, 1) \), \( \vec{L}(t) = \langle 2, 1, 3 \rangle + t \langle 2, 1, -2 \rangle \).
24. \( P = (2, 5, 6) \), \( \vec{L}(t) = \langle -1, 1, 1 \rangle + t \langle 1, 0, 1 \rangle \).
25. \( P = (0, 3) \), \( \vec{L}(t) = \langle 2, 0 \rangle + t \langle 1, 1 \rangle \).
26. \( P = (1, 1) \), \( \vec{L}(t) = \langle 4, 5 \rangle + t \langle -4, 3 \rangle \).

In Exercises 27–28, find the distance between the two lines.

27. \( \vec{L}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle \), \( \vec{L}_2(t) = \langle 3, 3, 3 \rangle + t \langle 4, 2, -2 \rangle \).
28. \( \vec{L}_1(t) = \langle 0, 0, 1 \rangle + t \langle 1, 0, 0 \rangle \), \( \vec{L}_2(t) = \langle 0, 0, 3 \rangle + t \langle 0, 1, 0 \rangle \).

Exercises 29–31 explore special cases of the distance formulas found in Key Idea 53.

29. Let \( Q \) be a point on the line \( \vec{L}(t) \). Show why the distance formula correctly gives the distance from the point to the line as 0.
30. Let lines \( \vec{L}_1(t) \) and \( \vec{L}_2(t) \) be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.
31. Let lines \( \vec{L}_1(t) \) and \( \vec{L}_2(t) \) be parallel.
   (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.
   (b) Show why letting \( \vec{e} = (\vec{P}_1\vec{P}_2 \times \vec{d}_2) \) allows one to use the formula.
   (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.
11.6 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point $P$ marked on it. One can take a nail and stick it into the cardboard at $P$ such that the nail is perpendicular to the cardboard; see Figure 11.55.

This nail provides a "handle" for the cardboard. Moving the cardboard around moves $P$ to different locations in space. Tilting the nail (but keeping $P$ fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of $P$ in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane “faces” (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a normal vector, that is orthogonal to the plane.

What exactly does “orthogonal to the plane” mean? Choose any two points $P$ and $Q$, in the plane, and consider the vector $\vec{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\vec{PQ}$ for all choices of $P$ and $Q$; that is, if $\vec{n} \cdot \vec{PQ} = 0$ for all $P$ and $Q$.

This gives us way of writing an equation describing the plane. Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = (a, b, c)$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in the plane defined by $P$ and $\vec{n}$ if, and only if, $\vec{PQ}$ is orthogonal to $\vec{n}$. Knowing $\vec{PQ} = (x - x_0, y - y_0, z - z_0)$, consider:

$$\vec{PQ} \cdot \vec{n} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(11.10)

Equation (11.10) defines an implicit function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with $d$:

$$ax + by + cz = d.$$  \hspace{1cm} (11.11)

As long as $c \neq 0$, we can solve for $z$:

$$z = \frac{1}{c}(d - ax - by).$$  \hspace{1cm} (11.12)

Notes:
Equation (11.12) is especially useful as many computer programs can graph functions in this form. Equations (11.10) and (11.11) have specific names, given next.

**Definition 66 Equations of a Plane in Standard and General Forms**
The plane passing through the point \( P = (x_0, y_0, z_0) \) with normal vector \( \vec{n} = (a, b, c) \) can be described by an equation with **standard form**

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;
\]

the equation’s **general form** is

\[
a x + b y + c z = d;
\]

the equation’s **vector form** is

\[
(x, y, z) \cdot \vec{n} = (x_0, y_0, z_0) \cdot \vec{n} = d.
\]

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

**Example 1 Finding the equation of a plane.**

Write the equation of the plane that passes through the points \( P = (1, 1, 0) \), \( Q = (1, 2, -1) \) and \( R = (0, 1, 2) \) in standard form.

**Solution** We need a vector \( \vec{n} \) that is orthogonal to the plane. Since \( P, Q \) and \( R \) are in the plane, so are the vectors \( \vec{PQ} \) and \( \vec{PR} \); \( \vec{PQ} \times \vec{PR} \) is orthogonal to \( \vec{PQ} \) and \( \vec{PR} \) and hence the plane itself.

It is straightforward to compute \( \vec{n} = \vec{PQ} \times \vec{PR} = (2, 1, 1) \). We can use any point we wish in the plane (any of \( P, Q \) or \( R \) will do) and we arbitrarily choose \( P \).

Notes:
Following Definition 66, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$ 

The plane is sketched in Figure 11.56.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not “rock;” it’s three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

**Example 2** Finding the equation of a plane.

Verify that lines $\ell_1$ and $\ell_2$, whose parametric equations are given below, intersect, then give the equation of the plane that contains these two lines in general form.

$\ell_1: \quad \begin{align*} x &= -5 + 2s \\ y &= 1 + s \\ z &= -4 + 2s \end{align*}$

$\ell_2: \quad \begin{align*} x &= 2 + 3t \\ y &= 1 - 2t \\ z &= 1 + t \end{align*}$

**SOLUTION** The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the $x$, $y$ and $z$ equations equal to each other and solve for $s$ and $t$:

$$\begin{align*}
-5 + 2s &= 2 + 3t \\
1 + s &= 1 - 2t \\
-4 + 2s &= 1 + t
\end{align*}$$

$$s = 2, \quad t = -1.$$

When $s = 2$ and $t = -1$, the lines intersect at the point $P = (-1, 3, 0)$.

Let $\vec{d}_1 = (2, 1, 2)$ and $\vec{d}_2 = (3, -2, 1)$ be the directions of lines $\ell_1$ and $\ell_2$, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d}_1$ and $\vec{d}_2$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n} = \vec{d}_1 \times \vec{d}_2 = (5, 4, -7)$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose $P$, the point of intersection.

We follow Definition 66 to write the plane’s equation in general form:

$$5(x + 1) + 4(y - 3) - 7z = 0$$
$$5x + 5 + 4y - 12 - 7z = 0$$
$$5x + 4y - 7z = 7.$$ 

The plane’s equation in general form is $5x + 4y - 7z = 7$; it is sketched in Figure 11.57.

Notes:
Example 3 Finding the equation of a plane
Give the equation, in standard form, of the plane that passes through the point $P = (-1, 0, 1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t) = (-1, 0, 1) + t\langle 1, 2, 2 \rangle$.

**SOLUTION**  As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d} = \langle 1, 2, 2 \rangle$. We use this as our normal vector. Thus the plane’s equation, in standard form, is

$$(x + 1) + 2y + 2(z - 1) = 0.$$  

The line and plane are sketched in Figure 11.58.

Example 4 Finding the intersection of two planes
Give the parametric equations of the line that is the intersection of the planes $p_1$ and $p_2$, where:

$p_1 : x - (y - 2) + (z - 1) = 0$
$p_2 : -2(x - 2) + (y + 1) + (z - 3) = 0$

**SOLUTION**  To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for $z$:

$p_1 : z = -x + y - 1$
$p_2 : z = 2x - y - 2$

We can now set these two equations equal to each other (i.e., we are finding values of $x$ and $y$ where the planes have the same $z$ value):

$-x + y - 1 = 2x - y - 2$
$2y = 3x - 1$
$y = \frac{1}{2}(3x - 1)$

We can choose any value for $x$; we choose $x = 1$. This determines that $y = 1$. We can now use the equations of either plane to find $z$: when $x = 1$ and $y = 1$, $z = -1$ on both planes. We have found a point $P$ on the line: $P = (1, 1, -1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for $p_1$ and $p_2$, we can quickly determine their normal vectors. For $p_1$,
\( \vec{n}_1 = (1, -1, 1) \) and for \( \vec{p}_2, \vec{n}_2 = (-2, 1, 1) \). A direction orthogonal to both of these directions is their cross product: \( \vec{d} = \vec{n}_1 \times \vec{n}_2 = (-2, -3, -1) \).

The parametric equations of the line through \( P = (1, 1, -1) \) in the direction of \( \vec{d} = (-2, -3, -1) \) is:

\[
\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.
\]

The planes and line are graphed in Figure 11.59.

**Example 5**  **Finding the intersection of a plane and a line**

Find the point of intersection, if any, of the line \( \ell(t) = (3, -3, -1) + t(1, 2, 1) \) and the plane with equation in general form \( 2x + y + z = 4 \).

**Solution**  
The equation of the plane shows that the vector \( \vec{n} = (2, 1, 1) \) is a normal vector to the plane, and the equation of the line shows that the line moves parallel to \( \vec{d} = (-1, 2, 1) \). Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a \( t \) value such that \( \ell(t) \) satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

\[
\ell(t) = \begin{cases} 
  x = 3 - t \\
  y = -3 + 2t \\
  z = -1 + t 
\end{cases}.
\]

Replacing \( x, y \) and \( z \) in the equation of the plane with the expressions containing \( t \) found in the equation of the line allows us to determine a \( t \) value that indicates the point of intersection:

\[
2x + y + z = 4 \\
2(3 - t) + (-3 + 2t) + (-1 + t) = 4 \\
t = 2.
\]

When \( t = 2 \), the point on the line satisfies the equation of the plane; that point is \( \ell(2) = (1, 1, 1) \). Thus the point \( (1, 1, 1) \) is the point of intersection between the plane and the line, illustrated in Figure 11.60.

---

Notes:
Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 11.61, where a plane with normal vector \( \vec{n} \) is sketched containing a point \( P \) and a point \( Q \), not on the plane, is given. We measure the distance from \( Q \) to the plane by measuring the length of the projection of \( \vec{PQ} \) onto \( \vec{n} \). That is, we want:

\[
\| \text{proj}_{\vec{n}} \vec{PQ} \| = \frac{\vec{n} \cdot \vec{PQ}}{||\vec{n}||^2} = \frac{\vec{n} \cdot \vec{PQ}}{||\vec{n}||} 
\]  

Equation (11.13) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (11.13) is important, we restate it as a Key Idea.

**Key Idea 54 Distance from a Point to a Plane**

Let a plane with normal vector \( \vec{n} \) be given, and let \( Q \) be a point. The distance \( h \) from \( Q \) to the plane is

\[
h = \frac{\vec{n} \cdot \vec{PQ}}{||\vec{n}||},
\]

where \( P \) is any point in the plane.

**Example 6 Distance between a point and a plane**

Find the distance between the point \( Q = (2, 1, 4) \) and the plane with equation \( 2x - 5y + 6z = 9 \).

**Solution** Using the equation of the plane, we find the normal vector \( \vec{n} = \langle 2, -5, 6 \rangle \). To find a point on the plane, we can let \( x \) and \( y \) be anything we choose, then let \( z \) be whatever satisfies the equation. Letting \( x \) and \( y \) be 0 seems simple; this makes \( z = 1.5 \). Thus we let \( P = \langle 0, 0, 1.5 \rangle \), and \( \vec{PQ} = \langle 2, 1, 2.5 \rangle \).
The distance $h$ from $Q$ to the plane is given by Key Idea 54:

$$h = \frac{\mathbf{n} \cdot \mathbf{PQ}}{||\mathbf{n}||}$$

$$= \frac{|(2, -5, 6) \cdot (2, 1, 2.5)|}{||2, -5, 6||}$$

$$= \frac{|14|}{\sqrt{65}}$$

$$= \frac{14}{\sqrt{65}}$$

We can use Key Idea 54 to find other distances. Given two parallel planes, we can find the distance between these planes by letting $P$ be a point on one plane and $Q$ a point on the other. If $l$ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let $P$ be a point in the plane and let $Q$ be any point on the line. (One can also use Key Idea 53.) The exercises contain several problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, approximated) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.
Exercises 11.6

Terms and Concepts

1. In order to find the equation of a plane, what two pieces of information must one have?
2. What is the relationship between a plane and one of its normal vectors?

Problems

In Exercises 3–6, give any two points in the given plane.

3. $2x - 4y + 7z = 2$
4. $3(x + 2) + 5(y - 9) - 4z = 0$
5. $x = 2$
6. $4(y + 2) - (z - 6) = 0$

In Exercises 7–20, give the equation of the described plane in standard and general forms.

7. Passes through $(2, 3, 4)$ and has normal vector $\vec{n} = (3, -1, 7)$.
8. Passes through $(1, 3, 5)$ and has normal vector $\vec{n} = (0, 2, 4)$.
9. Passes through the points $(1, 2, 3), (3, -1, 4)$ and $(1, 0, 1)$.
10. Passes through the points $(5, 3, 8), (6, 4, 9)$ and $(3, 3, 3)$.
11. Contains the intersecting lines $\vec{L}_1(t) = (2, 1, 2) + t (1, 2, 3)$ and $\vec{L}_2(t) = (2, 1, 2) + t (2, 5, 4)$.
12. Contains the intersecting lines $\vec{L}_1(t) = (5, 0, 3) + t (-1, 1, 1)$ and $\vec{L}_2(t) = (1, 4, 7) + t (3, 0, -3)$.
13. Contains the parallel lines $\vec{L}_1(t) = (1, 1, 1) + t (1, 2, 3)$ and $\vec{L}_2(t) = (1, 1, 2) + t (1, 2, 3)$.
14. Contains the parallel lines $\vec{L}_1(t) = (1, 1, 1) + t (4, 1, 3)$ and $\vec{L}_2(t) = (2, 2, 2) + t (4, 1, 3)$.
15. Contains the point $(2, -6, 1)$ and the line $x = 2 + 5t$
   $y = 2 + 2t$
   $z = -1 + 2t$
16. Contains the point $(5, 7, 3)$ and the line $x = t$
   $y = t$
   $z = t$
17. Contains the point $(5, 7, 3)$ and is orthogonal to the line $\vec{L}(t) = (4, 5, 6) + t (1, 1, 1)$.
18. Contains the point $(4, 1, 1)$ and is orthogonal to the line $\begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$
19. Contains the point $(-4, 7, 2)$ and is parallel to the plane $3(x - 2) + 8(y + 1) - 10z = 0$.
20. Contains the point $(1, 2, 3)$ and is parallel to the plane $x = 5$.

In Exercises 21–22, give the equation of the line that is the intersection of the given planes.

21. $p_1 : 3(x - 2) + (y - 1) + 4z = 0$, and $p_2 : 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$.
22. $p_1 : 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$, and $p_2 : 3x - 4(y - 1) + 2(z - 1) = 0$.

In Exercises 23–26, find the point of intersection between the line and the plane.

23. Line: $(5, 1, -1) + t (2, 2, 1)$, plane: $5x - y - z = -3$
24. Line: $(4, 1, 0) + t (1, 0, -1)$, plane: $3x + y - 2z = 8$
25. Line: $(1, 2, 3) + t (3, 5, -1)$, plane: $3x - 2y - z = 4$
26. Line: $(1, 2, 3) + t (3, 5, -1)$, plane: $3x - 2y - z = -4$

In Exercises 27–30, find the given distances.

27. The distance from the point $(1, 2, 3)$ to the plane $3(x - 1) + (y - 2) + 5(z - 2) = 0$.
28. The distance from the point $(2, 6, 2)$ to the plane $2(x - 1) - y + 4(z + 1) = 0$.
29. The distance between the parallel planes $x + y + z = 0$ and $(x - 2) + (y - 3) + (z + 4) = 0$
30. The distance between the parallel planes $2(x - 1) + 2(y + 1) + (z - 2) = 0$ and $2(x - 3) + 2(y - 1) + (z - 3) = 0$
31. Show why if the point $Q$ lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.
32. How is Exercise 30 in Section 11.5 easier to answer once we have an understanding of planes?
11.7 Curvilinear Coordinates

The Cartesian coordinates of a point \((x, y, z)\) are determined by following straight paths starting from the origin: first along the \(x\)-axis, then parallel to the \(y\)-axis, then parallel to the \(z\)-axis, as in Figure 11.62(a). In curvilinear coordinate systems, these paths can be curved. The two types of curvilinear coordinates which we will consider are cylindrical and spherical coordinates. Instead of referencing a point in terms of sides of a rectangular parallelepiped, as with Cartesian coordinates, we will think of the point as lying on a cylinder or sphere. Cylindrical coordinates are often used when there is symmetry around the \(z\)-axis; spherical coordinates are useful when there is symmetry about the origin.

Let \(P = (x, y, z)\) be a point in Cartesian coordinates in \(\mathbb{R}^3\), and let \(P_0 = (x, y, 0)\) be the projection of \(P\) upon the \(xy\)-plane. Treating \((x, y)\) as a point in \(\mathbb{R}^2\), let \((r, \theta)\) be its polar coordinates (see Figure 11.62(b)). Let \(\rho\) be the length of the line segment from the origin to \(P\), and let \(\phi\) be the angle between that line segment and the positive \(z\)-axis (see Figure 11.62(c)), which is called the zenith angle. Then the cylindrical coordinates \((r, \theta, z)\) and the spherical coordinates \((\rho, \theta, \phi)\) of \(P(x, y, z)\) are defined as follows:

**Key Idea 55 Cylindrical coordinates \((r, \theta, z)\)**

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

where \(0 \leq \theta \leq \pi\) if \(y \geq 0\) and \(\pi < \theta < 2\pi\) if \(y < 0\).

**Key Idea 56 Spherical coordinates \((\rho, \theta, \phi)\)**

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi
\end{align*}
\]

where \(0 \leq \theta \leq \pi\) if \(y \geq 0\) and \(\pi < \theta < 2\pi\) if \(y < 0\).

This “standard” definition of spherical coordinates used by mathematicians results in a left-handed system. For this reason, physicists usually switch the definitions of \(\theta\) and \(\phi\) to make \((\rho, \theta, \phi)\) a right-handed system.

Figure 11.62: Cartesian (top), cylindrical (middle), and spherical (bottom) coordinate systems
Chapter 11  Vectors

Both $\theta$ and $\phi$ are measured in radians. Note that $r \geq 0$, $0 \leq \theta < 2\pi$, $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Also, $\theta$ is undefined when $(x,y) = (0,0)$, and $\phi$ is undefined when $(x,y,z) = (0,0,0)$.

Watch the video:
Conversion From Rectangular Coordinates at https://youtu.be/w9wCdLuk1w8

Example 1  Converting Between Coordinate Systems
Convert the point $(-2,-2,1)$ from Cartesian coordinates to 1. cylindrical and 2. spherical coordinates.

**SOLUTION**
1. $r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$, since $y = -2 < 0$. Therefore $(r, \theta, z) = (2\sqrt{2}, \frac{\pi}{4}, 1)$.
2. $\rho = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$ and $\phi = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23$ radians. Therefore $(\rho, \theta, \phi) = (3, \frac{\pi}{4}, 1.23)$.

For cylindrical coordinates $(r, \theta, z)$, and constants $r_0$, $\theta_0$ and $z_0$, we see from Figure 11.63 that the surface $r = r_0$ is a cylinder of radius $r_0$ centered along the $z$-axis, the surface $\theta = \theta_0$ is a half-plane emanating from the $z$-axis, and the surface $z = z_0$ is a plane parallel to the $xy$-plane.

Figure 11.63: Cylindrical coordinate surfaces

Notes:
For spherical coordinates \((\rho, \theta, \phi)\), and constants \(\rho_0, \theta_0\) and \(\phi_0\), we see from Figure 11.64 that the surface \(\rho = \rho_0\) is a sphere of radius \(\rho_0\) centered at the origin, the surface \(\theta = \theta_0\) is a half-plane emanating from the \(z\)-axis, and the surface \(\phi = \phi_0\) is a circular cone whose vertex is at the origin.

Figure 11.64: Spherical coordinate surfaces

Figures 11.63(a) and 11.64(a) show how these coordinate systems got their names.

Sometimes the equation of a surface in Cartesian coordinates can be transformed into a simpler equation in some other coordinate system, as in the following example.

**Example 2**  Converting an Equation in Coordinate Systems

Write the equation of the cylinder \(x^2 + y^2 = 4\) in cylindrical coordinates.

**Solution** Since \(r = \sqrt{x^2 + y^2}\), then the equation in cylindrical coordinates is \(r = 2\).

Using spherical coordinates to write the equation of a sphere does not necessarily make the equation simpler, if the sphere is not centered at the origin.

**Example 3**  Converting an Equation to Spherical Coordinates

Write the equation \((x - 2)^2 + (y - 1)^2 + z^2 = 9\) in spherical coordinates.

**Solution** Multiplying the equation out gives

\[x^2 + y^2 + z^2 - 4x - 2y + 5 = 9, \text{ so we get}
\]
\[\rho^2 - 4\rho\sin\phi\cos\theta - 2\rho\sin\phi\sin\theta - 4 = 0, \text{ or}
\]
\[\rho^2 - 2\sin\phi(2\cos\theta - \sin\theta)\rho - 4 = 0
\]
after combining terms. Note that this actually makes it more difficult to figure out what the surface is, as opposed to the Cartesian equation where you could immediately identify the surface as a sphere of radius 3 centered at $(2, 1, 0)$.

Example 4 Identifying a Surface
Describe the surface given by $\theta = z$ in cylindrical coordinates.

**SOLUTION** This surface is called a *helicoid*. As the (vertical) $z$ coordinate increases, so does the angle $\theta$, while the radius $r$ is unrestricted. So this sweeps out a (ruled!) surface shaped like a spiral staircase, where the spiral has an infinite radius. Figure 11.65 shows a section of this surface restricted to $0 \leq z \leq 4\pi$ and $0 \leq r \leq 2$.

Notes:
Figure 11.65: Helicoid $\theta = z$
Exercises 11.7

Problems

In Exercises 1–4, find the (a) cylindrical and (b) spherical coordinates of the point whose Cartesian coordinates are given.

1. \((2, 2\sqrt{3}, -1)\)
2. \((-5, 5, 6)\)
3. \((\sqrt{21}, -\sqrt{7}, 0)\)
4. \((0, \sqrt{2}, 2)\)

In Exercises 5–7, write the given equation in (a) cylindrical and (b) spherical coordinates.

5. \(x^2 + y^2 + z^2 = 25\)
6. \(x^2 + y^2 = 2y\)
7. \(x^2 + y^2 + 9z^2 = 36\)

8. Describe the intersection of the surfaces whose equations in spherical coordinates are \(\theta = \frac{\pi}{2}\) and \(\phi = \frac{\pi}{2}\).

9. Show that for \(a \neq 0\), the equation \(\rho = 2a \sin \phi \cos \theta\) in spherical coordinates describes a sphere centered at \((a, 0, 0)\) with radius \(|a|\).

10. Let \(P = (a, \theta, \phi)\) be a point in spherical coordinates, with \(a > 0\) and \(0 < \phi < \pi\). Then \(P\) lies on the sphere \(\rho = a\). Since \(0 < \phi < \pi\), the line segment from the origin to \(P\) can be extended to intersect the cylinder given by \(r = a\) (in cylindrical coordinates). Find the cylindrical coordinates of that point of intersection.

11. Let \(P_1\) and \(P_2\) be points whose spherical coordinates are \((\rho_1, \theta_1, \phi_1)\) and \((\rho_2, \theta_2, \phi_2)\), respectively. Let \(\vec{v}_1\) be the vector from the origin to \(P_1\) and let \(\vec{v}_2\) be the vector from the origin to \(P_2\). For the angle \(\gamma\) between \(\vec{v}_1\) and \(\vec{v}_2\), show that

   \[\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos (\theta_2 - \theta_1)\]

   This formula is used in electrodynamics to prove the addition theorem for spherical harmonics, which provides a general expression for the electrostatic potential at a point due to a unit charge.

12. Show that the distance \(d\) between the points \(P_1\) and \(P_2\) with cylindrical coordinates \((r_1, \theta_1, z_1)\) and \((r_2, \theta_2, z_2)\), respectively, is

   \[d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_2 - \theta_1) + (z_2 - z_1)^2}\]

13. Show that the distance \(d\) between the points \(P_1\) and \(P_2\) with spherical coordinates \((\rho_1, \theta_1, \phi_1)\) and \((\rho_2, \theta_2, \phi_2)\), respectively, is

   \[d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \sin \phi_1 \sin \phi_2 \cos (\theta_2 - \theta_1) + \cos \phi_1 \cos \phi_2}\].
12: Vector Valued Functions

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we’ll build on this foundation to define functions whose input is a real number and whose output is a vector. We’ll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we’ll see why we are interested in doing this: we’ll see beautiful applications to the study of moving objects.

12.1 Vector–Valued Functions

We are very familiar with real valued functions, that is, functions whose output is a real number. This section introduces vector–valued functions — functions whose output is a vector.

Definition 67 Vector–Valued Functions
A vector–valued function is a function of the form
\[ \vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \]
where \( f, g \) and \( h \) are real valued functions.

The domain of \( \vec{r} \) is the set of all values of \( t \) for which \( \vec{r}(t) \) is defined. The range of \( \vec{r} \) is the set of all possible output vectors \( \vec{r}(t) \).

Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of \( t \) is straightforward; simply evaluate each component function at that value of \( t \). For instance, if \( \vec{r}(t) = \langle t^2, t^2 + t - 1 \rangle \), then \( \vec{r}(-2) = \langle 4, 1 \rangle \). We can sketch this vector, as is done in Figure 12.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The graph of a vector–valued function is the set of all terminal points of \( \vec{r}(t) \), where the initial point of each vector is always the origin. In Figure 12.1(b) we sketch the graph of \( \vec{r} \); we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points \((x(t), y(t))\) or \((x(t), y(t), z(t))\) to produce

Figure 12.1: Sketching the graph of a vector–valued function.
a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

Watch the video:
Domain of a Vector–Valued Function at https://youtu.be/DjtttmOC7zA

Example 1  Graphing vector–valued functions

Graph \( \vec{r}(t) = \langle t^3 - t, \frac{1}{t^2 + 1} \rangle \), for \(-2 \leq t \leq 2\). Sketch \( \vec{r}(-1) \) and \( \vec{r}(2) \).

**SOLUTION**

We start by making a table of \( t, x \) and \( y \) values as shown in Figure 12.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 12.2(b), we indicate these points and sketch the full graph. We also highlight \( \vec{r}(-1) \) and \( \vec{r}(2) \) on the graph.

Example 2  Graphing vector–valued functions.

Graph \( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \) for \( 0 \leq t \leq 4\pi \).

**SOLUTION**

We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the \( x \) and \( y \) components trace out a circle of radius 1 centered at the origin. Noticing that the \( z \) component is \( t \), we see that as the graph winds around the \( z \)-axis, it is also increasing at a constant rate in the positive \( z \) direction, forming a spiral. This is graphed in Figure 12.3. In the graph \( \vec{r}(7\pi/4) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{7\pi}{4} \rangle \) is highlighted to help us understand the graph.

**Algebra of Vector–Valued Functions**

Notes:
Definition 68  Operations on Vector–Valued Functions
Let \( \mathbf{r}_1(t) = (f_1(t), g_1(t)) \) and \( \mathbf{r}_2(t) = (f_2(t), g_2(t)) \) be vector–valued functions in \( \mathbb{R}^2 \) and let \( c \) be a scalar. Then:

1. \( \mathbf{r}_1(t) \pm \mathbf{r}_2(t) = (f_1(t) \pm f_2(t), g_1(t) \pm g_2(t)). \)
2. \( c\mathbf{r}_1(t) = (cf_1(t), cg_1(t)). \)

A similar definition holds for vector–valued functions in \( \mathbb{R}^3 \).

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 3  Adding and scaling vector–valued functions.
Let \( \mathbf{r}_1(t) = (0.2t, 0.3t) \), \( \mathbf{r}_2(t) = (\cos t, \sin t) \) and \( \mathbf{r}(t) = \mathbf{r}_1(t) + \mathbf{r}_2(t) \). Graph \( \mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}(t) \) and \( 5\mathbf{r}(t) \) on \(-10 \leq t \leq 10 \).

**Solution**  We can graph \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) easily by plotting points (or just using technology). Let’s think about each for a moment to better understand how vector–valued functions work.

We can rewrite \( \mathbf{r}_1(t) = (0.2t, 0.3t) \) as \( \mathbf{r}_1(t) = t(0.2, 0.3) \). That is, the function \( \mathbf{r}_1 \) scales the vector \((0.2, 0.3)\) by \( t \). This scaling of a vector produces a line in the direction of \((0.2, 0.3)\).

We are familiar with \( \mathbf{r}_2(t) = (\cos t, \sin t) \); it traces out a circle, centered at the origin, of radius 1. Figure 12.4(a) graphs \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(t) \).

Adding \( \mathbf{r}_1(t) \) to \( \mathbf{r}_2(t) \) produces \( \mathbf{r}(t) = (\cos t + 0.2t, \sin t + 0.3t) \), graphed in Figure 12.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of \((0.2, 0.3)\). (We encourage the reader to experiment by changing \( \mathbf{r}_1(t) \) to \((2t, 3t)\), etc., and observe the effects on the loops.)

Multiplying \( \mathbf{r}(t) \) by 5 scales the function by 5, producing \( 5\mathbf{r}(t) = (5\cos t + 1.5\sin t + 1.5) \), which is graphed in Figure 12.4(c) along with \( \mathbf{r}(t) \). The new function is “5 times bigger” than \( \mathbf{r}(t) \). Note how the graph of \( 5\mathbf{r}(t) \) in (c) looks identical to the graph of \( \mathbf{r}(t) \) in (b). This is due to the fact that the \( x \) and \( y \) bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

Example 4  Adding and scaling vector–valued functions.
A **cycloid** is a graph traced by a point \( p \) on a rolling circle, as shown in Figure 12.6. Find an equation describing the cycloid, where the circle has radius 1.
This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point $p$ on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = (\cos t, -\sin t)$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the $y$ component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The $x$ component of $\vec{c}(t)$ is a linear function of $t$: $f(t) = mt$ for some scalar $m$. When $t = 0$, $f(t) = 0$ (the circle starts centered on the $y$-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its circumference, which is also $2\pi$. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = (t, 1)$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = (\cos t + t, -\sin t + 1)$, which is graphed in Figure 12.5.

**Displacement**

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time $t$. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$, give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the displacement.

**Definition 69 Displacement**

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The displacement $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

Notes:
When the displacement vector is drawn with initial point at \( \vec{r}(t_0) \), its terminal point is \( \vec{r}(t_1) \). We think of it as the vector which points from a starting position to an ending position.

**Example 5** Finding and graphing displacement vectors

Let \( \vec{r}(t) = \langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \rangle \). Graph \( \vec{r}(t) \) on \(-1 \leq t \leq 1\), and find the displacement of \( \vec{r}(t) \) on this interval.

**Solution** The function \( \vec{r}(t) \) traces out the unit circle, though at a different rate than the “usual” \( \langle \cos t, \sin t \rangle \) parametrization. At \( t_0 = -1 \), we have \( \vec{r}(t_0) = \langle 0, -1 \rangle \); at \( t_1 = 1 \), we have \( \vec{r}(t_1) = \langle 0, 1 \rangle \). The displacement of \( \vec{r}(t) \) on \([-1, 1]\) is thus \( \vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle \).

A graph of \( \vec{r}(t) \) on \([-1, 1]\) is given in Figure 12.6, along with the displacement vector \( \vec{d} \) on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute \( \vec{d} = \langle 0, 2 \rangle \). However, measuring distance from the starting point is different from measuring distance traveled. Being a semi–circle, we can measure the distance traveled by this object as \( \pi \) units. Knowing distance from the starting point allows us to compute **average rate of change**.

**Definition 70** Average Rate of Change

Let \( \vec{r}(t) \) be a vector–valued function, where each of its component functions is continuous on its domain, and let \( t_0 < t_1 \). The **average rate of change** of \( \vec{r}(t) \) on \([t_0, t_1]\) is

\[
\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.
\]

**Example 6** Average rate of change

Let \( \vec{r}(t) = \langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \rangle \) as in Example 5. Find the average rate of change of \( \vec{r}(t) \) on \([-1, 1]\) and on \([-1, 5]\).

**Solution** We computed in Example 5 that the displacement of \( \vec{r}(t) \) on \([-1, 1]\) was \( \vec{d} = \langle 0, 2 \rangle \). Thus the average rate of change of \( \vec{r}(t) \) on \([-1, 1]\) is:

\[
\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle.
\]
We interpret this as follows: the object followed a semi-circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. On average, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle.$$

As it took “3 times as long” to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector-valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.
Terms and Concepts

1. Vector–valued functions are closely related to _______ of graphs.
2. When sketching vector–valued functions, technically one isn't graphing points, but rather _______.
3. It can be useful to think of _______ as a vector that points from a starting position to an ending position.

Problems

In Exercises 4–11, sketch the vector–valued function on the given interval.

4. \( \vec{r}(t) = \langle t^2, t^3 - 1 \rangle \), for \(-2 \leq t \leq 2\).
5. \( \vec{r}(t) = \langle t^2, t^3 \rangle \), for \(-2 \leq t \leq 2\).
6. \( \vec{r}(t) = \langle 1/t, 1/t^2 \rangle \), for \(-2 \leq t \leq 2\).
7. \( \vec{r}(t) = \langle \frac{1}{5} t^2, \sin t \rangle \), for \(-2\pi \leq t \leq 2\pi\).
8. \( \vec{r}(t) = \langle \frac{1}{5} t^2, \sin t \rangle \), for \(-2\pi \leq t \leq 2\pi\).
9. \( \vec{r}(t) = \langle 3\sin (\pi t), 2\cos (\pi t) \rangle \), on \([0, 2]\).
10. \( \vec{r}(t) = \langle 3\cos t, 2\sin (2t) \rangle \), on \([0, 2\pi]\).
11. \( \vec{r}(t) = \langle 2 \sec t, \tan t \rangle \), on \([-\pi, \pi]\).

In Exercises 12–15, sketch the vector–valued function on the given interval in \(\mathbb{R}^3\). Technology may be useful in creating the sketch.

12. \( \vec{r}(t) = \langle 2\cos t, t, 2\sin t \rangle \), on \([0, 2\pi]\).
13. \( \vec{r}(t) = \langle 3\cos t, \sin t, t/\pi \rangle \) on \([0, 2\pi]\).
14. \( \vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle \) on \([0, 2\pi]\).
15. \( \vec{r}(t) = \langle \cos t, \sin t, \sin (2t) \rangle \) on \([0, 2\pi]\).

In Exercises 16–19, find \( |\vec{r}(t)| \).

16. \( \vec{r}(t) = \langle t, t^2 \rangle \).
17. \( \vec{r}(t) = \langle 5\cos t, 3\sin t \rangle \).
18. \( \vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle \).
19. \( \vec{r}(t) = \langle \cos t, t, t^2 \rangle \).

In Exercises 20–27, create a vector–valued function whose graph matches the given description.

20. A circle of radius 2, centered at \((1, 2)\), traced counterclockwise once on \([0, 2\pi]\).
21. A circle of radius 3, centered at \((5, 5)\), traced clockwise once on \([0, 2\pi]\).
22. An ellipse, centered at \((0, 0)\) with vertical major axis of length 10 and minor axis of length 3, traced once clockwise on \([0, 2\pi]\).
23. An ellipse, centered at \((3, -2)\) with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on \([0, 2\pi]\).
24. A line through \((2, 3)\) with a slope of 5.
25. A line through \((1, 5)\) with a slope of \(-1/2\).
26. A vertically oriented helix with radius of 2 that starts at \((2, 0, 0)\) and ends at \((2, 0, 4\pi)\) after 1 revolution on \([0, 2\pi]\).
27. A vertically oriented helix with radius of 3 that starts at \((3, 0, 0)\) and ends at \((3, 0, 3)\) after 2 revolutions on \([0, 1]\).

In Exercises 28–31, find the average rate of change of \( \vec{r}(t) \) on the given interval.

28. \( \vec{r}(t) = \langle t, t^2 \rangle \) on \([-2, 2]\).
29. \( \vec{r}(t) = \langle t, t + \sin t \rangle \) on \([0, 2\pi]\).
30. \( \vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle \) on \([0, 2\pi]\).
31. \( \vec{r}(t) = \langle t, t^2, t^3 \rangle \) on \([-1, 3]\).
12.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition 1. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 71 Limits of Vector–Valued Functions
Let $I$ be an open interval containing $c$, and let $\vec{r}(t)$ be a vector–valued function defined on $I$, except possibly at $c$. The limit of $\vec{r}(t)$, as $t$ approaches $c$, is $\vec{L}$, expressed as

$$\lim_{t \to c} \vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \neq c$, if $|t - c| < \delta$, we have $||\vec{r}(t) - \vec{L}|| < \varepsilon$.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Notes:
12.2 Calculus and Vector–Valued Functions

Theorem 91 Limits of Vector–Valued Functions

1. Let \( \mathbf{r}(t) = \langle f(t), g(t) \rangle \) be a vector–valued function in \( \mathbb{R}^2 \) defined on an open interval \( I \) containing \( c \). Then
   \[
   \lim_{t \to c} \mathbf{r}(t) = \langle \lim_{t \to c} f(t), \lim_{t \to c} g(t) \rangle.
   \]

2. Let \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) be a vector–valued function in \( \mathbb{R}^3 \) defined on an open interval \( I \) containing \( c \). Then
   \[
   \lim_{t \to c} \mathbf{r}(t) = \langle \lim_{t \to c} f(t), \lim_{t \to c} g(t), \lim_{t \to c} h(t) \rangle.
   \]

Theorem 91 states that we compute limits component–wise.

Example 1 Finding limits of vector–valued functions

Let \( \mathbf{r}(t) = \langle \frac{\sin t}{t}, t^2 - 3t + 3, \cos t \rangle \). Find \( \lim_{t \to 0} \mathbf{r}(t) \).

Solution We apply the theorem and compute limits component–wise.

\[
\lim_{t \to 0} \mathbf{r}(t) = \langle \lim_{t \to 0} \frac{\sin t}{t}, \lim_{t \to 0} t^2 - 3t + 3, \lim_{t \to 0} \cos t \rangle = \langle 1, 3, 1 \rangle.
\]

Continuity

Definition 72 Continuity of Vector–Valued Functions

Let \( \mathbf{r}(t) \) be a vector–valued function defined on an open interval \( I \) containing \( c \).

1. \( \mathbf{r}(t) \) is continuous at \( c \) if \( \lim_{t \to c} \mathbf{r}(t) = \mathbf{r}(c) \).

2. If \( \mathbf{r}(t) \) is continuous at all \( c \) in \( I \), then \( \mathbf{r}(t) \) is continuous on \( I \).

We again have a theorem that lets us evaluate continuity component–wise.

Notes:
Chapter 12  Vector Valued Functions

**Theorem 92  Continuity of Vector–Valued Functions**
Let \( \vec{r}(t) \) be a vector–valued function defined on an open interval \( I \) containing \( c \). Then \( \vec{r}(t) \) is continuous at \( c \) if, and only if, each of its component functions is continuous at \( c \).

**Example 2  Evaluating continuity of vector–valued functions**
Let \( \vec{r}(t) = \left\langle \frac{\sin t}{t}, t^2 - 3t + 3, \cos t \right\rangle \). Determine whether \( \vec{r} \) is continuous at \( t = 0 \) and \( t = 1 \).

**Solution**  While the second and third components of \( \vec{r}(t) \) are defined at \( t = 0 \), the first component, \( (\sin t)/t \), is not. Since the first component is not even defined at \( t = 0 \), \( \vec{r}(t) \) is not defined at \( t = 0 \), and hence it is not continuous at \( t = 0 \).
At \( t = 1 \) each of the component functions is continuous. Therefore \( \vec{r}(t) \) is continuous at \( t = 1 \).

**Derivatives**
Consider a vector–valued function \( \vec{r} \) defined on an open interval \( I \) containing \( t_0 \) and \( t_1 \). We can compute the displacement of \( \vec{r} \) on \( [t_0, t_1] \), as shown in Figure 12.7(a). Recall that dividing the displacement vector by \( t_1 - t_0 \) gives the average rate of change on \( [t_0, t_1] \), as shown in (b).

![Diagram of vector-valued function](image)

Figure 12.7: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the _instantaneous_ rate of change, measured by taking the limit as the length of \( [t_0, t_1] \) goes to 0. Instead of thinking of an interval as \( [t_0, t_1] \), we think of it as \( [c, c + h] \) for some
value of $h$ (hence the interval has length $h$). The average rate of change is

$$\frac{\vec{r}(c + h) - \vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

**Definition 73 Derivative of a Vector–Valued Function**

Let $\vec{r}(t)$ be continuous on an open interval $I$ containing $c$.

1. The derivative of $\vec{r}$ at $t = c$ is

$$\vec{r}'(c) = \lim_{h \to 0} \frac{\vec{r}(c + h) - \vec{r}(c)}{h}.$$  

2. The derivative of $\vec{r}$ is

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t + h) - \vec{r}(t)}{h}.$$  

If a vector–valued function has a derivative for all $c$ in an open interval $I$, we say that $\vec{r}(t)$ is **differentiable** on $I$.

Once again we might view this definition as intimidating, but recall that we can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

**Theorem 93 Derivatives of Vector–Valued Functions**

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$. Then

$$\vec{r}'(t) = \langle f'(t), g'(t) \rangle.$$  

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$  

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}'(t) = \frac{d}{dt}(\vec{r}(t)) = \frac{d\vec{r}}{dt}.$$
Chapter 12  Vector Valued Functions

Example 3  Derivatives of vector–valued functions
Let \( \vec{r}(t) = \langle t^2, t \rangle \).

1. Sketch \( \vec{r}(t) \) and \( \vec{r}'(t) \) on the same axes.

2. Compute \( \vec{r}'(1) \) and sketch this vector with its initial point at the origin and at \( \vec{r}(1) \).

SOLUTION

1. Theorem 93 allows us to compute derivatives component–wise, so
   \[
   \vec{r}'(t) = \langle 2t, 1 \rangle.
   \]

   \( \vec{r}(t) \) and \( \vec{r}'(t) \) are graphed together in Figure 12.8(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real–valued functions, plotting \( f(x) \) with \( f'(x) \) gave us useful information as we were able to compare \( f \) and \( f' \) at the same \( x \)-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of \( \vec{r}' \) correspond to which points on the graph of \( \vec{r} \).

2. We easily compute \( \vec{r}'(1) = \langle 2, 1 \rangle \), which is drawn in Figure 12.8 with its initial point at the origin, as well as at \( \vec{r}(1) = \langle 1, 1 \rangle \). These are sketched in Figure 12.8(b).

Example 4  Derivatives of vector–valued functions
Let \( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \). Compute \( \vec{r}'(t) \) and \( \vec{r}'(\pi/2) \). Sketch \( \vec{r}'(\pi/2) \) with its initial point at the origin and at \( \vec{r}(\pi/2) \).

SOLUTION

We compute \( \vec{r}' \) as \( \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \). At \( t = \pi/2 \), we have \( \vec{r}'(\pi/2) = \langle -1, 0, 1 \rangle \). Figure 12.9 shows two graphs of \( \vec{r}(t) \), from different perspectives, with \( \vec{r}'(\pi/2) \) plotted with its initial point at the origin and at \( \vec{r}(\pi/2) \).

Notes:
In Examples 3 and 4, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be tangent to the graph. We have not yet defined what “tangent” means in terms of curves in space; in fact, we use the derivative to define this term.

**Definition 7.4 Tangent Vector, Tangent Line**

Let \( \vec{r}(t) \) be a differentiable vector–valued function on an open interval \( I \) containing \( c \), where \( \vec{r}'(c) \neq \vec{0} \).

1. A vector \( \vec{v} \) is **tangent to the graph of** \( \vec{r}(t) \) at \( t = c \) if \( \vec{v} \) is parallel to \( \vec{r}'(c) \).

2. The **tangent line** to the graph of \( \vec{r}(t) \) at \( t = c \) is the line through \( \vec{r}(c) \) with direction parallel to \( \vec{r}'(c) \). An equation of the tangent line is
   \[ \vec{L}(t) = \vec{r}(c) + t\vec{r}'(c). \]

**Example 5 Finding tangent lines to curves in space**

Let \( \vec{r}(t) = \langle t, t^2, t^3 \rangle \) on \([-1.5, 1.5]\). Find the vector equation of the line tangent to the graph of \( \vec{r} \) at \( t = -1 \).

**Solution** To find the equation of a line, we need a point on the line and the line’s direction. The point is given by \( \vec{r}(-1) = \langle -1, 1, -1 \rangle \). (To be clear, \( \langle -1, 1, -1 \rangle \) is a vector, not a point, but we use the point “pointed to” by this vector.)

The direction comes from \( \vec{r}'(-1) \). We compute, component–wise, \( \vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle \). Thus \( \vec{r}'(-1) = \langle 1, -2, 3 \rangle \).

The vector equation of the line is \( \vec{L}(t) = \langle -1, 1, -1 \rangle + t \langle 1, -2, 3 \rangle \). This line and \( \vec{r}(t) \) are sketched, from two perspectives, in Figure 12.10 (a) and (b).

**Example 6 Finding tangent lines to curves**

Find the equations of the lines tangent to \( \vec{r}(t) = \langle t^2, t^3 \rangle \) at \( t = -1 \) and \( t = 0 \).

**Solution** We find that \( \vec{r}'(t) = \langle 2t, 3t^2 \rangle \). At \( t = -1 \), we have

\( \vec{r}(-1) = \langle -1, 1 \rangle \) and \( \vec{r}'(-1) = \langle 3, -2 \rangle \).

Notes:
so the equation of the line tangent to the graph of \( \vec{r}(t) \) at \( t = -1 \) is

\[
\ell(t) = (-1, 1) + t (3, -2)
\]

This line is graphed with \( \vec{r}(t) \) in Figure 12.11.

At \( t = 0 \), we have \( \vec{r}'(0) = (0, 0) = \vec{0} \). This implies that the tangent line “has no direction.” We cannot apply Definition 74, hence cannot find the equation of the tangent line.

We were unable to compute the equation of the tangent line to \( \vec{r}(t) = \langle t^3, t^2 \rangle \) at \( t = 0 \) because \( \vec{r}'(0) = \vec{0} \). The graph in Figure 12.11 shows that there is a cusp at this point. This leads us to another definition of smooth, previously defined by Definition 48 in Section 10.2.

**Definition 75  Smooth Vector–Valued Functions**

Let \( \vec{r}(t) \) be a differentiable vector–valued function on an open interval \( I \). Then \( \vec{r}(t) \) is smooth on \( I \) if \( \vec{r}'(t) \) is continuous and \( \vec{r}'(t) \neq \vec{0} \) on \( I \).

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

**Theorem 94  Properties of Derivatives of Vector–Valued Functions**

Let \( \vec{r} \) and \( \vec{s} \) be differentiable vector–valued functions, let \( f \) be a differentiable real–valued function, and let \( c \) be a real number.

1. \( \frac{d}{dt} \left( \vec{r}(t) \pm \vec{s}(t) \right) = \vec{r}'(t) \pm \vec{s}'(t) \)
2. \( \frac{d}{dt} (c \vec{r}(t)) = c \vec{r}'(t) \)
3. \( \frac{d}{dt} (f(t) \vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}'(t) \)  \quad \text{Product Rule}
4. \( \frac{d}{dt} (\vec{r}(t) \cdot \vec{s}(t)) = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t) \)  \quad \text{Product Rule}
5. \( \frac{d}{dt} (\vec{r}(t) \times \vec{s}(t)) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t) \)  \quad \text{Product Rule}
6. \( \frac{d}{dt} (\vec{r}(f(t))) = \vec{r}'(f(t))f'(t) \)  \quad \text{Chain Rule}

**Note:** Because the order is important when computing a cross product, we must maintain the correct order of the functions in rule 5.

Notes:
Example 7 Using derivative properties of vector–valued functions

Let \( \mathbf{r}(t) = \langle t, t^2 - 1 \rangle \) and let \( \mathbf{u}(t) \) be the unit vector that points in the direction of \( \mathbf{r}(t) \).

1. Graph \( \mathbf{r}(t) \) and \( \mathbf{u}(t) \) on the same axes, on \([-2, 2] \).

2. Find \( \mathbf{u}'(t) \) and sketch \( \mathbf{u}'(-2) \), \( \mathbf{u}'(-1) \) and \( \mathbf{u}'(0) \). Sketch each with initial point the corresponding point on the graph of \( \mathbf{u} \).

**Solution**

1. To form the unit vector that points in the direction of \( \mathbf{r} \), we need to divide \( \mathbf{r}(t) \) by its magnitude.

\[
\|\mathbf{r}(t)\| = \sqrt{t^2 + (t^2 - 1)^2} \quad \Rightarrow \quad \mathbf{u}(t) = \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} \langle t, t^2 - 1 \rangle.
\]

\( \mathbf{r}(t) \) and \( \mathbf{u}(t) \) are graphed in Figure 12.12. Note how the graph of \( \mathbf{u}(t) \) forms part of a circle; this must be the case, as the length of \( \mathbf{u}(t) \) is 1 for all \( t \).

2. To compute \( \mathbf{u}'(t) \), we use Theorem 94, writing

\[
\mathbf{u}(t) = f(t)\mathbf{r}(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} = \left(t^2 + (t^2 - 1)^2\right)^{-1/2}.
\]

(We could write

\[
\mathbf{u}(t) = \left\langle \frac{t}{\sqrt{t^2 + (t^2 - 1)^2}}, \frac{t^2 - 1}{\sqrt{t^2 + (t^2 - 1)^2}} \right\rangle
\]

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find \( f'(t) \) using the Chain Rule:

\[
f'(t) = -\frac{1}{2} \left(t^2 + (t^2 - 1)^2\right)^{-3/2} \left(2t + 2(t^2 - 1)(2t)\right)
= -\frac{2t(2t^2 - 1)}{2\left(\sqrt{t^2 + (t^2 - 1)^2}\right)^3}
\]

We now find \( \mathbf{u}'(t) \) using part 3 of Theorem 94:

\[
\mathbf{u}'(t) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)
= -\frac{2t(2t^2 - 1)}{2\left(\sqrt{t^2 + (t^2 - 1)^2}\right)^3} \langle t, t^2 - 1 \rangle + \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} \langle 1, 2t \rangle.
\]

Notes:
Chapter 12 Vector Valued Functions

This is admittedly very “messy;” such is usually the case when we deal with unit vectors. We can use this formula to compute \( \vec{u}'(-2) \), \( \vec{u}'(-1) \) and \( \vec{u}'(0) \):

\[
\vec{u}'(-2) = \left\langle \frac{-15}{13\sqrt{13}}, \frac{10}{13\sqrt{13}} \right\rangle \\
\vec{u}'(-1) = \langle 0, -2 \rangle \\
\vec{u}'(0) = \langle 1, 0 \rangle
\]

Each of these is sketched in Figure 12.13. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When \( t = -2 \), the circle is being drawn relatively slow; when \( t = -1 \), the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point \( P \) is perpendicular to the line passing through the center of the circle and \( P \). This is illustrated in Figure 12.13; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: \( \vec{u}'(t) \) is orthogonal to \( \vec{u}(t) \).

Recall that the dot product serves as a test for orthogonality: if \( \vec{u} \cdot \vec{v} = 0 \), then \( \vec{u} \) is orthogonal to \( \vec{v} \). Thus in the above example, \( \vec{u}(t) \cdot \vec{u}'(t) = 0 \).

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as Exercise 42.)

**Theorem 95 Vector–Valued Functions of Constant Length**

Let \( \vec{r}(t) \) be a differentiable vector–valued function on an open interval \( I \) of constant length. That is, \( ||\vec{r}(t)|| = c \) for all \( t \) in \( I \) (equivalently, \( \vec{r}(t) \cdot \vec{r}(t) = c^2 \) for all \( t \) in \( I \)). Then \( \vec{r}(t) \cdot \vec{r}'(t) = 0 \) for all \( t \) in \( I \).

**Integration**

Indefinite and definite integrals of vector–valued functions are also evaluated component–wise.

Notes:
### Theorem 96  Indefinite and Definite Integrals of Vector–Valued Functions
Let \( \mathbf{r}(t) = (f(t), g(t)) \) be a vector–valued function in \( \mathbb{R}^2 \).

1. \( \int \mathbf{r}(t) \, dt = \left( \int f(t) \, dt, \int g(t) \, dt \right) \)

2. \( \int_a^b \mathbf{r}(t) \, dt = \left( \int_a^b f(t) \, dt, \int_a^b g(t) \, dt \right) \)

Let \( \mathbf{r}(t) = (f(t), g(t), h(t)) \) be a vector–valued function in \( \mathbb{R}^3 \).

1. \( \int \mathbf{r}(t) \, dt = \left( \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right) \)

2. \( \int_a^b \mathbf{r}(t) \, dt = \left( \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right) \)

### Example 8  Evaluating a definite integral of a vector–valued function
Let \( \mathbf{r}(t) = (e^{2t}, \sin t) \). Evaluate \( \int_0^1 \mathbf{r}(t) \, dt \).

**SOLUTION**  We follow Theorem 96.

\[
\int_0^1 \mathbf{r}(t) \, dt = \int_0^1 (e^{2t}, \sin t) \, dt = \left( \int_0^1 e^{2t} \, dt, \int_0^1 \sin t \, dt \right)
\]

\[
= \left( \frac{1}{2} e^{2t} \bigg|_0^1, -\cos t \bigg|_0^1 \right)
\]

\[
= \left( \frac{1}{2} (e^2 - 1), -\cos(1) + 1 \right).
\]

### Example 9  Solving an initial value problem
Let \( \mathbf{r}''(t) = (2, \cos t, 12t) \). Find \( \mathbf{r}(t) \) where:
- \( \mathbf{r}(0) = (-7, -1, 2) \) and
- \( \mathbf{r}'(0) = (5, 3, 0) \).

---

Notes:
Knowing \( \vec{r}''(t) = \langle 2, \cos t, 12t \rangle \), we find \( \vec{r}'(t) \) by evaluating the indefinite integral.

\[
\int \vec{r}''(t) \, dt = \left\langle \int 2 \, dt, \int \cos t \, dt, \int 12t \, dt \right\rangle = \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3 \right\rangle = \left\langle 2t, \sin t, 6t^2 \right\rangle + \vec{C}.
\]

Note how each indefinite integral creates its own constant which we collect as one constant vector \( \vec{C} \). Knowing \( \vec{r}'(0) = \langle 5, 3, 0 \rangle \) allows us to solve for \( \vec{C} \):

\[
\vec{r}'(0) = \langle 2t, \sin t, 6t^2 \rangle + \vec{C}
\]

\[
\langle 5, 3, 0 \rangle = \vec{C}.
\]

So \( \vec{r}'(t) = \langle 2t, \sin t, 6t^2 \rangle + \langle 5, 3, 0 \rangle = \langle 2t + 5, \sin t + 3, 6t^2 \rangle \). To find \( \vec{r}(t) \), we integrate once more.

\[
\int \vec{r}'(t) \, dt = \left\langle \int 2 \, dt + 5 \, dt, \int \sin t + 3 \, dt, \int 6t^2 \, dt \right\rangle = \left\langle t^2 + 5t, -\cos t + 3t, 2t^3 \right\rangle + \vec{C}.
\]

With \( \vec{r}(0) = \langle -7, -1, 2 \rangle \), we solve for \( \vec{C} \):

\[
\vec{r}(t) = \langle t^2 + 5t, -\cos t + 3t, 2t^3 \rangle + \vec{C}
\]

\[
\vec{r}(0) = \langle 0, -1, 0 \rangle + \vec{C}
\]

\[
\langle -7, -1, 2 \rangle = \langle 0, -1, 0 \rangle + \vec{C}
\]

\[
\langle -7, 0, 2 \rangle = \vec{C}.
\]

Therefore,

\[
\vec{r}(t) = \langle t^2 + 5t, -\cos t + 3t, 2t^3 \rangle + \langle -7, 0, 2 \rangle = \langle t^2 + 5t - 7, -\cos t + 3t, 2t^3 + 2 \rangle.
\]

What does the integration of a vector–valued function mean? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

Notes:
A key understanding for us comes from considering the integral of a derivative:
\[ \int_a^b \vec{r}'(t) \, dt = \vec{r}(t) \bigg|_a^b = \vec{r}(b) - \vec{r}(a). \]

Integrating a rate of change function gives displacement.

Noting that vector-valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector-valued function as an integral. Given parametric equations \( x = f(t), y = g(t) \), the arc length on \([a, b]\) of the graph is
\[ \text{Arc Length} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \, dt, \]

as stated in Theorem 84 in Section 10.3. If \( \vec{r}(t) = (f(t), g(t)) \), note that \( \sqrt{f'(t)^2 + g'(t)^2} = \|\vec{r}'(t)\| \). Therefore we can express the arc length of the graph of a vector-valued function as an integral of the magnitude of its derivative.

**Theorem 97  Arc Length of a Vector–Valued Function**

Let \( \vec{r}(t) \) be a vector–valued function where \( \vec{r}'(t) \) is continuous on \([a, b]\).
The arc length \( L \) of the graph of \( \vec{r}(t) \) is
\[ L = \int_a^b \|\vec{r}'(t)\| \, dt. \]

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let \( \vec{r}(t) \) describe the path of an object in the plane or in space and will discover the information provided by \( \vec{r}'(t) \) and \( \vec{r}''(t) \).
Exercises 12.2

Terms and Concepts

1. Limits, derivatives and integrals of vector-valued functions are all evaluated \( \) -wise.
2. The definite integral of a rate of change function gives \( \) .
3. Why is it generally not useful to graph both \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \) on the same axes?

Problems

In Exercises 4–7, evaluate the given limit.
4. \( \lim_{t \to 5} \left( 2t + 1, 3t^2 - 1, \sin t \right) \)
5. \( \lim_{t \to 3} \left( e^t, t^2 - 9, \frac{9}{1 + 3} \right) \)
6. \( \lim_{t \to 0} \left( \frac{t}{\sin t}, (1 + t)^3 \right) \)
7. \( \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \), where \( \mathbf{r}(t) = (t^2, t, 1) \).

In Exercises 8–9, identify the interval(s) on which \( \mathbf{r}(t) \) is continuous.
8. \( \mathbf{r}(t) = (t^2, 1/t) \)
9. \( \mathbf{r}(t) = (\cos t, e^t, \ln t) \)

In Exercises 10–14, find the derivative of the given function.
10. \( \mathbf{r}(t) = (\cos t, t, e^t) \)
11. \( \mathbf{r}(t) = \left( \frac{1}{2}, \frac{2t - 1}{3t + 1}, \tan t \right) \)
12. \( \mathbf{r}(t) = (t^2) \) on \( [0, 2t + 5] \)
13. \( \mathbf{r}(t) = (t^2 + 1, t - 1) \) on \( [0, 2t + 5] \)
14. \( \mathbf{r}(t) = (t^2 + 1, t - 1) \) on \( [0, 2t + 5] \)

In Exercises 15–18, find \( \mathbf{r}'(t) \). Sketch \( \mathbf{r}(t) \) and \( \mathbf{r}'(1) \), with the initial point of \( \mathbf{r}'(1) \) at \( \mathbf{r}(1) \).
15. \( \mathbf{r}(t) = (t^2 + t, t^2 - t) \)
16. \( \mathbf{r}(t) = (t^2 - 2t + 2, t^3 - 3t^2 + 2t) \)
17. \( \mathbf{r}(t) = (t^2 + t, t^2 - t) \)
18. \( \mathbf{r}(t) = (t^2 - 4t + 5, t^3 - 6t^2 + 11t - 6) \)

In Exercises 19–22, give the equation of the line tangent to the graph of \( \mathbf{r}(t) \) at the given \( t \) value.
19. \( \mathbf{r}(t) = (t^2 + t, t^2 - t) \) at \( t = 1 \).
20. \( \mathbf{r}(t) = (3 \cos t, \sin t) \) at \( t = \pi/4 \).
21. \( \mathbf{r}(t) = (3 \cos t, 3 \sin t) \) at \( t = \pi \).
22. \( \mathbf{r}(t) = (e^t, \tan t, t) \) at \( t = 0 \).

In Exercises 23–26, find the value(s) of \( t \) for which \( \mathbf{r}(t) \) is not smooth.
23. \( \mathbf{r}(t) = (\cos t, \sin t - t) \)
24. \( \mathbf{r}(t) = (t^2 - 2t + 1, t^3 + t^2 - 5t + 3) \)
25. \( \mathbf{r}(t) = (\cos t - \sin t, \sin t - t + \cos t \cdot \cos 4t) \)
26. \( \mathbf{r}(t) = (t^3 - 3t + 2, -e^{4t} \pi t, \sin^2(\pi t)) \)

Exercises 27–29 ask you to verify parts of Theorem 94. In each case, let \( f(t) = t^2, \mathbf{r}(t) = (t^2, t - 1, 1) \) and \( \mathbf{s}(t) = (\sin t, e^t, t) \). Compute the various derivatives as indicated.
27. Simplify \( f(t)\mathbf{r}(t) \), then find its derivative; show this is the same as \( f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t) \).
28. Simplify \( \mathbf{r}(t) \cdot \mathbf{s}(t) \), then find its derivative; show this is the same as \( \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t) \).
29. Simplify \( \mathbf{r}(t) \times \mathbf{s}(t) \), then find its derivative; show this is the same as \( \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t) \).

In Exercises 30–33, evaluate the given definite or indefinite integral.
30. \( \int (t^3, \cos t, te^t) \ dt \)
31. \( \int \left( \frac{1}{1 + t^2}, \sec^2 t \right) \ dt \)
32. \( \int_0^\pi (-\sin t, \cos t) \ dt \)
33. \( \int_{-2}^2 (2t + 1, 2t - 1) \ dt \)

In Exercises 34–37, solve the given initial value problems.
34. Find \( \mathbf{r}(t) \), given that \( \mathbf{r}'(t) = (t, \sin t) \) and \( \mathbf{r}(0) = (2, 2) \).
35. Find \( \mathbf{r}(t) \), given that \( \mathbf{r}'(t) = (1/(t + 1), \tan t) \) and \( \mathbf{r}(0) = (1, 2) \).
36. Find \( \mathbf{r}(t) \), given that \( \mathbf{r}''(t) = (t^2, t, 1) \), \( \mathbf{r}'(0) = (1, 2, 3) \) and \( \mathbf{r}(0) = (4, 5, 6) \).
37. Find \( \mathbf{r}(t) \), given that \( \mathbf{r}''(t) = (3 \cos t, t, e^t) \), \( \mathbf{r}'(0) = (0, 0, 0) \) and \( \mathbf{r}(0) = (0, 0, 0) \).

In Exercises 38–41, find the arc length of \( \mathbf{r}(t) \) on the indicated interval.
38. \( \mathbf{r}(t) = (2 \cos t, 2 \sin t, 3t) \) on \( [0, 2\pi] \).
39. \( \mathbf{r}(t) = (5 \cos t, 3 \sin t, 4 \sin t) \) on \( [0, 2\pi] \).
40. \( \mathbf{r}(t) = (t^3, \sqrt[3]{t}, t^4) \) on \( [0, 1] \).
41. \( \mathbf{r}(t) = (e^{-t}, \cos t, e^{-t} \sin t) \) on \( [0, 1] \).
42. Prove Theorem 95; that is, show if \( \mathbf{r}(t) \) has constant length and is differentiable, then \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \). (Hint: use the Product Rule to compute \( \frac{d}{dt} \langle \mathbf{r}(t), \mathbf{r}(t) \rangle \).)
12.3 The Calculus of Motion

A common use of vector-valued functions is to describe the motion of an object in the plane or in space. A position function \( \mathbf{r}(t) \) gives the position of an object at time \( t \). This section explores how derivatives and integrals are used to study the motion described by such a function.

**Definition 76 Velocity, Speed and Acceleration**

Let \( \mathbf{r}(t) \) be a position function in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

1. **Velocity**, denoted \( \mathbf{v}(t) \), is the instantaneous rate of position change; that is, \( \mathbf{v}(t) = \mathbf{r}'(t) \).
2. **Speed** is the magnitude of velocity, \( \|\mathbf{v}(t)\| \).
3. **Acceleration**, denoted \( \mathbf{a}(t) \), is the instantaneous rate of velocity change; that is, \( \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \).

---

Watch the video:
Example of Position, Velocity and Acceleration in Three Space at [https://youtu.be/gD2R4Jqu6dQ](https://youtu.be/gD2R4Jqu6dQ)

---

**Example 1 Finding velocity and acceleration**

An object is moving with position function \( \mathbf{r}(t) = \langle t^2 - t, t^2 + t \rangle \), \(-3 \leq t \leq 3\), where distances are measured in feet and time is measured in seconds.

1. Find \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \).
2. Sketch \( \mathbf{r}(t) \); plot \( \mathbf{v}(-1), \mathbf{a}(-1), \mathbf{v}(1) \) and \( \mathbf{a}(1) \), each with their initial point at their corresponding point on the graph of \( \mathbf{r}(t) \).
3. When is the object’s speed minimized?

**SOLUTION**

---

Notes:
Chapter 12  Vector Valued Functions

1. Taking derivatives, we find
   \[ \vec{v}(t) = \vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}''(t) = \langle 2, 2 \rangle. \]

   Note that acceleration is constant.

2. \[ \vec{v}(-1) = (-3, -1), \quad \vec{a}(-1) = \langle 2, 2 \rangle; \quad \vec{v}(1) = \langle 1, 3 \rangle, \quad \vec{a}(1) = \langle 2, 2 \rangle. \]
   These are plotted with \( \vec{r}(t) \) in Figure 12.14(a).

   We can think of acceleration as “pulling” the velocity vector in a certain direction. At \( t = -1 \), the velocity vector points down and to the left; at \( t = 1 \), the velocity vector has been pulled in the \( \langle 2, 2 \rangle \) direction and is now pointing up and to the right. In Figure 12.14(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

   Since \( \vec{a}(t) \) is constant in this example, as \( t \) grows large \( \vec{v}(t) \) becomes almost parallel to \( \vec{a}(t) \). For instance, when \( t = 10 \), \( \vec{v}(10) = \langle 19, 21 \rangle \), which is nearly parallel to \( \langle 2, 2 \rangle \).

3. The object’s speed is given by
   \[ \| \vec{v}(t) \| = \sqrt{(2t - 1)^2 + (2t + 1)^2} = \sqrt{8t^2 + 2}. \]

   To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for \( t \), etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when \( t = 0 \). Thus the speed is minimized at \( t = 0 \), with a speed of \( \sqrt{2} \) ft/s.

   The graph in Figure 12.14(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of \( t \) between \(-3 \) and \( 3 \). Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near \( t = 0 \), implying the speed is minimized near that value.

---

**Example 2  Analyzing Motion**

Two objects follow an identical path at different rates on \([-1, 1]\). The position function for Object 1 is \( \vec{r}_1(t) = \langle t, t^2 \rangle \); the position function for Object 2 is \( \vec{r}_2(t) = \langle t^3, t^4 \rangle \), where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

---

Notes:
12.3 The Calculus of Motion

We begin by computing the velocity and acceleration functions for each object:

\[
\vec{v}_1(t) = \langle 1, 2t \rangle \\
\vec{v}_2(t) = \langle 3t^3, 6t^5 \rangle \\
\vec{a}_1(t) = \langle 0, 2 \rangle \\
\vec{a}_2(t) = \langle 6t, 30t^4 \rangle
\]

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At \( t = -1 \), we have \( \vec{v}_1(-1) = \langle 1, -2 \rangle \) and \( \vec{v}_2(-1) = \langle 3, -6 \rangle \); the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 (\( 3\sqrt{5} \) \( \text{ft/s} \) compared to \( \sqrt{5} \) \( \text{ft/s} \)).

At \( t = 0 \), the velocity of Object 1 is \( \vec{v}(1) = \langle 1, 0 \rangle \) and the velocity of Object 2 is \( \vec{0} \). This tells us that Object 2 comes to a complete stop at \( t = 0 \).

In Figure 12.15, we see the velocity and acceleration vectors for Object 1 plotted for \( t = -1, -1/2, 0, 1/2 \) and \( t = 1 \). Note again how the constant acceleration vector seems to “pull” the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large \( \vec{a}_2(-1) = \langle -6, 30 \rangle \).

Instead, we simply plot the locations of Object 1 and 2 on intervals of \( 1/5 \) of a second, shown in Figure 12.16(a) and (b). Note how the \( x \)-values of Object 1 increase at a steady rate. This is because the \( x \)-component of \( \vec{a}(t) = 0 \); there is no acceleration in the \( x \)-component. The dots are not evenly spaced; the object is moving faster near \( t = -1 \) and \( t = 1 \) than near \( t = 0 \).

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from \( t = -1 \) to \( t = -0.8 \); the object starts moving very quickly. However, it slows considerably as it approaches the origin, and comes to a complete stop at \( t = 0 \). While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, they have the same displacement. Since they begin and end at the same time, with the same displacement, they have the same average rate of change (i.e., they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

Example 3 Analyzing the motion of a whirling ball on a string

A young boy whirls a ball, attached to a string, above his head in a counterclockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function \( \vec{r}(t) \) that describes this situation.

Notes:
2. Find the acceleration of the ball and derive a physical interpretation of it.

3. A tree stands 10ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

**Solution**

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function \( \mathbf{r}(t) = (2 \cos t, 2 \sin t) \) describes a circle with radius 2, centered at the origin, but makes a full revolution every \( 2\pi \) seconds, not two revolutions per second. We modify the period of the trigonometric functions to be 1/2 by multiplying \( t \) by \( 4\pi \). The final position function is thus

\[
\mathbf{r}(t) = (2 \cos(4\pi t), 2 \sin(4\pi t)).
\]

(Plot this for \( 0 \leq t \leq 1/2 \) to verify that one revolution is made in 1/2 a second.)

2. To find \( \mathbf{a}(t) \), we derive \( \mathbf{r}(t) \) twice.

\[
\mathbf{v}(t) = \mathbf{r}'(t) = (-8\pi \sin(4\pi t), 8\pi \cos(4\pi t))
\]

\[
\mathbf{a}(t) = \mathbf{r}''(t) = (-32\pi^2 \cos(4\pi t), -32\pi^2 \sin(4\pi t))
\]

\[
= -32\pi^2 (\cos(4\pi t), \sin(4\pi t)).
\]

Note how \( \mathbf{a}(t) \) is parallel to \( \mathbf{r}(t) \), but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, “Force = mass \times\) acceleration.” A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a weight). Thus force and acceleration are closely related. A moving ball “wants” to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy’s hand by a string. The string applies a force to the ball, affecting it’s motion: the string accelerates the ball. This is not acceleration in the sense of “it travels faster;” rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy’s hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be “large.”

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is 0 and the ball can now move in a straight line in the direction of \( \mathbf{v}(t) \).
Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, traveling in the direction of $\vec{v}(t_0)$. We want to find $t_0$ so that this line contains the point $(0, 10)$ (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 12.17, the vector from the release point to the tree is $\langle 0; 10 \rangle$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.

$$\vec{r}(t_0) \cdot (\langle 0, 10 \rangle - \vec{r}(t_0)) = 0$$

$$\langle 2 \cos(4\pi t_0), 2 \sin(4\pi t_0) \rangle \cdot (-2 \cos(4\pi t_0), 10 - 2 \sin(4\pi t_0)) = 0$$

$$-4 \cos^2(4\pi t_0) + 20 \sin(4\pi t_0) - 4 \sin^2(4\pi t_0) = 0$$

$$20 \sin(4\pi t_0) - 4 = 0$$

$$\sin(4\pi t_0) = 1/5$$

$$4\pi t_0 = \sin^{-1}(1/5)$$

$$4\pi t_0 \approx 0.2 + 2\pi n$$

$$t_0 \approx 0.016 + n/2$$

where $n$ is an integer. This means that every 1/2 second after $t = 0.016s$ the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

### Example 4 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, where distances are measured in meters and time is in minutes. Describe the object’s speed and acceleration at time $t$.

**SOLUTION** With $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, we have:

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle \quad \text{and}$$

$$\vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle .$$

The speed of the object is $||\vec{v}(t)|| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2} \text{m/min}$; it moves at a constant speed. Note that the object does not accelerate in the $z$-direction, but rather moves up at a constant rate of 1m/min.
The objects in Examples 3 and 4 traveled at a constant speed. That is, \( \| \mathbf{v}(t) \| = c \) for some constant \( c \). Recall Theorem 95, which states that if a vector–valued function \( \mathbf{r}(t) \) has constant length, then \( \mathbf{r}(t) \) is perpendicular to its derivative: \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \). In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: \( \mathbf{v}(t) \cdot \mathbf{a}(t) = 0 \). A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object’s speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

### Key Idea 57 Objects With Constant Speed

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, \( \mathbf{v}(t) \cdot \mathbf{a}(t) = 0 \).

---

**Projectile Motion**

An important application of vector–valued position functions is projectile motion: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it is and where it is going.)

Suppose an object has initial position \( \mathbf{r}(0) = \langle x_0, y_0 \rangle \) and initial velocity \( \mathbf{v}(0) = \langle v_x, v_y \rangle \). It is customary to rewrite \( \mathbf{v}(0) \) in terms of its speed \( v_0 \) and direction \( \mathbf{u} \), where \( \mathbf{u} \) is a unit vector. Recall all unit vectors in \( \mathbb{R}^2 \) can be written as \( \langle \cos \theta, \sin \theta \rangle \), where \( \theta \) is an angle measure counter–clockwise from the x-axis. (We refer to \( \theta \) as the angle of elevation.) Thus \( \mathbf{v}(0) = v_0 \langle \cos \theta, \sin \theta \rangle \).

Since the acceleration of the object is known, namely \( \mathbf{a}(t) = (0, -g) \), where \( g \) is the gravitational constant, we can find \( \mathbf{r}(t) \) knowing our two initial conditions. We first find \( \mathbf{v}(t) \):

---

**Notes:**

In this text we use \( g = 32 \text{ft/s}^2 \) when using Imperial units, and \( g = 9.8 \text{m/s}^2 \) when using SI units.
The Calculus of Motion

\[ \vec{v}(t) = \int \vec{a}(t) \, dt \]

\[ \vec{v}(t) = \int (0, -g) \, dt \]

\[ \vec{v}(t) = (0, -gt) + \vec{C}. \]

Knowing \( \vec{v}(0) = v_0 \langle \cos \theta, \sin \theta \rangle \), we have \( \vec{C} = v_0 \langle \cos \theta, \sin \theta \rangle \) and so

\[ \vec{v}(t) = \langle v_0 \cos \theta, -gt + v_0 \sin \theta \rangle. \]

We integrate once more to find \( \vec{r}(t) \):

\[ \vec{r}(t) = \int \vec{v}(t) \, dt \]

\[ \vec{r}(t) = \int \langle v_0 \cos \theta, -gt + v_0 \sin \theta \rangle \, dt \]

\[ \vec{r}(t) = \left( \langle v_0 \cos \theta \rangle t, -\frac{1}{2}gt^2 + (v_0 \sin \theta) t \right) + \vec{C}. \]

Knowing \( \vec{r}(0) = \langle x_0, y_0 \rangle \), we conclude \( \vec{C} = \langle x_0, y_0 \rangle \) and

\[ \vec{r}(t) = \left( \langle v_0 \cos \theta \rangle t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin \theta) t + y_0 \right). \tag{12.1} \]

We demonstrate how to use this position function in the next two examples.

Example 5 Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

Solution A direct application of Equation (12.1) gives

\[ \vec{r}(t) = \left( \langle 350 \cos 5^\circ \rangle t, -16t^2 + (350 \sin 5^\circ) t + 4 \right) \]

\[ \approx \left( 346.67t, -16t^2 + 30.50t + 4 \right), \]

where we set her initial position to be \( \langle 0, 4 \rangle \). We need to find when the bb lands, then we can find where. We accomplish this by setting the \( y \)-component equal to 0 and solving for \( t \):

\[ -16t^2 + 30.50t + 4 = 0 \]

\[ t = \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32} \]

\[ t \approx 2.03 \text{s}. \]

Notes:
(We discarded a negative solution that resulted from our quadratic equation.) We have found that the bb lands 2.03s after firing; with $t = 2.03$, we find the $x$-component of our position function is $346.67(2.03) = 703.74\text{ft}$. The bb lands about 704 feet away.

Example 6 Projectile Motion
Alex holds his sister’s bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

SOLUTION The position function for the path of Alex’s bb is

$$\vec{r}(t) = \langle (350 \cos \theta) t, -16t^2 + (350 \sin \theta) t + 3 \rangle.$$ 

We need to find $\theta$ so that $\vec{r}(t) = \langle 25, 6 \rangle$ for some value of $t$. That is, we want to find $\theta$ and $t$ such that

$$(350 \cos \theta) t = 25 \quad \text{and} \quad -16t^2 + (350 \sin \theta) t + 3 = 6.$$ 

This is not trivial (though not “hard”). We start by solving each equation for $\cos \theta$ and $\sin \theta$, respectively.

$$\cos \theta = \frac{25}{350t} \quad \text{and} \quad \sin \theta = \frac{3 + 16t^2}{350t}.$$ 

Using the Pythagorean Identity $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\left( \frac{25}{350t} \right)^2 + \left( \frac{3 + 16t^2}{350t} \right)^2 = 1$$

Multiply both sides by $(350t)^2$:

$$25^2 + (3 + 16t^2)^2 = 350^2t^2$$

$$256t^4 - 122,404t^2 + 634 = 0.$$ 

This is a quadratic in $t^2$. That is, we can apply the quadratic formula to find $t^2$, then solve for $t$ itself.

$$t^2 = \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512}$$

$$t^2 = 0.0052, 478.135$$

$$t = \pm 0.072, \pm 21.866$$

Notes:
Clearly the negative $t$ values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos \theta = 25/(350t)$, we can solve for $\theta$:

$$\theta = \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right)$$

$$\theta = 7.03^\circ \quad \text{and} \quad 89.8^\circ.$$

Alex has two choices of angle. He can hold the rifle at an angle of about $7^\circ$ with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of $89.8^\circ$, where he’ll hit his target about 22s later. The first option is clearly the option he should choose.

**Distance Traveled**

Consider a driver who sets her cruise–control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?

2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by $||\vec{v}(t)||$.

---

**Theorem 98 Distance Traveled**

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b ||\vec{v}(t)|| \, dt.$$

Note that this is just a restatement of Theorem 97: arc length is the same as distance traveled, just viewed in a different context.

**Example 7 Distance Traveled, Displacement, and Average Speed**

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t) \rangle$ on $[-2, 2]$, where $t$ is measured in seconds and distances are in meters. Find:

---

Notes:
1. The distance traveled by the particle on \([-2, 2]\).
2. The displacement of the particle on \([-2, 2]\).
3. The particle’s average speed.

**Solution**

1. We use Theorem 98 to establish the integral:

   \[
   \text{distance traveled} = \int_{-2}^{2} \|\vec{v}(t)\| \, dt
   \]

   \[
   = \int_{-2}^{2} \sqrt{1 + (2t)^2 + \pi^2 \cos^2(\pi t)} \, dt.
   \]

   This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88 m.

2. The displacement is the vector

   \[
   \vec{r}(2) - \vec{r}(-2) = (2, 4, 0) - (-2, 4, 0) = (4, 0, 0).
   \]

   That is, the particle ends with an x-value increased by 4 and with y- and z-values the same (see Figure 12.18).

3. We found above that the particle traveled 12.88 m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22 m/s.

   We should also consider Definition 25 of Section 5.4, which says that the average value of a function \(f\) on \([a, b]\) is

   \[
   \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
   \]

   In our context, the average speed is

   \[
   \text{average speed} = \frac{1}{2 - (-2)} \int_{-2}^{2} \|\vec{v}(t)\| \, dt \approx \frac{1}{4} \cdot 12.88 = 3.22 \text{ m/s}.
   \]

   Note how in Example 7 we computed the average speed as

   \[
   \frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2 - (-2)} \int_{-2}^{2} \|\vec{v}(t)\| \, dt;
   \]

   In Definition 25 of Chapter 5 we defined the average value of a function \(f(x)\) on \([a, b]\) to be

   \[
   \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
   \]

   Note how in Example 7 we computed the average speed as

   \[
   \frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2 - (-2)} \int_{-2}^{2} \|\vec{v}(t)\| \, dt;
   \]

   Notes:
that is, we just found the average value of $\|\vec{v}(t)\|$ on $[-2, 2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a, b]$ is

$$
\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a} \int_a^b \vec{r}'(t) \, dt = \frac{\vec{r}(b) - \vec{r}(a)}{b-a};
$$

that is, it is the average value of $\vec{r}'(t)$, or $\vec{v}(t)$, on $[a, b]$.

**Key Idea 58 Average Speed, Average Velocity**

Let $\vec{r}(t)$ be a continuous position function on an open interval $I$ containing $a < b$.

The **average speed** is:

$$
\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\vec{v}(t)\| \, dt}{b-a} = \frac{1}{b-a} \int_a^b \|\vec{v}(t)\| \, dt.
$$

The **average velocity** is:

$$
\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}'(t) \, dt}{b-a} = \frac{1}{b-a} \int_a^b \vec{r}'(t) \, dt.
$$

The next two sections investigate more properties of the graphs of vector--valued functions and we’ll apply these new ideas to what we just learned about motion.
Exercises 12.3

Terms and Concepts

1. How is velocity different from speed?
2. What is the difference between displacement and distance traveled?
3. What is the difference between average velocity and average speed?
4. Distance traveled is the same as __________, just viewed in a different context.
5. Describe a scenario where an object’s average speed is a large number, but the magnitude of the average velocity is not a large number.
6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7–10, a position function \( \vec{r}(t) \) is given. Find \( \vec{v}(t) \) and \( \vec{a}(t) \).

7. \( \vec{r}(t) = (2t + 1, 5t^2 - 2, 7) \)
8. \( \vec{r}(t) = (3t^2 - 2t + 1, -t^2 + t + 14) \)
9. \( \vec{r}(t) = (\cos t, \sin t) \)
10. \( \vec{r}(t) = (t/10, -\cos t, \sin t) \)

In Exercises 11–14, a position function \( \vec{r}(t) \) is given. Sketch \( \vec{r}(t) \) on the indicated interval. Find \( \vec{v}(t) \) and \( \vec{a}(t) \), then add \( \vec{v}(t_0) \) and \( \vec{a}(t_0) \) to your sketch, with their initial points at \( \vec{r}(t_0) \), for the given value of \( t_0 \).

11. \( \vec{r}(t) = (t, \sin t) \) on \([0, \pi/2] \); \( t_0 = \pi/4 \)
12. \( \vec{r}(t) = (t^2, \sin t^2) \) on \([0, \pi/2] \); \( t_0 = \sqrt{\pi}/4 \)
13. \( \vec{r}(t) = (t^2 + t, -t^2 + 2t) \) on \([-2, 2] \); \( t_0 = 1 \)
14. \( \vec{r}(t) = \left(\frac{2t + 3}{2}, \frac{1}{1 + t^2}\right) \) on \([-1, 1] \); \( t_0 = 0 \)

In Exercises 15–24, a position function \( \vec{r}(t) \) of an object is given. Find the speed of the object in terms of \( t \), and find where the speed is minimized/maximized on the indicated interval.

15. \( \vec{r}(t) = (t^2, t) \) on \([-1, 1] \)
16. \( \vec{r}(t) = (t^2, t^2 - t^3) \) on \([-1, 1] \)
17. \( \vec{r}(t) = (5 \cos t, 5 \sin t) \) on \([0, 2\pi]\)
18. \( \vec{r}(t) = (2 \cos t, 5 \sin t) \) on \([0, 2\pi]\)
19. \( \vec{r}(t) = (\sec t, \tan t) \) on \([0, \pi/4]\)
20. \( \vec{r}(t) = (t + \cos t, 1 - \sin t) \) on \([0, 2\pi]\)
21. \( \vec{r}(t) = (12t, 5 \cos t, 5 \sin t) \) on \([0, 4\pi]\)
22. \( \vec{r}(t) = (t^2 - t, t^2 + t, t) \) on \([0, 1]\)
23. \( \vec{r}(t) = (t, t^2, \sqrt{1 - t^2}) \) on \([-1, 1] \)
24. Projectile Motion: \( \vec{r}(t) = \left(v_0 \cos \theta t, -\frac{1}{2}gt^2 + (v_0 \sin \theta t)\right) \) on \([0, \frac{2v_0 \sin \theta}{g}]\)

In Exercises 25–28, position functions \( r_1(t) \) and \( r_2(s) \) for two objects are given that follow the same path on the respective intervals.

(a) Show that the positions are the same at the indicated \( t_0 \) and \( s_0 \) values; i.e., show \( r_1(t_0) = r_2(s_0) \).
(b) Find the velocity, speed and acceleration of the two objects at \( t_0 \) and \( s_0 \), respectively.

25. \( r_1(t) = (t, t^2) \) on \([0, 1] \); \( t_0 = 1 \)
\( r_2(s) = (s^2, s^3) \) on \([0, 1] \); \( s_0 = 1 \)
26. \( r_1(t) = (3 \cos t, 3 \sin t) \) on \([0, 2\pi] \); \( t_0 = \pi/2 \)
\( r_2(s) = (3 \cos(4s), 3 \sin(4s)) \) on \([0, \pi/2] \); \( s_0 = \pi/8 \)
27. \( r_1(t) = (3t, 2t) \) on \([0, 2] \); \( t_0 = 2 \)
\( r_2(s) = (6s - 6, 4s - 4) \) on \([1, 2] \); \( s_0 = 2 \)
28. \( r_1(t) = (t, \sqrt{t}) \) on \([0, 1] \); \( t_0 = 1 \)
\( r_2(s) = (\sin t, \sqrt{\sin t}) \) on \([0, \pi/2] \); \( s_0 = \pi/2 \)

In Exercises 29–32, find the position function of an object given its acceleration and initial velocity and position.

29. \( \vec{a}(t) = (2, 3) \); \( \vec{v}(0) = (1, 2) \); \( \vec{r}(0) = (5, -2) \)
30. \( \vec{a}(t) = (2, 3) \); \( \vec{v}(1) = (1, 2) \); \( \vec{r}(1) = (5, -2) \)
31. \( \vec{a}(t) = (\cos t, -\sin t) \); \( \vec{v}(0) = (0, 0) \); \( \vec{r}(0) = (0, 0) \)
32. \( \vec{a}(t) = (0, -32) \); \( \vec{v}(0) = (10, 50) \); \( \vec{r}(0) = (0, 0) \)

In Exercises 33–36, find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

33. An object with position function \( \vec{r}(t) = (2 \cos t, 2 \sin t, 3t) \) where distances are measured in feet and time is in seconds, on \([0, 2\pi]\).
34. An object with position function \( \vec{r}(t) = (5 \cos t, -5 \sin t) \), where distances are measured in feet and time is in seconds, on \([0, \pi]\).
35. An object with velocity function \( \vec{v}(t) = (\cos t, \sin t) \), where distances are measured in feet and time is in seconds, on \([0, 2\pi]\).
36. An object with velocity function \( \vec{v}(t) = (1, 2, -1) \), where distances are measured in feet and time is in seconds, on \([0, 10]\).

Exercises 37–42 ask you to solve a variety of problems based on the principles of projectile motion.

37. A boy whirls a ball, attached to a 3ft string, above his head in a counter-clockwise circle. The ball makes 2 revolutions per second. At what \( t \)-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?
38. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

(a) At what t-values must David release the stone in his sling in order to hit Goliath?

(b) What is the speed at which the stone is traveling when released?

(c) Assume David releases the stone from a height of 6ft and Goliath’s forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath’s head?

39. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

(a) At what angle of elevation should she hold the crossbow to hit her target?

(b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

40. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of 20°, at Boston’s Fenway Park. The ball flies towards the famed “Green Monster,” a wall 37ft high located 310ft from home plate.

(a) Show that as hit, the ball hits the wall.

(b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

41. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

42. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

(a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

(b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?
12.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 74 that any vector parallel to $\vec{r}'(t_0)$ is tangent to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the direction of $\vec{r}'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}'(t)$. This leads to a definition.

**Definition 77 Unit Tangent Vector**

Let $\vec{r}(t)$ be a smooth function on an open interval $I$. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t).$$

Example 1 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

**SOLUTION** We apply Definition 77 to find $\vec{T}(t)$.

$$\vec{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t) = \frac{1}{\sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2}} \langle -3 \sin t, 3 \cos t, 4 \rangle = \langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \rangle.$$

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \langle 0, \frac{3}{5}, \frac{4}{5} \rangle; \quad \vec{T}(1) = \langle -\frac{3}{5} \sin 1, \frac{3}{5} \cos 1, \frac{4}{5} \rangle.$$
These are plotted in Figure 12.19 with their initial points at \( \vec{r}(0) \) and \( \vec{r}(1) \), respectively. (They look rather “short” since they are only length 1.)

In many ways, the previous example was “too nice.” It turned out that \( \vec{r}'(t) \) was always of length 5. In the next example the length of \( \vec{r}'(t) \) is variable, leaving us with a formula that is not as clean.

**Example 2** Computing the unit tangent vector

Let \( \vec{r}(t) = \langle t^2 - t, t^2 + t \rangle \). Find \( \vec{T}(t) \) and compute \( \vec{T}(0) \) and \( \vec{T}(1) \).

**Solution** We find \( \vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle \), and

\[
\|\vec{r}'(t)\| = \sqrt{(2t - 1)^2 + (2t + 1)^2} = \sqrt{8t^2 + 2}.
\]

Therefore

\[
\vec{T}(t) = \frac{1}{\sqrt{8t^2 + 2}} \langle 2t - 1, 2t + 1 \rangle = \left\langle \frac{2t - 1}{\sqrt{8t^2 + 2}}, \frac{2t + 1}{\sqrt{8t^2 + 2}} \right\rangle.
\]

When \( t = 0 \), we have \( \vec{T}(0) = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle \); when \( t = 1 \), we have \( \vec{T}(1) = \langle 1/\sqrt{10}, 3/\sqrt{10} \rangle \). We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 12.20

**Unit Normal Vector**

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given \( \vec{r}(t) \) in \( \mathbb{R}^2 \), we have 2 directions perpendicular to the tangent vector, as shown in Figure 12.21. It is good to wonder “Is one of these two directions preferable over the other?”

Given \( \vec{r}(t) \) in \( \mathbb{R}^3 \), there are infinite vectors orthogonal to the tangent vector at a given point. Again, we might wonder “Is one of these infinite choices preferable over the others? Is one of these the ‘right’ choice?”

The answer in both \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) is “Yes, there is one vector that is preferable.”

Recall Theorem 95, which states that if \( \vec{r}(t) \) has constant length, then \( \vec{r}(t) \) is orthogonal to \( \vec{r}'(t) \) for all \( t \). We know \( \vec{T}(t) \), the unit tangent vector, has constant length. Therefore \( \vec{T}(t) \) is orthogonal to \( \vec{T}'(t) \).

We’ll see that \( \vec{T}'(t) \) is more than just a convenient choice of vector that is orthogonal to \( \vec{r}'(t) \); rather, it is the “right” choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

**Note:** \( \vec{T}(t) \) is a unit vector, by definition. This *does not* imply that \( \vec{T}'(t) \) is also a unit vector.
Definition 78  Unit Normal Vector
Let \( \vec{r}(t) \) be a vector–valued function where the unit tangent vector, \( \vec{T}(t) \), is smooth on an open interval \( I \). The unit normal vector \( \vec{N}(t) \) is

\[
\vec{N}(t) = \frac{1}{\| \vec{T}'(t) \|} \vec{T}'(t).
\]

Example 3  Computing the unit normal vector
Let \( \vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle \) as in Example 1. Sketch both \( \vec{T}(\pi/2) \) and \( \vec{N}(\pi/2) \) with initial points at \( \vec{r}(\pi/2) \).

**SOLUTION**

In Example 1, we found \( \vec{T}(t) = \langle (-3/5) \sin t, (3/5) \cos t, 4/5 \rangle \).

Therefore

\[
\vec{T}'(t) = \langle -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \rangle \quad \text{and} \quad \| \vec{T}'(t) \| = \frac{3}{5}.
\]

Thus

\[
\vec{N}(t) = \frac{\vec{T}'(t)}{\| \vec{T}'(t) \|} = \langle -\cos t, -\sin t, 0 \rangle.
\]

We compute \( \vec{T}(\pi/2) = \langle -3/5, 0, 4/5 \rangle \) and \( \vec{N}(\pi/2) = \langle 0, -1, 0 \rangle \). These are sketched in Figure 12.22.

The previous example was once again “too nice.” In general, the expression for \( \vec{T}(t) \) contains fractions of square–roots, hence the expression of \( \vec{T}'(t) \) is very messy. We demonstrate this in the next example.

Example 4  Computing the unit normal vector
Let \( \vec{r}(t) = \langle t^2 - t, t^2 + t \rangle \) as in Example 2. Find \( \vec{N}(t) \) and sketch \( \vec{r}(t) \) with the unit tangent and normal vectors at \( t = -1, 0 \) and 1.

**SOLUTION**

In Example 2, we found

\[
\vec{T}(t) = \left\langle \frac{2t - 1}{\sqrt{8t^2 + 2}}, \frac{2t + 1}{\sqrt{8t^2 + 2}} \right\rangle.
\]

Finding \( \vec{T}'(t) \) requires two applications of the Quotient Rule:

Notes:
\[ \vec{T}'(t) = \left( \frac{\sqrt{8t^2 + 2} - (2t - 1) \left( \frac{1}{2} (8t^2 + 2)^{-1/2} (16t) \right)}{8t^2 + 2}, \frac{\sqrt{8t^2 + 2} - (2t + 1) \left( \frac{1}{2} (8t^2 + 2)^{-1/2} (16t) \right)}{8t^2 + 2} \right) \]

\[ = \left( \frac{4(2t + 1)}{(8t^2 + 2)^{3/2}}, \frac{4(1 - 2t)}{(8t^2 + 2)^{3/2}} \right) \]

This is not a unit vector; to find \( \vec{N}(t) \), we need to divide \( \vec{T}'(t) \) by its magnitude.

\[ \| \vec{T}'(t) \| = \sqrt{\frac{16t^2 + 1}{(8t^2 + 2)^{3}} + \frac{16(1 - 2t)^2}{(8t^2 + 2)^{3}}} \]

\[ = \frac{4}{8t^2 + 2} \]

Finally,

\[ \vec{N}(t) = \frac{1}{4/(8t^2 + 2)} \left( \frac{4(2t + 1)}{(8t^2 + 2)^{3/2}}, \frac{4(1 - 2t)}{(8t^2 + 2)^{3/2}} \right) \]

\[ = \left( \frac{2t + 1}{\sqrt{8t^2 + 2}}, -\frac{2t - 1}{\sqrt{8t^2 + 2}} \right) \]

Because we are normalizing \( \vec{T}'(t) \), it is usually easier to scale it first. We see that \( \vec{T}'(t) \) is parallel to \( (2t + 1, 1 - 2t) \), which has length \( \sqrt{(2t + 1)^2 + (1 - 2t)^2} = \sqrt{8t^2 + 2} \), leading to the same \( \vec{N}(t) \).

Using this formula for \( \vec{N}(t) \), we compute the unit tangent and normal vectors for \( t = -1, 0 \) and sketch them in Figure 12.23.

The final result for \( \vec{N}(t) \) in Example 4 is suspiciously similar to \( \vec{T}(t) \). There is a clear reason for this. If \( \vec{u} = (u_1, u_2) \) is a unit vector in \( \mathbb{R}^2 \), then the only unit vectors orthogonal to \( \vec{u} \) are \( (u_2, -u_1) \) and \( -(u_2, u_1) \). Given \( \vec{T}(t) \), we can quickly determine \( \vec{N}(t) \) if we know which term to multiply by \(-1\).

Consider again Figure 12.23, where we have plotted some unit tangent and normal vectors. Note how \( \vec{N}(t) \) always points “inside” the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that \( \vec{r}'(t) \) “turns” allows us to quickly find \( \vec{N}(t) \).

Notes:
Chapter 12  Vector Valued Functions

Theorem 99  Unit Normal Vectors in $\mathbb{R}^2$
Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}'(t)$ is smooth on an open interval $I$. Let $t_0$ be in $I$ and $\vec{T}(t_0) = (t_1, t_2)$. Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = (-t_2, t_1) \quad \text{or} \quad \vec{N}(t_0) = (t^2, -t_1),$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 100) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalars $\alpha_T$ and $\alpha_N$ such that

$$\vec{a}(t) = \alpha_T \vec{T}(t) + \alpha_N \vec{N}(t).$$

The scalar $\alpha_T$ measures “how much” acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar $\alpha_N$ measures “how much” acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find $\alpha_T$ using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 62 in Section 11.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t) \cdot \vec{T}(t) = 1$, so we have

$$\text{proj}_{\vec{T}(t)} \vec{a}(t) = \frac{\vec{a}(t) \cdot \vec{T}(t)}{\vec{T}(t) \cdot \vec{T}(t)} \vec{T}(t) = \frac{(\vec{a}(t) \cdot \vec{T}(t))}{\alpha_T} \vec{T}(t).$$

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $\alpha_T = \vec{a}(t) \cdot \vec{T}(t)$. The same logic gives $\alpha_N = \vec{a}(t) \cdot \vec{N}(t)$.

While this is a fine way of computing $\alpha_T$, there are simpler ways of finding $\alpha_N$ (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for $\alpha_T$ and $\alpha_N$.

Note: Keep in mind that both $\alpha_T$ and $\alpha_N$ are functions of $t$; that is, the scalar changes depending on $t$. It is convention to drop the “$(t)$” notation from $\alpha_T(t)$ and simply write $\alpha_T$.  

Notes:
Theorem 100  Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars $a_T$ and $a_N$ such that

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t).$$

Moreover,

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{d}{dt} \left( \|\vec{v}(t)\| \right)$$

$$a_N = \vec{a}(t) \cdot \vec{N}(t) = \sqrt{\|\vec{a}(t)\|^2 - a_T^2} = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{\|\vec{v}(t)\|} = \|\vec{v}(t)\| \|\vec{T}'(t)\|.$$ 

Note the second formula for $a_T$: $\frac{d}{dt} \left( \|\vec{v}(t)\| \right)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Proof

We see that

$$\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = \frac{d}{dt} \left( \|\vec{v}(t)\| \vec{T}(t) \right) = \left( \frac{d}{dt} \|\vec{v}(t)\| \right) \vec{T}(t) + \|\vec{v}(t)\| \vec{T}'(t)$$

$$= \left( \frac{d}{dt} \|\vec{v}(t)\| \right) \vec{T}(t) + \|\vec{v}(t)\| \|\vec{T}'(t)\| \vec{N}(t).$$

Since $\vec{T}(t)$ and $\vec{N}(t)$ are not parallel, this decomposition is unique and the coefficients tell us $a_T$ and $a_N$.

Because $\|\vec{T}\| = 1$, Theorem 95 tells us that $\vec{T}$ and $\vec{T}' = \|\vec{T}'\| \vec{N}$ are orthogonal. This means that

$$\|\vec{a}(t) \times \vec{v}(t)\| = \|a_N \vec{N}(t) \times \|\vec{v}(t)\| \vec{T}(t)\| = a_N \|\vec{v}(t)\|.$$ 

Also, the Pythagorean theorem tells us that

$$\|\vec{a}(t)\|^2 = \|a_T \vec{T}(t)\|^2 + \|a_N \vec{N}(t)\|^2 = a_T^2 + a_N^2.$$ 

\[ \square \]

Example 5  Computing $a_T$ and $a_N$

Let $\vec{r}(t) = (3 \cos t, 3 \sin t, 4t)$ as in Examples 1 and 3. Find $a_T$ and $a_N$. 

Notes:
The previous examples give \( \vec{a}(t) = \langle -3 \cos t, -3 \sin t, 0 \rangle \) and
\[
\vec{T}(t) = \left\langle \frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle \quad \text{and} \quad \vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle.
\]
We can find \( a_T \) and \( a_N \) directly with dot products:
\[
a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{9}{5} \cos t \sin t - \frac{9}{5} \cos t \sin t + 0 = 0.
\]
\[
a_N = \vec{a}(t) \cdot \vec{N}(t) = 3 \cos^2 t + 3 \sin^2 t + 0 = 3.
\]
Thus \( \vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t) \), which is clearly the case.

What is the practical interpretation of these numbers? \( a_T = 0 \) means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 6 Computing \( a_T \) and \( a_N \)
Let \( \vec{r}(t) = \langle t^2 - t, t^2 + t \rangle \) as in Examples 2 and 4. Find \( a_T \) and \( a_N \).

The previous examples give \( \vec{a}(t) = \langle 2, 2 \rangle \) and
\[
\vec{T}(t) = \left\langle \frac{2t - 1}{\sqrt{8t^2 + 2}}, \frac{2t + 1}{\sqrt{8t^2 + 2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t + 1}{\sqrt{8t^2 + 2}}, -\frac{2t - 1}{\sqrt{8t^2 + 2}} \right\rangle.
\]
While we can compute \( a_N \) using \( \vec{N}(t) \), we instead demonstrate using another formula from Theorem 100.
\[
a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{4t - 2}{\sqrt{8t^2 + 2}} + \frac{4t + 2}{\sqrt{8t^2 + 2}} = \frac{8t}{\sqrt{8t^2 + 2}}.
\]
\[
a_N = \sqrt{||\vec{a}(t)||^2 - a_T^2} = \sqrt{8 - \left( \frac{8t}{\sqrt{8t^2 + 2}} \right)^2} = \frac{4}{\sqrt{8t^2 + 2}}.
\]
When \( t = 2 \), \( a_T = \frac{16}{\sqrt{34}} \) and \( a_N = \frac{4}{\sqrt{34}} \). We interpret this to mean that at \( t = 2 \), the particle is acculturing mostly by increasing speed, not by changing direction. As the path near \( t = 2 \) is relatively straight, this should make intuitive sense. Figure 12.24 gives a graph of the path for reference.

Contrast this with \( t = 0 \), where \( a_T = 0 \) and \( a_N = 4/\sqrt{2} = 2\sqrt{2} \). Here the particle’s speed is not changing and all acceleration is in the form of direction change.

Figure 12.24: Graphing \( \vec{r}(t) \) in Example 6.

Notes:
Example 7 Analyzing projectile motion

A ball is thrown from a height of 240 ft with an initial speed of 64 ft/s and an angle of elevation of 30°. Find the position function \( \mathbf{r}(t) \) of the ball and analyze \( a_T \) and \( a_N \).

**Solution**

Using Equation (12.1) of Section 12.3 we form the position function of the ball:

\[
\mathbf{r}(t) = \left( (64 \cos 30^\circ) t, -16t^2 + (64 \sin 30^\circ) t + 240 \right),
\]

which we plot in Figure 12.25.

From this we find \( \mathbf{v}(t) = (64 \cos 30^\circ, -32t + 64 \sin 30^\circ) \) and \( \mathbf{a}(t) = (0, -32) \).

Computing \( \mathbf{T}(t) \) is not difficult, and with some simplification we find

\[
\mathbf{T}(t) = \left( \frac{\sqrt{3}}{\sqrt{t^2 - 2t + 4}}, \frac{1 - t}{\sqrt{t^2 - 2t + 4}} \right).
\]

With \( \mathbf{a}(t) \) as simple as it is, finding \( a_T \) is also simple:

\[
a_T = \mathbf{a}(t) \cdot \mathbf{T}(t) = \frac{32t - 32}{\sqrt{t^2 - 2t + 4}}.
\]

We choose to not find \( \mathbf{N}(t) \) and find \( a_N \) through the formula

\[
a_N = \sqrt{a^2}.
\]

Figure 12.26 gives a table of values of \( a_T \) and \( a_N \). When \( t = 0 \), we see the ball's speed is decreasing; when \( t = 1 \) the speed of the ball is unchanged. This corresponds to the fact that at \( t = 1 \) the ball reaches its highest point.

After \( t = 1 \) we see that \( a_N \) is decreasing in value. This is because as the ball falls, its path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 7 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance (“The gas station is about 2 miles ahead, on the left.”). The arc length parameter allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of “curviness.” The arc length parameter provides a way for us to compute curvature, a quantitative measurement of how curvy a curve is.

---

Notes:
Exercises 12.4

Terms and Concepts

1. If \( \vec{T}(t) \) is a unit tangent vector, what is \( \| \vec{T}(t) \| \)?
2. If \( \vec{N}(t) \) is a unit normal vector, what is \( \vec{N}(t) \cdot \vec{T}(t) \)?
3. The acceleration vector \( \vec{a}(t) \) lies in the plane defined by what two vectors?
4. \( \alpha_r \) measures how much the acceleration is affecting the __________ of an object.

Problems

In Exercises 5–8, given \( \vec{r}(t) \), find \( \vec{T}(t) \) and evaluate it at the indicated value of \( t \).

5. \( \vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1 \)
6. \( \vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4 \)
7. \( \vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4 \)
8. \( \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi \)

In Exercises 9–12, find the equation of the line tangent to the curve at the indicated \( t \)-value using the unit tangent vector. Note: these are the same problems as in Exercises 5–8.

9. \( \vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1 \)
10. \( \vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4 \)
11. \( \vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4 \)
12. \( \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi \)

In Exercises 13–16, find \( \vec{N}(t) \) using Definition 78. Confirm the result using Theorem 99.

13. \( \vec{r}(t) = \langle 3 \cos t, 3 \sin t \rangle \)
14. \( \vec{r}(t) = \langle t, t^2 \rangle \)
15. \( \vec{r}(t) = \langle \cos t, 2 \sin t \rangle \)
16. \( \vec{r}(t) = \langle e^t, e^{-t} \rangle \)

In Exercises 17–20, a position function \( \vec{r}(t) \) is given along with its unit tangent vector \( \vec{T}(t) \) evaluated at \( t = a \), for some value of \( a \).

a. Confirm that \( \vec{T}(a) \) is as stated.

b. Using a graph of \( \vec{T}(t) \) and Theorem 99, find \( \vec{N}(a) \).

17. \( \vec{r}(t) = \langle 3 \cos t, 5 \sin t \rangle; \quad \vec{T}(\pi/4) = \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle \)
18. \( \vec{r}(t) = \langle t, \frac{1}{t^2 + 1} \rangle; \quad \vec{T}(1) = \langle 2\sqrt{5}, -\frac{1}{\sqrt{5}} \rangle \)
19. \( \vec{r}(t) = \langle 1 + 2 \sin t, \cos t, \sin t \rangle; \quad \vec{T}(0) = \langle 2\sqrt{5}, \frac{1}{\sqrt{5}} \rangle \)
20. \( \vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle; \quad \vec{T}(\pi/4) = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \)

In Exercises 21–24, find \( \vec{N}(t) \).

21. \( \vec{r}(t) = \langle 4t, 2 \sin t, 2 \cos t \rangle \)
22. \( \vec{r}(t) = \langle 5 \cos t, 3 \sin t, 4 \sin t \rangle \)
23. \( \vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle; \quad a > 0 \)
24. \( \vec{r}(t) = \langle \cos(at), \sin(at), t \rangle \)

In Exercises 25–30, find \( \alpha_r \) and \( \alpha_n \) given \( \vec{r}(t) \). Sketch \( \vec{T}(t) \) on the indicated interval, and comment on the relative sizes of \( \alpha_r \) and \( \alpha_n \) at the indicated \( t \) values.

25. \( \vec{r}(t) = \langle t, t^2 \rangle \) on \([-1, 1] \); consider \( t = 0 \) and \( t = 1 \).
26. \( \vec{r}(t) = \langle t, 1/t \rangle \) on \([0, 4]\) \); consider \( t = 1 \) and \( t = 2 \).
27. \( \vec{r}(t) = \langle 2 \cos t, 2 \sin t \rangle \) on \([0, 2\pi]\); consider \( t = 0 \) and \( t = \pi/2 \).
28. \( \vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle \) on \([0, 2\pi]\); consider \( t = \sqrt{\pi/2} \) and \( t = \sqrt{\pi} \).
29. \( \vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle \) on \([0, 2\pi]\), where \( a, b > 0 \); consider \( t = 0 \) and \( t = \pi/2 \).
30. \( \vec{r}(t) = \langle 5 \cos t, 4 \sin t, 3 \sin t \rangle \) on \([0, 2\pi]\); consider \( t = 0 \) and \( t = \pi/2 \).
12.5 The Arc Length Parameter and Curvature

In normal conversation we describe position in terms of both time and distance. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer “I am 20 minutes from your house,” or you might say “I am 10 miles from your house.” Both answers provide your friend with a general idea of where you are.

Currently, our vector–valued functions have defined points with a parameter $t$, which we often take to represent time. Consider Figure 12.27(a), where $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$ is graphed and the points corresponding to $t = 0$, 1 and 2 are shown. Note how the arc length between $t = 0$ and $t = 1$ is smaller than the arc length between $t = 1$ and $t = 2$; if the parameter $t$ is time and $\vec{r}$ is position, we can say that the particle traveled faster on $[1, 2]$ than on $[0, 1]$.

Now consider Figure 12.27(b), where the same graph is parametrized by a different variable $s$. Points corresponding to $s = 0$ through $s = 6$ are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter $s$ as distance; that is, the arc length of the graph from $s = 0$ to $s = 3$ is 3, the arc length from $s = 2$ to $s = 6$ is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e., $s = 0$), one would compute $\vec{r}(2.5)$. This parameter $s$ is very useful, and is called the arc length parameter.

How do we find the arc length parameter?

Start with any parametrization of $\vec{r}$. We can compute the arc length of the graph of $\vec{r}$ on the interval $[0, t]$ with

$$\text{arc length} = \int_0^t \| \vec{r}'(u) \| \, du.$$ 

We can turn this into a function: as $t$ varies, we find the arc length $s$ from 0 to $t$. This function is

$$s(t) = \int_0^t \| \vec{r}'(u) \| \, du. \quad (12.2)$$

This establishes a relationship between $s$ and $t$. Knowing this relationship explicitly, we can rewrite $\vec{r}(t)$ as a function of $s$: $\vec{r}(s)$. We demonstrate this in an example.

Watch the video:
Parametrize a Curve with Respect to Arc Length at https://youtu.be/G0R-qia1KlE

Notes:
Example 1  Finding the arc length parameter
Let \( \vec{r}(t) = (3t - 1, 4t + 2) \). Parametrize \( \vec{r} \) with the arc length parameter \( s \).

**Solution**  Using Equation (12.2), we write
\[
s(t) = \int_0^t \| \vec{r}'(u) \| \, du.
\]
We can integrate this, explicitly finding a relationship between \( s \) and \( t \):
\[
s(t) = \int_0^t \sqrt{3^2 + 4^2} \, du = \int_0^t 5 \, du = 5t.
\]
Since \( s = 5t \), we can write \( t = s/5 \) and replace \( t \) in \( \vec{r}(t) \) with \( s/5 \):
\[
\vec{r}(s) = \langle 3(\frac{s}{5}) - 1, \frac{4s}{5} + 2 \rangle = \left\langle \frac{3s}{5} - 1, \frac{4s}{5} + 2 \right\rangle.
\]
Clearly, as shown in Figure 12.28, the graph of \( \vec{r} \) is a line, where \( t = 0 \) corresponds to the point \((-1, 2)\). What point on the line is 2 units away from this initial point? We find it with \( s(2) = \frac{1}{5}, \frac{18}{5} \).

Is the point \((\frac{1}{5}, \frac{18}{5})\) really 2 units away from \((-1, 2)\)? We use the Distance Formula to check:
\[
d = \sqrt{\left(\frac{1}{5} - (-1)\right)^2 + \left(\frac{18}{5} - 2\right)^2} = \sqrt{\frac{36}{25} + \frac{64}{25}} = \sqrt{4} = 2.
\]
Yes, \( s(2) \) is indeed 2 units away, in the direction of travel, from the initial point.

Things worked out very nicely in Example 1; we were able to establish directly that \( s = 5t \). Usually, the arc length parameter is much more difficult to describe in terms of \( t \), a result of integrating a square–root. There are a number of things that we can learn about the arc length parameter from Equation (12.2), though, that are incredibly useful.

First, take the derivative of \( s \) with respect to \( t \). The Fundamental Theorem of Calculus (see Theorem 38) states that
\[
\frac{ds}{dt} = s'(t) = \| \vec{r}'(t) \|.
\]
Letting \( t \) represent time and \( \vec{r}(t) \) represent position, we see that the rate of change of \( s \) with respect to \( t \) is speed; that is, the rate of change of “distance traveled” is speed, which should match our intuition.

The Chain Rule states that

\[
\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt},
\]

\[
\vec{r}'(t) = \vec{r}'(s) \cdot \|\vec{r}'(t)\|.
\]

Solving for \( \vec{r}'(s) \), we have

\[
\vec{r}'(s) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \vec{T}(t),
\]

where \( \vec{T}(t) \) is the unit tangent vector. Equation (12.4) is often misinterpreted, as one is tempted to think it states \( \vec{r}'(t) = \vec{T}(t) \), but there is a big difference between \( \vec{r}'(s) \) and \( \vec{r}'(t) \). The key to take from it is that \( \vec{r}'(s) \) is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

**Theorem 101 Arc Length Parameter**

Let \( \vec{r}(s) \) be a vector–valued function. The parameter \( s \) is the arc length parameter if, and only if, \( \|\vec{r}'(s)\| = 1 \).

**Curvature**

Consider points \( A \) and \( B \) on the curve graphed in Figure 12.29(a). One can readily argue that the curve curves more sharply at \( A \) than at \( B \). It is useful to use a number to describe how sharply the curve bends; that number is the curvature of the curve.

We derive this number in the following way. Consider Figure 12.29(b), where unit tangent vectors are graphed around points \( A \) and \( B \). Notice how the direction of the unit tangent vector changes quite a bit near \( A \), whereas it does not change as much around \( B \). This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.
Definition 79  Curvature
Let \( \vec{r}(s) \) be a vector–valued function where \( s \) is the arc length parameter. The curvature \( \kappa \) of the graph of \( \vec{r}(s) \) is

\[
\kappa = \frac{d\vec{T}}{ds} = \| \vec{T}'(s) \|.
\]

If \( \vec{r}(s) \) is parametrized by the arc length parameter, then

\[
\vec{T}(s) = \frac{\vec{r}'(s)}{\| \vec{r}'(s) \|} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{T}'(s)}{\| \vec{T}'(s) \|}.
\]

Having defined \( \| \vec{T}'(s) \| = \kappa \), we can rewrite the second equation as

\[
\vec{T}'(s) = \kappa \vec{N}(s). \tag{12.5}
\]

We already knew that \( \vec{T}'(s) \) is in the same direction as \( \vec{N}(s) \); that is, we can think of \( \vec{T}'(s) \) as being “pulled” in the direction of \( \vec{N}(s) \). How “hard” is it being pulled? By a factor of \( \kappa \). When the curvature is large, \( \vec{T}(s) \) is being “pulled hard” and the direction of \( \vec{T}(s) \) changes rapidly. When \( \kappa \) is small, \( \vec{T}(s) \) is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 79 to find the curvature of the line in Example 1.

Example 2  Finding the curvature of a line
Use Definition 79 to find the curvature of \( \vec{r}(t) = (3t - 1, 4t + 2) \).

\[ \text{SOLUTION} \quad \text{In Example 1, we found that the arc length parameter was defined by } s = 5t, \text{ so } \vec{r}(s) = (3s/5 - 1, 4s/5 + 2) \text{ parametrized } \vec{r} \text{ with the arc length parameter. To find } \kappa, \text{ we need to find } \vec{T}'(s). \quad \]

\[ \vec{T}(s) = \frac{\vec{r}'(s)}{\| \vec{r}'(s) \|} \text{ (recall this is a unit vector)} \]

\[ = \left( \frac{3}{5}, \frac{4}{5} \right). \]

Therefore

\[
\vec{T}'(s) = (0, 0)
\]

and

\[
\kappa = \| \vec{T}'(s) \| = 0.
\]

It probably comes as no surprise that the curvature of a line is 0. (How “curvy” is a line? It is not curvy at all.)
While the definition of curvature is a useful mathematical concept, it is nearly impossible to use most of the time; writing $\vec{r}$ in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a trade-off: the definition is “easy” to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

**Theorem 102 Formulas for Curvature**
Let $C$ be a smooth curve on an open interval $I$ in the plane or in space.

1. If $C$ is defined by $y = f(x)$, then
   \[
   \kappa = \frac{|f''(x)|}{\left(1 + \left(f'(x)^2\right)^{3/2}\right)}.
   \]

2. If $C$ is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t) \rangle$, then
   \[
   \kappa = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}.
   \]

3. If $C$ is defined in space by a vector–valued function $\vec{r}(t)$, then
   \[
   \kappa = \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\|\vec{v}(t)\|^2}.
   \]

**Proof**
We’ll prove statement 3; statements 1 and 2 are applications that we leave to the exercises. By the chain rule and then Equation (12.3),
\[
\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \frac{ds}{dt} = \frac{d\vec{T}}{ds} \|\vec{r}'(t)\|,
\]
so that
\[
\kappa = \frac{d\vec{T}(s)}{ds} = \frac{d\vec{T}(t)}{\|\vec{r}'(t)\|} = \frac{d\vec{T}(t)}{\|\vec{r}'(t)\|} = \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|}.
\]

**Notes:**
Chapter 12  Vector Valued Functions

Now, \( \vec{r}'(t) = \| \vec{r}'(t) \| \vec{T}(t) \) so that
\[
\vec{r}''(t) = \left( \| \vec{r}'(t) \| \right)' \vec{T}(t) + \| \vec{r}'(t) \| \vec{T}'(t)
\]
and
\[
\vec{r}'(t) \times \vec{r}''(t) = \| \vec{r}'(t) \| \vec{T}(t) \times \| \vec{r}'(t) \| \vec{T}'(t) = \| \vec{r}'(t) \|^2 \vec{T}(t) \times \vec{T}'(t)
\]
Because \( \| \vec{T} \| = 1 \), Theorem 95 tells us that \( \vec{T} \) and \( \vec{T}' \) are orthogonal. This means that
\[
\| \vec{r}'(t) \times \vec{r}''(t) \| = \| \vec{r}'(t) \|^2 \| \vec{T}(t) \| \| \vec{T}'(t) \| = \| \vec{r}'(t) \|^2 \| \vec{T}'(t) \|
\]
and
\[
\kappa = \frac{\| \vec{T}'(t) \|}{\| \vec{r}'(t) \|^2} = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|^3}.
\]

Theorem 100 tells us that \( \vec{r}'''(t) = a_T \vec{T}(t) + (\vec{r}''(t) \cdot \vec{N}(t))\vec{N}(t) \). Since \( \vec{r}'(t) = \vec{v}(t) \) and \( \vec{T}(t) \) are parallel, their cross product is zero and
\[
\kappa = \frac{\| \vec{v}(t) \times [a_T \vec{T}(t) + (\vec{r}''(t) \cdot \vec{N}(t))\vec{N}(t)] \|}{\| \vec{v}(t) \|^3} = \frac{\| \vec{v}(t) \times (\vec{a}(t) \cdot \vec{N}(t))\vec{N}(t) \|}{\| \vec{v}(t) \|^3}.
\]

Since \( \vec{v}(t) \) and \( \vec{N}(t) \) are orthogonal, the norm of their cross product is the product of their norms, and
\[
\kappa = \frac{\| \vec{v}(t) \| (\vec{a}(t) \cdot \vec{N}(t)) \| \vec{N}(t) \|}{\| \vec{v}(t) \|^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\| \vec{v}(t) \|^2}.
\]

We practice using these formulas.

Example 3  Finding the curvature of a circle
Find the curvature of a circle with radius \( r \), defined by \( \vec{c}(t) = (r \cos t, r \sin t) \).

SOLUTION  Before we start, we should expect the curvature of a circle to be constant, and not dependent on \( t \). (Why?)

We compute \( \kappa \) using the second part of Theorem 102.
\[
\kappa = \frac{|(-r \sin t)(-r \sin t) - (r \cos t)(r \cos t)|}{((-r \sin t)^2 + (r \cos t)^2)^{3/2}}
\]
\[
= \frac{r^2 (\sin^2 t + \cos^2 t)}{(r^2 (\sin^2 t + \cos^2 t))^{3/2}}
\]
\[
= \frac{r^2}{r^3} = \frac{1}{r}.
\]

We have found that a circle with radius \( r \) has curvature \( \kappa = 1/r \).
Example 3 gives a great result. Before this example, if we were told "The curve has a curvature of 5 at point A," we would have no idea what this really meant. Is 5 "big" – does it correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point $P$ on a smooth curve $C$ be given, and let $\kappa$ be the curvature of the curve at $P$. A circle that:

- passes through $P$,
- lies on the concave side of $C$,
- has a common tangent line as $C$ at $P$ and
- has radius $r = 1/\kappa$ (hence has curvature $\kappa$)

is the osculating circle, or circle of curvature, to $C$ at $P$, and $r$ is the radius of curvature. Figure 12.30 shows the graph of the curve seen earlier in Figure 12.29 and its osculating circles at $A$ and $B$. A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many useful ideas in mathematics have come from studying the osculating circles to a curve.)

**Example 4  Finding curvature**

Find the curvature of the parabola defined by $y = x^2$ at the vertex and at $x = 1$.

**SOLUTION** We use the first formula found in Theorem 102.

$$\kappa(x) = \frac{|2|}{(1 + (2x)^2)^{3/2}}$$

$$= \frac{2}{(1 + 4x^2)^{3/2}}.$$

At the vertex ($x = 0$), the curvature is $\kappa = 2$. At $x = 1$, the curvature is $\kappa = 2/(5)^{3/2}$. So at $x = 0$, the curvature of $y = x^2$ is that of a circle of radius $1/2$; at $x = 1$, the curvature is that of a circle with radius $(5)^{3/2}/2 \approx 5.59$. This is illustrated in Figure 12.31. At $x = 3$, the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

---

Notes:
Example 5  Finding curvature

Find where the curvature of \( \vec{r}(t) = \langle t, t^2, 2t^3 \rangle \) is maximized.

**Solution**

We use the third formula in Theorem 102 as \( \vec{r}(t) \) is defined in space. We leave it to the reader to verify that

\[ \vec{r}'(t) = \langle 1, 2t, 6t^2 \rangle, \quad \vec{r}''(t) = \langle 0, 2, 12t \rangle, \quad \text{and} \quad \vec{r}' \times \vec{r}''(t) = \langle 12t^2, -12t, 2 \rangle. \]

Thus

\[
\kappa(t) = \frac{\|\vec{r}' \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\|\langle 12t^2, -12t, 2 \rangle\|}{\|\langle 1, 2t, 6t^2 \rangle\|^3} = \frac{\sqrt{144t^4 + 144t^2 + 4}}{(\sqrt{1 + 4t^2 + 36t^4})^3}
\]

While this is not a particularly "nice" formula, it does explicitly tell us what the curvature is at a given \( t \) value. To maximize \( \kappa(t) \), we should solve \( \kappa'(t) = 0 \) for \( t \). This is doable, but very time consuming. Instead, consider the graph of \( \kappa(t) \) as given in Figure 12.32(a). We see that \( \kappa \) is maximized at two \( t \) values; using a numerical solver, we find these values are \( t \approx \pm 0.189 \). In part (b) of the figure we graph \( \vec{r}(t) \) and indicate the points where curvature is maximized.

Curvature and Motion

Let \( \vec{r}(t) \) be a position function of an object, with velocity \( \vec{v}(t) = \vec{r}'(t) \) and acceleration \( \vec{a}(t) = \vec{r}''(t) \). In Section 12.4 we established that acceleration is in the plane formed by \( \vec{T}(t) \) and \( \vec{N}(t) \), and that we can find scalars \( a_T \) and \( a_N \) such that

\[ \vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t). \]

Theorem 100 gives formulas for \( a_T \) and \( a_N \):

\[
a_T = \frac{d}{dt} \left( \frac{\|\vec{v}(t)\|}{\|\vec{v}(t)\|} \right) \quad \text{and} \quad a_N = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|}. \]

We understood that the amount of acceleration in the direction of \( \vec{T} \) relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of \( \vec{N} \) relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed, \( a_T = 0 \); if the object travels in a constant direction, \( a_N = 0 \).)

Notes:
In Equation (12.3) at the beginning of this section, we found \( s'(t) = \| \vec{v}(t) \| \).
We can combine this fact with the above formula for \( \alpha_T \) to write
\[
\alpha_T = \frac{d}{dt} \left( \| \vec{v}(t) \| \right) = \frac{d}{dt} (s'(t)) = s''(t).
\]
Since \( s'(t) \) is speed, \( s''(t) \) is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for \( a_N \) above to the formula for curvature in Theorem 102:
\[
a_N = \frac{\| \vec{v}(t) \times \vec{a}(t) \|}{\| \vec{v}(t) \|} \quad \text{and} \quad \kappa = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|^3} = \frac{\| \vec{v}(t) \times \vec{a}(t) \|}{\| \vec{v}(t) \|^3}.
\]
Thus
\[
a_N = \kappa \| \vec{v}(t) \|^2 = \kappa \left( s'(t) \right)^2 \tag{12.6}
\]

This last equation shows that the component of acceleration that changes the object’s direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing \( s'(t) \)), the door pushes harder against you (\( a_N \) has increased). If you keep your speed constant but tighten the turn (i.e., increase \( \kappa \)), once again the door will push harder against you.

Putting our new formulas for \( \alpha_T \) and \( a_N \) together, we have
\[
\vec{a}(t) = s''(t) \vec{T}(t) + \kappa \| \vec{v}(t) \|^2 \vec{N}(t).
\]
This is not a particularly practical way of finding \( \alpha_T \) and \( a_N \), but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

**Example 6 Curvature and road design**
The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 12.33. For each curve and speed, compute \( a_N \).

**SOLUTION** Using Equation (12.6), we can compute the acceleration normal to the curve in each case. We start by converting each speed from “miles per hour” to “feet per second” by multiplying by 5280/3600.

<table>
<thead>
<tr>
<th>Operating Speed (mph)</th>
<th>Minimum Radius (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>310</td>
</tr>
<tr>
<td>40</td>
<td>430</td>
</tr>
<tr>
<td>45</td>
<td>540</td>
</tr>
</tbody>
</table>

Figure 12.33: Operating speed and minimum radius in highway cloverleaf design.
Chapter 12  Vector Valued Functions

\[ a_N = \kappa \|\vec{v}(t)\|^2 \]
\[ = \frac{1}{310} (51.33)^2 \]
\[ = 8.50 \text{ft/s}^2. \]

\[ a_N = \frac{1}{430} (58.67)^2 \]
\[ = 8.00 \text{ft/s}^2. \]

\[ a_N = \frac{1}{540} (66)^2 \]
\[ = 8.07 \text{ft/s}^2. \]

Note that each acceleration is similar; this is by design. Considering the classic “Force = mass $\times$ acceleration” formula, this acceleration must be kept small in order for the tires of a vehicle to keep a “grip” on the road. If one travels on a turn of radius 310ft at a rate of 50mph, the acceleration is double, at 17.35ft/s\(^2\). If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a “safe” design speed, then subtract 5-10mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we’ve covered. We started with vector–valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behavior of functions and the study of motion. Vector–valued position functions convey displacement, distance traveled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.
Exercises 12.5

Terms and Concepts

1. It is common to describe position in terms of both _______ and/or _______.
2. A measure of the “curviness” of a curve is _______.
3. Give two shapes with constant curvature.
4. Describe in your own words what an “osculating circle” is.
5. Complete the identity: $\mathbf{T}'(s) =$ _______ $\mathbf{N}(s)$.
6. Given a position function $\mathbf{r}(t)$, how are $\alpha$ and $\beta$ affected by the curvature?

Problems

In Exercises 7–10, a position function $\mathbf{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter $s$, and rewrite $\mathbf{r}(t)$ in terms of $s$; that is, find $\mathbf{r}(s)$.

7. $\mathbf{r}(t) = (2t, t, -2t)$
8. $\mathbf{r}(t) = (7 \cos t, 7 \sin t)$
9. $\mathbf{r}(t) = (3 \cos t, 3 \sin t, 2t)$
10. $\mathbf{r}(t) = (5 \cos t, 13 \sin t, 12 \cos t)$

In Exercises 11–22, a curve $C$ is described along with 2 points on $C$.

(a) Using a sketch, determine at which of these points the curvature is greater.
(b) Find the curvature $\kappa$ of $C$, and evaluate $\kappa$ at each of the 2 given points.

11. $C$ is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.
12. $C$ is defined by $y = \frac{1}{x^2 + 1}$; points given at $x = 0$ and $x = 2$.
13. $C$ is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.
14. $C$ is defined by $y = \sqrt{1 - x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.
15. $C$ is defined by $\mathbf{r}(t) = (\cos t, \sin(2t))$; points given at $t = 0$ and $t = \pi/4$.
16. $C$ is defined by $\mathbf{r}(t) = (\cos^2 t, \sin t \cos t)$; points given at $t = 0$ and $t = \pi/3$.
17. $C$ is defined by $\mathbf{r}(t) = (t^2 - 1, t^3 - t)$; points given at $t = 0$ and $t = 5$.
18. $C$ is defined by $\mathbf{r}(t) = (\tan t, \sec t)$; points given at $t = 0$ and $t = \pi/6$.
19. $C$ is defined by $\mathbf{r}(t) = (4t + 2, 3t - 1, 2t + 5)$; points given at $t = 0$ and $t = 1$.
20. $C$ is defined by $\mathbf{r}(t) = (t^3 - t, t^2 - 4, t^2 - 1)$; points given at $t = 0$ and $t = 1$.
21. $C$ is defined by $\mathbf{r}(t) = (3 \cos t, 3 \sin t, 2t)$; points given at $t = 0$ and $t = \pi/2$.
22. $C$ is defined by $\mathbf{r}(t) = (5 \cos t, 13 \sin t, 12 \cos t)$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23–26, find the value of $x$ or $t$ where curvature is maximized.

23. $y = \frac{1}{6} x^3$
24. $y = \sin x$
25. $\mathbf{r}(t) = (t^2 + 2t, 3t - t^2)$
26. $\mathbf{r}(t) = (t, 4/t, 3/t)$

In Exercises 27–30, find the radius of curvature at the indicated value.

27. $y = \tan x$, at $x = \pi/4$
28. $y = x^2 + x - 3$, at $x = \pi/4$
29. $\mathbf{r}(t) = (\cos t, \sin(3t))$, at $t = 0$
30. $\mathbf{r}(t) = (5 \cos(3t), t)$, at $t = 0$

In Exercises 31–34, find the equation of the osculating circle to the curve at the indicated $t$-value.

31. $\mathbf{r}(t) = (t, t^3)$, at $t = 0$
32. $\mathbf{r}(t) = (3 \cos t, \sin t)$, at $t = 0$
33. $\mathbf{r}(t) = (3 \cos t, \sin t)$, at $t = \pi/2$
34. $\mathbf{r}(t) = (t^2 - t, t^2 + t)$, at $t = 0$

35. For Theorem 102, use part 3 to prove part 2: If $\mathbf{r}(t) = (x(t), y(t))$ is a vector–valued function in the plane, then

$$\kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}.$$  

36. For Theorem 102, use part 2 or 3 to prove part 1: If $y = f(x)$, then

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$
13: FUNCTIONS OF SEVERAL VARIABLES

A function of the form $y = f(x)$ is a function of a single variable; given a value of $x$, we can find a value $y$. Even the vector–valued functions of Chapter 12 are single–variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player’s batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies multivariable functions, that is, functions with more than one input.

13.1 Introduction to Multivariable Functions

Definition 80 Function of Two Variables
Let $D$ be a subset of $\mathbb{R}^2$. A function $f$ of two variables is a rule that assigns each pair $(x, y)$ in $D$ a value $z = f(x, y)$ in $\mathbb{R}$. The set $D$ is the domain of $f$; the set of all outputs of $f$ is the range.

Example 1 Understanding a function of two variables
Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2), f(2, 1), \text{ and } f(-2, 4)$; find the domain and range of $f$. 

Watch the video:
Finding and Sketching the Domain of a Multivariable Function at https://youtu.be/q8ictFvAHLk
Using the definition \( f(x, y) = x^2 - y \), we have:

\[
\begin{align*}
  f(1, 2) &= 1^2 - 2 = -1 \\
  f(2, 1) &= 2^2 - 1 = 3 \\
  f(-2, 4) &= (-2)^2 - 4 = 0
\end{align*}
\]

The domain is not specified, so we take it to be all possible pairs in \( \mathbb{R}^2 \) for which \( f \) is defined. In this example, \( f \) is defined for all pairs \((x, y)\), so the domain \( D \) of \( f \) is \( \mathbb{R}^2 \).

The output of \( f \) can be made as large or small as possible; any real number \( r \) can be the output. (In fact, given any real number \( r \), \( f(0, r) = r \).) So the range \( R \) of \( f \) is \( \mathbb{R} \).

Example 2 Understanding a function of two variables

Let \( f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \). Find the domain and range of \( f \).

**Solution**

The domain is all pairs \((x, y)\) allowable as input in \( f \). Because of the square–root, we need \((x, y)\) such that \( 0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4} \):

\[
\frac{x^2}{9} + \frac{y^2}{4} \leq 1
\]

The above equation describes an ellipse and its interior as shown in Figure 13.1. We can represent the domain \( D \) graphically with the figure; in set notation, we can write \( D = \{(x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \} \).

The range is the set of all possible output values. The square–root ensures that all output is \( \geq 0 \). Since the \( x \) and \( y \) terms are squared, then subtracted, inside the square-root, the largest output value comes at \( x = 0, y = 0 \): \( f(0, 0) = 1 \). Thus the range \( R \) is the interval \([0, 1]\).

Graphing Functions of Two Variables

The **graph** of a function \( f \) of two variables is the set of all points \((x, y, f(x, y))\) where \((x, y)\) is in the domain of \( f \). This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 13.2(a) where 25 points have been plotted of \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \).

More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading...
to create a graph like Figure 13.2(b) which does a far better job of illustrating the behavior of \( f \).

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching level curves.

**Level Curves**

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don’t realize it). Topographical maps, like the one shown in Figure 13.3, represent the surface of Earth by indicating points with the same elevation with contour lines. The elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near “Aspen Campground,” elevation changes more gradually as one has to walk farther to rise 50ft.

Given a function \( z = f(x, y) \), we can draw a “topographical map” of \( f \) by drawing level curves (or, contour lines). A level curve at \( z = c \) is a curve in the \( x-y \) plane such that for all points \((x, y)\) on the curve, \( f(x, y) = c \).

When drawing level curves, it helps to evenly space the \( c \) values as that gives the best insight to how quickly the “elevation” is changing. Examples will help one understand this concept.

**Example 3** Drawing Level Curves

Let \( f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \). Find the level curves of \( f \) for \( c = 0, 0.2, 0.4, 0.6, 0.8 \) and 1.

**Solution** Consider first \( c = 0 \). The level curve for \( c = 0 \) is the set of all points \((x, y)\) such that \( 0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \). Squaring both sides gives us

\[
\frac{x^2}{9} + \frac{y^2}{4} = 1,
\]

an ellipse centered at \((0, 0)\) with horizontal major axis of length 6 and minor axis of length 4. Thus for any point \((x, y)\) on this curve, \( f(x, y) = 0 \).
Now consider the level curve for \( c = 0.2 \)
\[
0.2 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}
\]
\[
0.04 = 1 - \frac{x^2}{9} - \frac{y^2}{4}
\]
\[
\frac{x^2}{9} + \frac{y^2}{4} = 0.96
\]
\[
\frac{x^2}{8.64} + \frac{y^2}{3.84} = 1.
\]
This is also an ellipse, where \( a = \sqrt{8.64} \approx 2.94 \) and \( b = \sqrt{3.84} \approx 1.96 \).

In general, for \( z = c \), the level curve is:
\[
c = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}
\]
\[
c^2 = 1 - \frac{x^2}{9} - \frac{y^2}{4}
\]
\[
\frac{x^2}{9} + \frac{y^2}{4} = 1 - c^2
\]
\[
\frac{x^2}{9(1 - c^2)} + \frac{y^2}{4(1 - c^2)} = 1,
\]
ellipses that are decreasing in size as \( c \) increases. A special case is when \( c = 1 \); there the ellipse is just the point \((0, 0)\).

The level curves are shown in Figure 13.4(a). Note how the level curves for \( c = 0 \) and \( c = 0.2 \) are very, very close together: this indicates that \( f \) is growing rapidly along those curves.

In Figure 13.4(b), the curves are drawn on a graph of \( f \) in space. Note how the elevations are evenly spaced. Near the level curves of \( c = 0 \) and \( c = 0.2 \) we can see that \( f \) indeed is growing quickly.

**Example 4 Analyzing Level Curves**

Let \( f(x, y) = \frac{x + y}{x^2 + y^2 + 1} \). Find the level curves for \( z = c \).

**SOLUTION** We begin by setting \( f(x, y) = c \) for an arbitrary \( c \) and seeing if algebraic manipulation of the equation reveals anything significant.
\[
\frac{x + y}{x^2 + y^2 + 1} = c
\]
\[
x + y = c(x^2 + y^2 + 1).
\]
We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

\[
\left( x - \frac{1}{2c} \right)^2 + \left( y - \frac{1}{2c} \right)^2 = \frac{1}{2c^2} - 1,
\]

a circle centered at \( (1/(2c), 1/(2c)) \) with radius \( \sqrt{1/(2c^2) - 1} \), where \( |c| < 1/\sqrt{2} \). The level curves for \( c = \pm 0.2 \), \( \pm 0.4 \) and \( \pm 0.6 \) are sketched in Figure 13.5(a). To help illustrate “elevation,” we use thicker lines for \( c \) values near 0, and dashed lines indicate where \( c < 0 \).

There is one special level curve, when \( c = 0 \). The level curve in this situation is \( x + y = 0 \), the line \( y = -x \).

In Figure 13.5(b) we see a graph of the surface. Note how the \( y \)-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can “walk” along the line \( y = -x \) without elevation change, though the level curve does.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables so that we are able to view the domain without exceeding three dimensions.)

**Definition 81 Function of Three Variables**

Let \( D \) be a subset of \( \mathbb{R}^3 \). A function \( f \) of three variables is a rule that assigns each triple \( (x, y, z) \) in \( D \) a value \( w = f(x, y, z) \) in \( \mathbb{R} \). The set \( D \) is the domain of \( f \); the set of all outputs of \( f \) is the range.

Note how this definition closely resembles that of Definition 80.

**Example 5 Understanding a function of three variables**

Let \( f(x, y, z) = \frac{x^2 + z + 3 \sin y}{x + 2y - z} \). Evaluate \( f \) at the point \( (3, 0, 2) \) and find the domain and range of \( f \).

Notes:
Chapter 13  Functions of Several Variables

SOLUTION \[ f(3, 0, 2) = \frac{3^2 + 2 + 3 \sin 0}{3 + 2(0) - 2} = 11. \]

As the domain of \( f \) is not specified, we take it to be the set of all triples \((x, y, z)\) for which \( f(x, y, z) \) is defined. As we cannot divide by 0, we find the domain \( D \) is

\[ D = \{(x, y, z) \mid x + 2y - z \neq 0\}. \]

We recognize that the set of all points in \( \mathbb{R}^3 \) that are not in \( D \) form a plane in space that passes through the origin (with normal vector \( \langle 1, 2, -1 \rangle \)).

We determine the range \( R \) is \( \mathbb{R} \); that is, all real numbers are possible outputs of \( f \). There is no set way of establishing this. Rather, to get numbers near 0 we can let \( y = 0 \) and choose \( z \approx -x^2 \). To get numbers of arbitrarily large magnitude, we can let \( z \approx x + 2y \).

**Level Surfaces**

It is very difficult to produce a meaningful graph of a function of three variables. A function of one variable is a curve drawn in 2 dimensions; a function of two variables is a surface drawn in 3 dimensions; a function of three variables is a hypersurface drawn in 4 dimensions.

There are a few techniques one can employ to try to “picture” a graph of three variables. One is an analogue of level curves: level surfaces. Given \( w = f(x, y, z) \), the level surface at \( w = c \) is the surface in space formed by all points \((x, y, z)\) where \( f(x, y, z) = c \).

**Example 6** Finding level surfaces

If a point source \( S \) is radiating energy, the intensity \( I \) at a given point \( P \) in space is inversely proportional to the square of the distance between \( S \) and \( P \). That is, when \( S = (0, 0, 0) \), \( I(x, y, z) = \frac{k}{x^2 + y^2 + z^2} \) for some constant \( k \).

Let \( k = 1 \); find the level surfaces of \( I \).

**SOLUTION** We can (mostly) answer this question using “common sense.” If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at \( I = c \) is defined by

\[ c = \frac{1}{x^2 + y^2 + z^2}. \]
taking reciprocals reveals

\[ x^2 + y^2 + z^2 = \frac{1}{c}. \]

Given an intensity \( c \), the level surface \( I = c \) is a sphere of radius \( 1/\sqrt{c} \), centered at the origin.

Figure 13.6 gives a table of the radii of the spheres for given \( c \) values. Normally one would use equally spaced \( c \) values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

In the next section we apply the concepts of limits to functions of two or more variables.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.</td>
<td>0.25</td>
</tr>
<tr>
<td>8.</td>
<td>0.35</td>
</tr>
<tr>
<td>4.</td>
<td>0.5</td>
</tr>
<tr>
<td>2.</td>
<td>0.71</td>
</tr>
<tr>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>0.5</td>
<td>1.41</td>
</tr>
<tr>
<td>0.25</td>
<td>2.</td>
</tr>
<tr>
<td>0.125</td>
<td>2.83</td>
</tr>
<tr>
<td>0.0625</td>
<td>4.</td>
</tr>
</tbody>
</table>

Figure 13.6: A table of \( c \) values and the corresponding radius \( r \) of the spheres of constant value in Example 6.


**Terms and Concepts**

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.
2. The graph of a function of two variables is a _________.
3. Most people are familiar with the concept of level curves in the context of ________ maps.
4. T/F: Along a level curve, the output of a function does not change.
5. The analogue of a level curve for functions of three variables is a level _________.
6. What does it mean when level curves are close together? Far apart?

**Problems**

In Exercises 7–14, give the domain and range of the multivariable function.

7. \( f(x, y) = x^2 + y^2 + 2 \)
8. \( f(x, y) = x + 2y \)
9. \( f(x, y) = x - 2y \)
10. \( f(x, y) = \frac{1}{x + 2y} \)
11. \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \)
12. \( f(x, y) = \sin x \cos y \)
13. \( f(x, y) = \sqrt{9 - x^2 - y^2} \)
14. \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2} - 9} \)

In Exercises 15–22, describe in words and sketch the level curves for the function and given \( c \) values.

15. \( f(x, y) = 3x - 2y; \ c = -2, 0, 2 \)
16. \( f(x, y) = x^2 - y^2; \ c = -1, 0, 1 \)
17. \( f(x, y) = x - y^2; \ c = -2, 0, 2 \)
18. \( f(x, y) = \frac{1 - x^2 - y^2}{2y - 2x}; \ c = -2, 0, 2 \)
19. \( f(x, y) = \frac{2x - 2y}{x^2 + y^2 + 1}; \ c = -1, 0, 1 \)
20. \( f(x, y) = \frac{y - x^3 - 1}{x}; \ c = -3, -1, 0, 1, 3 \)
21. \( f(x, y) = \sqrt{x^2 + 4y^2}; \ c = 1, 2, 3, 4 \)
22. \( f(x, y) = x^3 + 4y^2; \ c = 1, 2, 3, 4 \)

In Exercises 23–26, give the domain and range of the functions of three variables.

23. \( f(x, y, z) = \frac{x}{x + 2y - 4z} \)
24. \( f(x, y, z) = \frac{1}{1 - x^3 - y^2 - z^2} \)
25. \( f(x, y, z) = \sqrt{z - x^2 + y^2} \)
26. \( f(x, y, z) = z^2 \sin x \cos y \)

In Exercises 27–30, describe the level surfaces of the given functions of three variables.

27. \( f(x, y, z) = x^2 + y^2 + z^2 \)
28. \( f(x, y, z) = z - x^2 + y^2 \)
29. \( f(x, y, z) = \frac{x^2 + y^2}{z} \)
30. \( f(x, y, z) = \frac{z}{x - y} \)

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.
13.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be “continuous.”

We begin with a series of definitions. We are used to “open intervals” such as \((1, 3)\), which represents the set of all \(x\) such that \(1 < x < 3\), and “closed intervals” such as \([1, 3]\), which represents the set of all \(x\) such that \(1 \leq x \leq 3\). We need analogous definitions for open and closed sets in the \(x-y\) plane.

Figure 13.7 shows several sets in the \(x-y\) plane. In each set, point \(P_1\) lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point \(P_2\) is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 13.7(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.
Example 1 Determining open/closed, bounded/unbounded
Determine if the domain of the function \( f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \) is open, closed, or neither, and if it is bounded.

Solution This domain of this function was found in Example 13.1.2 to be \( D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \leq 1\} \), the region bounded by the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \). Since the region includes the boundary (indicated by the use of “\( \leq \)”), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains \( D \).

Example 2 Determining open/closed, bounded/unbounded
Determine if the domain of \( f(x, y) = \frac{1}{x-y} \) is open, closed, or neither.

Solution As we cannot divide by 0, we find the domain to be \( D = \{(x, y) | x - y \neq 0\} \). In other words, the domain is the set of all points \((x, y)\) not on the line \( y = x \).

The domain is sketched in Figure 13.8. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line \( y = x \). We conclude the domain is an open set. The set is unbounded.

Limits
Recall a pseudo–definition of the limit of a function of one variable: “\( \lim_{x \to c} f(x) = L \)” means that if \( x \) is “really close” to \( c \), then \( f(x) \) is “really close” to \( L \). A similar pseudo–definition holds for functions of two variables. We’ll say that

\[
\text{“} \lim_{(x,y) \to (x_0, y_0)} f(x, y) = L \text{”}
\]

means “if the point \((x, y)\) is really close to the point \((x_0, y_0)\), then \( f(x, y) \) is really close to \( L \).” The formal definition is given below.
13.2 Limits and Continuity of Multivariable Functions

**Definition 83 Limit of a Function of Two Variables**

Let $S$ be an open set containing $(x_0, y_0)$, and let $f$ be a function of two variables defined on $S$, except possibly at $(x_0, y_0)$. The limit of $f(x, y)$ as $(x, y)$ approaches $(x_0, y_0)$ is $L$, denoted

\[ \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L, \]

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \neq (x_0, y_0)$, if $(x, y)$ is in the open disk centered at $(x_0, y_0)$ with radius $\delta$, then $|f(x, y) - L| < \varepsilon$.

The concept behind Definition 83 is sketched in Figure 13.9. Given $\varepsilon > 0$, find $\delta > 0$ such that if $(x, y)$ is any point in the open disk centered at $(x_0, y_0)$ in the $x$-$y$ plane with radius $\delta$, then $f(x, y)$ should be within $\varepsilon$ of $L$.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

---

Notes:
Chapter 13  Functions of Several Variables

Theorem 103  Basic Limit Properties of Functions of Two Variables
Let \( b, x_0, y_0, L \) and \( K \) be real numbers, let \( n \) be a positive integer, and let \( f \) and \( g \) be functions with the following limits:

\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} g(x,y) = K.
\]

The following limits hold.

1. Constants: \( \lim_{(x,y) \to (x_0,y_0)} b = b \)
2. Identity \( \lim_{(x,y) \to (x_0,y_0)} x = x_0; \quad \lim_{(x,y) \to (x_0,y_0)} y = y_0 \)
3. Sums/Differences: \( \lim_{(x,y) \to (x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm K \)
4. Scalar Multiples: \( \lim_{(x,y) \to (x_0,y_0)} b \cdot f(x,y) = bL \)
5. Products: \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \cdot g(x,y) = LK \)
6. Quotients: \( \lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{K}, \quad (K \neq 0) \)
7. Powers: \( \lim_{(x,y) \to (x_0,y_0)} f(x,y)^n = L^n \)
8. Roots: \( \lim_{(x,y) \to (x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \quad (\text{when } n \text{ is odd or } L \geq 0) \)

This theorem can be proved by the same arguments as the analogous results for functions of one variable in Theorem 1. Combined with Theorems 3 and 4 of Section 1.3, this allows us to evaluate many limits.

Example 3  Evaluating a limit
Evaluate the following limits:

1. \( \lim_{(x,y) \to (1,\pi)} \frac{y}{x} + \cos(xy) \)
2. \( \lim_{(x,y) \to (0,0)} \frac{3xy}{x^2 + y^2} \)

**Solution**

1. The aforementioned theorems allow us to simply evaluate \( \frac{y}{x} + \cos(xy) \) when \( x = 1 \) and \( y = \pi \). If an indeterminate form is returned, we must do

Notes:

776
13.2 Limits and Continuity of Multivariable Functions

more work to evaluate the limit; otherwise, the result is the limit. Therefore

\[
\lim_{(x,y) \to (1,\pi)} \frac{y}{x} + \cos(xy) = \frac{\pi}{1} + \cos \pi = \pi - 1.
\]

2. We attempt to evaluate the limit by substituting 0 in for \( x \) and \( y \), but the result is the indeterminate form “0/0.” To evaluate this limit, we must “do more work,” but we have not yet learned what “kind” of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one-sided limits and stated

\[
\lim_{x \to c} f(x) = L \quad \text{if and only if both} \quad \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.
\]

That is, the limit is \( L \) if and only if \( f(x) \) approaches \( L \) when \( x \) approaches \( c \) from either direction, the left or the right.

In the plane, there are infinite directions from which \( (x, y) \) might approach \( (x_0, y_0) \). In fact, we do not have to restrict ourselves to approaching \( (x_0, y_0) \) from a particular direction, but rather we can approach that point along any possible path. It is possible to arrive at different limiting values by approaching \( (x_0, y_0) \) along different paths. If this happens, we say that \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \) does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Watch the video:
Showing a Limit Does Not Exist at
https://youtu.be/q9xF93ql8

Notes:
Example 4  

Showing limits do not exist

1. Show \( \lim_{(x,y) \to (0,0)} \frac{3xy}{x^2 + y^2} \) does not exist by finding the limits along the lines \( y = mx \).

2. Show \( \lim_{(x,y) \to (0,0)} \frac{\sin(xy)}{x+y} \) does not exist by finding the limit along the path \( y = -\sin x \).

SOLUTION

1. Evaluating \( \lim_{(x,y) \to (0,0)} \frac{3xy}{x^2 + y^2} \) along the lines \( y = mx \) means replace all \( y \)'s with \( mx \) and evaluating the resulting limit:

\[
\lim_{(x,mx) \to (0,0)} \frac{3x(mx)}{x^2 + (mx)^2} = \lim_{x \to 0} \frac{3mx^2}{x^2(m^2 + 1)} = \lim_{x \to 0} \frac{3m}{m^2 + 1} = \frac{3m}{m^2 + 1}.
\]

While the limit exists for each choice of \( m \), we get a different limit for each choice of \( m \). That is, along different lines we get differing limiting values, meaning the limit does not exist.

2. Let \( f(x, y) = \frac{\sin(xy)}{x+y} \). We are to show that \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist by finding the limit along the path \( y = -\sin x \). First, however, consider the limits found along the lines \( y = mx \) as done above.

\[
\lim_{(x,mx) \to (0,0)} \frac{\sin(x(mx))}{x + mx} = \lim_{x \to 0} \frac{\sin(mx^2)}{x(m + 1)} = \lim_{x \to 0} \frac{\sin(mx^2)}{x} \cdot \frac{1}{m + 1}.
\]

By applying L'Hôpital's Rule, we can show this limit is 0 except when \( m = -1 \), that is, along the line \( y = -x \). This line is not in the domain of \( f \), so we have found the following fact: along every line \( y = mx \) in the domain of \( f \), \( \lim_{(x,y) \to (0,0)} f(x, y) = 0 \).

Now consider the limit along the path \( y = -\sin x \):

\[
\lim_{(x,-\sin x) \to (0,0)} \frac{\sin(-x \sin x)}{x - \sin x} = \lim_{x \to 0} \frac{\sin(-x \sin x)}{x - \sin x}.
\]
Now apply L'Hôpital's Rule twice:
\[
\lim_{x \to 0} \frac{\cos(-x \sin x) (-\sin x - x \cos x)}{1 - \cos x} = 0
\]
\[
= \lim_{x \to 0} \frac{-\sin(-x \sin x) (-\sin x - x \cos x)^2 + \cos(-x \sin x) (-2 \cos x + x \sin x)}{\sin x}
\]
\[
= -2/0 = \text{the limit does not exist.}
\]

Step back and consider what we have just discovered. Along any line \( y = mx \) in the domain of the function \( f(x, y) \), the limit is 0. However, along the path \( y = -\sin x \), which lies in the domain of \( f(x, y) \) for all \( x \neq 0 \), the limit does not exist. Since the limit is not the same along every path to \( (0, 0) \), we say the limit does not exist.

**Example 5 Finding a limit**

Let \( f(x, y) = \frac{5x^2 y^2}{x^2 + y^2} \). Find \( \lim_{(x,y) \to (0,0)} f(x, y) \).

**SOLUTION**

It is relatively easy to show that along any line \( y = mx \), the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 83. Let \( \varepsilon > 0 \) be given. We want to find \( \delta > 0 \) such that if \( \sqrt{(x - 0)^2 + (y - 0)^2} < \delta \), then \( |f(x, y) - 0| < \varepsilon \).

Set \( \delta < \sqrt{\varepsilon/5} \). Note that \( \left| \frac{5y^2}{x^2 + y^2} \right| < 5 \) for all \( (x, y) \neq (0, 0) \), and that if \( \sqrt{x^2 + y^2} < \delta \), then \( x^2 < \delta^2 \).

Let \( \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} < \delta \). Consider \( |f(x, y) - 0| \):

\[
|f(x, y) - 0| = \left| \frac{5x^2 y^2}{x^2 + y^2} - 0 \right| = \frac{5x^2 y^2}{x^2 + y^2} \leq \delta^2 \cdot 5 \\
= \frac{\varepsilon}{5} \cdot 5 = \varepsilon.
\]

Thus if \( \sqrt{(x - 0)^2 + (y - 0)^2} < \delta \) then \( |f(x, y) - 0| < \varepsilon \), which is what we wanted to show. Thus \( \lim_{(x,y) \to (0,0)} \frac{5x^2 y^2}{x^2 + y^2} = 0 \).

Notes:
Chapter 13  Functions of Several Variables

Continuity

Definition 7 defines what it means for a function of one variable to be continuous. In brief, it meant that the function always equaled its limit. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

**Definition 84  Continuous**

Let a function \( f(x, y) \) be defined on an open disk \( B \) containing the point \((x_0, y_0)\).

1. \( f \) is **continuous** at \((x_0, y_0)\) if \( \lim_{(x,y) \to (x_0,y_0)} f(x, y) = f(x_0, y_0) \).
2. \( f \) is **continuous on an open set** \( S \) if \( f \) is continuous at each point in \( S \). (We say that \( f \) is **continuous everywhere** if \( f \) is continuous on \( \mathbb{R}^2 \).)

**Example 6  Continuity of a function of two variables**

Let \( f(x, y) = \begin{cases} \cos y \sin x & x \neq 0 \\ \frac{\cos y}{x} & x = 0 \end{cases} \). Is \( f \) continuous at \((0, 0)\)? Is \( f \) continuous everywhere?

**Solution**

To determine if \( f \) is continuous at \((0, 0)\), we need to compare \( \lim_{(x,y) \to (0,0)} f(x, y) \) to \( f(0, 0) \).

Applying the definition of \( f \), we see that \( f(0, 0) = \cos 0 = 1 \).

We now consider the limit \( \lim_{(x,y) \to (0,0)} f(x, y) \). Substituting \( 0 \) for \( x \) and \( y \) in \((\cos y \sin x)/x \) returns the indeterminate form “0/0”, so we need to do more work to evaluate this limit.

Consider two related limits: \( \lim_{(x,y) \to (0,0)} \cos y \) and \( \lim_{(x,y) \to (0,0)} \frac{\sin x}{x} \). The first limit does not contain \( x \), and since \( \cos y \) is continuous,

\[ \lim_{(x,y) \to (0,0)} \cos y = \lim_{y \to 0} \cos y = \cos 0 = 1. \]

The second limit does not contain \( y \). By Theorem 6 we can say

\[ \lim_{(x,y) \to (0,0)} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

**Notes:**
Finally, Theorem 103 of this section states that we can combine these two limits as follows:

\[
\lim_{(x,y) \to (0,0)} \frac{\cos y \sin x}{x} = \lim_{(x,y) \to (0,0)} (\cos y) \left( \frac{\sin x}{x} \right)
\]

\[
= \left( \lim_{(x,y) \to (0,0)} \cos y \right) \left( \lim_{(x,y) \to (0,0)} \frac{\sin x}{x} \right)
\]

\[
= (1)(1) = 1.
\]

We have found that \( \lim_{(x,y) \to (0,0)} \frac{\cos y \sin x}{x} = f(0,0) \), so \( f \) is continuous at \((0,0)\).

A similar analysis shows that \( f \) is continuous at all points in \( \mathbb{R}^2 \). As long as \( x \neq 0 \), we can evaluate the limit directly; when \( x = 0 \), a similar analysis shows that the limit is \( \cos y \). Thus we can say that \( f \) is continuous everywhere. A graph of \( f \) is given in Figure 13.10. Notice how it has no breaks, jumps, etc.

The following theorems are very similar to Theorems 10 and 11, giving us ways to combine continuous functions to create other continuous functions.

---

**Theorem 104  Properties of Continuous Functions**

Let \( f \) and \( g \) be continuous on an open set \( S \), let \( c \) be a real number, and let \( n \) be a positive integer. The following functions are continuous on \( S \).

1. Sums/Differences: \( f \pm g \)
2. Constant Multiples: \( c \cdot f \)
3. Products: \( f \cdot g \)
4. Quotients: \( f/g \) (as longs as \( g \neq 0 \) on \( B \))
5. Powers: \( f^n \)
6. Roots: \( \sqrt{f} \) (if \( f \geq 0 \) on \( B \) or \( n \) is odd)

Notes:
**Theorem 105  Continuity of Compositions**

Let \( f \) be continuous on \( S \), where the range of \( f \) on \( S \) is \( J \), and let \( g \) be a single variable function that is continuous on \( J \). Then

\[
(g \circ f)(x, y) = g(f(x, y)),
\]

is continuous on \( S \).

**Example 7  Establishing continuity of a function**

Let \( f(x, y) = \sin(x^2 \cos y) \). Show \( f \) is continuous everywhere.

**SOLUTION**

We will apply Theorems 10, 104, and 105. Let \( f_1(x, y) = x^2 \). Since \( y \) is not actually used in the function, and polynomials are continuous (by Theorem 10), we conclude \( f_1 \) is continuous everywhere. A similar statement can be made about \( f_2(x, y) = \cos y \). Part 3 of Theorem 104 states that \( f_3 = f_1 \cdot f_2 \) is continuous everywhere, and Theorem 105 states the composition of sine with \( f_3 \) is continuous: that is, \( \sin(f_3) = \sin(x^2 \cos y) \) is continuous everywhere.

**Functions of Three Variables**

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 82 and 84 are not redefined but their analogous meanings should be clear to the reader.
13.2 Limits and Continuity of Multivariable Functions

Definition 85  Open Balls, Limit, Continuous

1. An open ball in \( \mathbb{R}^3 \) centered at \((x_0, y_0, z_0)\) with radius \( r \) is the set of all points \((x, y, z)\) such that \( \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r \).

2. Let \( D \) be an open set in \( \mathbb{R}^3 \) containing \((x_0, y_0, z_0)\), and let \( f(x, y, z) \) be a function of three variables defined on \( D \), except possibly at \((x_0, y_0, z_0)\). The limit of \( f(x, y, z) \) as \((x, y, z)\) approaches \((x_0, y_0, z_0)\) is \( L \), denoted

\[
\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x, y, z) = L,
\]

means that given any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that for all \((x, y, z) \neq (x_0, y_0, z_0)\), if \((x, y, z)\) is in the open ball centered at \((x_0, y_0, z_0)\) with radius \( \delta \), then \(|f(x, y, z) - L| < \varepsilon\).

3. Let \( f(x, y, z) \) be defined on an open ball \( B \) containing \((x_0, y_0, z_0)\). Then \( f \) is continuous at \((x_0, y_0, z_0)\) if

\[
\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0).
\]

These definitions can also be extended naturally to apply to functions of four or more variables. Theorems 104 and 105 also applies to function of three or more variables, allowing us to say that the function

\[
f(x, y, z) = \frac{e^{x^2 + y} \sqrt{y^2 + z^2 + 3}}{\sin(xyz) + 5}
\]

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivariable context.

Notes:
Exercises 13.2

Terms and Concepts

1. Describe in your own words the difference between boundary and interior point of a set.
2. Use your own words to describe (informally) what \( \lim_{(x,y) \to (1,2)} f(x,y) = 17 \) means.
3. Give an example of a closed, bounded set.
4. Give an example of a closed, unbounded set.
5. Give an example of an open, bounded set.
6. Give an example of an open, unbounded set.

Problems

In Exercises 7–10, a set \( S \) is given.

(a) Give one boundary point and one interior point, when possible, of \( S \).
(b) State whether \( S \) is open, closed, or neither.
(c) State whether \( S \) is bounded or unbounded.

7. \( S = \left\{ (x,y) \mid \frac{(x-1)^2}{4} + \frac{(y-3)^2}{9} \leq 1 \right\} \)
8. \( S = \left\{ (x,y) \mid y \neq x^2 \right\} \)
9. \( S = \left\{ (x,y) \mid x^2 + y^2 = 1 \right\} \)
10. \( S = \left\{ (x,y) \mid y > \sin x \right\} \)

In Exercises 11–14:

(a) Find the domain \( D \) of the given function.
(b) State whether \( D \) is an open or closed set.
(c) State whether \( D \) is bounded or unbounded.

11. \( f(x,y) = \sqrt{9 - x^2 - y^2} \)
12. \( f(x,y) = \sqrt{y - x^2} \)
13. \( f(x,y) = \frac{1}{\sqrt{y - x^2}} \)
14. \( f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \)

In Exercises 15–20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \)
(a) Along the path \( y = 0 \).
(b) Along the path \( x = 0 \).
16. \( \lim_{(x,y) \to (0,0)} \frac{x + y}{x - y} \)
(a) Along the path \( y = mx \).
(b) Along the path \( x = 0 \).
17. \( \lim_{(x,y) \to (0,0)} \frac{xy - y^2}{y^2 + x} \)
(a) Along the path \( y = mx \).
(b) Along the path \( x = 0 \).
18. \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2)}{y} \)
(a) Along the path \( y = mx \).
(b) Along the path \( y = x^2 \).
19. \( \lim_{(x,y) \to (1,2)} \frac{x + y - 3}{x^2 - 1} \)
(a) Along the path \( y = 2 \).
(b) Along the path \( y = x + 1 \).
20. \( \lim_{(x,y) \to (\pi, \pi/2)} \frac{\sin x}{\cos y} \)
(a) Along the path \( x = \pi \).
(b) Along the path \( y = x - \pi/2 \).
13.3 Partial Derivatives

Let $y$ be a function of $x$. We have studied in great detail the derivative of $y$ with respect to $x$, that is, $\frac{dy}{dx}$, which measures the rate at which $y$ changes with respect to $x$. Consider now $z = f(x, y)$. It makes sense to want to know how $z$ changes with respect to $x$ and/or $y$. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 13.11(a). By fixing $y = 2$, we focus our attention to all points on the surface where the $y$-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of $z$ with respect to $x$ along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating $y$ as constant (it does not vary) we can consider how $z$ changes with respect to $x$. In a similar fashion, we can hold $x$ constant and consider how $z$ changes with respect to $y$. This is the underlying principle of partial derivatives. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.

**Definition 86 Partial Derivative**
Let $z = f(x, y)$ be a continuous function on an open set $S$ in $\mathbb{R}^2$.

1. The partial derivative of $f$ with respect to $x$ is:
   \[
   f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.
   \]

2. The partial derivative of $f$ with respect to $y$ is:
   \[
   f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.
   \]

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and} \quad z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated $f_x$. 

**Notes:**

Watch the video:
Partial Derivatives at https://youtu.be/SbfRDBmyAMI

Figure 13.11: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.
Chapter 13  Functions of Several Variables

Example 1  Computing partial derivatives with the limit definition
Let \( f(x, y) = x^2 y + 2x + y^3 \). Find \( f_x(x, y) \) using the limit definition.

**SOLUTION**

Using Definition 86, we have:

\[
\begin{align*}
  f_x(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
  &= \lim_{h \to 0} \frac{(x+h)^2 y + 2(x+h) + y^3 - (x^2 y + 2x + y^3)}{h} \\
  &= \lim_{h \to 0} \frac{x^2 y + 2xh + y^3 + h(2x + y^3) - (x^2 y + 2x + y^3)}{h} \\
  &= \lim_{h \to 0} \frac{2xh}{h} \\
  &= \lim_{h \to 0} 2xy + hy + 2 \\
  &= 2xy + 2.
\end{align*}
\]

We have found \( f_x(x, y) = 2xy + 2 \).

Example 1 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing \( f_x(x, y) \), we hold \( y \) fixed – it does not vary. Therefore we can compute the derivative with respect to \( x \) by treating \( y \) as a constant or coefficient.

Just as \( \frac{d}{dx} (5x^2) = 10x \), we compute \( \frac{\partial}{\partial x} (x^2 y) = 2xy \). Here we are treating \( y \) as a coefficient. Just as \( \frac{d}{dx} (5^3) = 0 \), we compute \( \frac{\partial}{\partial x} (y^3) = 0 \). Here we are treating \( y \) as a constant. More examples will help make this clear.

Example 2  Finding partial derivatives
Find \( f_x(x, y) \) and \( f_y(x, y) \) in each of the following.

1. \( f(x, y) = x^3 y^2 + 5y^2 - x + 7 \)
2. \( f(x, y) = \cos(xy^2) + \sin x \)
3. \( f(x, y) = e^{x^2} y \sqrt{x^2 + 1} \)

**SOLUTION**

1. We have \( f(x, y) = x^3 y^2 + 5y^2 - x + 7 \).
   Begin with \( f_x(x, y) \). Keep \( y \) fixed, treating it as a constant or coefficient, as appropriate:
   \[ f_x(x, y) = 3x^2 y^2 - 1. \]
Note how the $5y^2$ and 7 terms go to zero.

To compute $f_y(x, y)$, we hold $x$ fixed:

$$f_y(x, y) = 2x^3y + 10y.$$  

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x, y) = \cos(xy^2) + \sin x$.

Begin with $f_x(x, y)$. We need to apply the Chain Rule with the cosine term; $y^2$ is the coefficient of the $x$-term inside the cosine function.

$$f_x(x, y) = -\sin(xy^2)(y^2) + \cos x = -y^2 \sin(xy^2) + \cos x.$$  

To find $f_y(x, y)$, note that $x$ is the coefficient of the $y^2$ term inside of the cosine term; also note that since $x$ is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy \sin(xy^2).$$  

3. We have $f(x, y) = e^{x^2y^3} \sqrt{x^2 + 1}$.

Beginning with $f_x(x, y)$, note how we need to apply the Product Rule.

$$f_x(x, y) = e^{x^2y^3}(2xy^3) \sqrt{x^2 + 1} + e^{x^2y^3} \frac{1}{2}(x^2 + 1)^{-1/2}(2x)$$

$$= 2xy^3 e^{x^2y^3} \sqrt{x^2 + 1} + \frac{xe^{x^2y^3}}{\sqrt{x^2 + 1}}.$$  

Note that when finding $f_y(x, y)$ we do not have to apply the Product Rule; since $\sqrt{x^2 + 1}$ does not contain $y$, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x, y) = e^{x^2y^3}(3x^2y^2) \sqrt{x^2 + 1} = 3x^2y^2 e^{x^2y^3} \sqrt{x^2 + 1}.$$  

We have shown how to compute a partial derivative, but it may still not be clear what a partial derivative means. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which $z$ changes as only $x$ varies: $y$ is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring $z_x$: you are moving only east (in the “$x$”-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the “$y$”-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: $z$ does not change with respect to $y$. We can see that $z_x$ and $z_y$ do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

The following example helps us visualize this more.

Notes:
Example 3  Evaluating partial derivatives
Let \( z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10 \). Find \( f_x(2, 1) \) and \( f_y(2, 1) \) and interpret their meaning.

**SOLUTION** We begin by computing \( f_x(x, y) = -2x + y \) and \( f_y(x, y) = -y + x \). Thus

\[
\begin{align*}
    f_x(2, 1) &= -3 \\
    f_y(2, 1) &= 1
\end{align*}
\]

It is also useful to note that \( f(2, 1) = 7.5 \). What does each of these numbers mean?

Consider \( f_x(2, 1) = -3 \), along with Figure 13.12(a). If one “stands” on the surface at the point \((2, 1, 7.5)\) and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is \(-3\). Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider \( f_y(2, 1) = 1 \), illustrated in Figure 13.12(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of \(1\). Increasing the y-value by \(1\) would increase the z-value by approximately \(1\).

Since the magnitude of \( f_x \) is greater than the magnitude of \( f_y \) at \((2, 1)\), it is “steeper” in the x-direction than in the y-direction.

**Second Partial Derivatives**

Let \( z = f(x, y) \). We have learned to find the partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \), which are each functions of \( x \) and \( y \). Therefore we can take partial derivatives of these, each with respect to \( x \) and \( y \). We define these “second partials” along with the notation, give examples, then discuss their meaning.
13.3 Partial Derivatives

Definition 87 Second Partial Derivative, Mixed Partial Derivative
Let \( z = f(x, y) \) be continuous on an open set \( S \).

1. The second partial derivative of \( f \) with respect to \( x \) then \( x \) is
   \[
   \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}
   \]

2. The second partial derivative of \( f \) with respect to \( x \) then \( y \) is
   \[
   \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}
   \]
   Similar definitions hold for \( \frac{\partial^2 f}{\partial y^2} = f_{yy} \) and \( \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \).

The second partial derivatives \( f_{xy} \) and \( f_{yx} \) are mixed partial derivatives.

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If \( y = f(x) \), then \( f''(x) = \frac{d^2y}{dx^2} \). The “\( d^2y \)” portion means “take the derivative of \( y \) twice,” while “\( dx^2 \)” means “with respect to \( x \) both times.” When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 4 Second partial derivatives
For each of the following, find all six first and second partial derivatives. That is, find

\[ f_x, \ f_y, \ f_{xx}, \ f_{yy}, \ f_{xy} \text{ and } f_{yx}. \]

1. \( f(x, y) = x^3y^2 + 2xy^3 + \cos x \)
2. \( f(x, y) = \frac{x^3}{y^2} \)

**Solution** In each, we give \( f_x \) and \( f_y \) immediately and then spend time deriving the second partial derivatives.

Note: The terms in Definition 87 all depend on limits, so each definition comes with the caveat “where the limit exists.”

The way to keep track of the order is to start with the variable nearest to the function. Unfortunately, this means that while \( \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \) both mean “differentiate with respect to \( x \) first”, the \( x \) and \( y \) appear in a different order. Fortunately Theorem 106 will soon tell us that the order doesn’t usually matter.
Chapter 13  Functions of Several Variables

1. \[ f(x, y) = x^3 y^2 + 2xy^3 + \cos x \]
   \[ f_x(x, y) = 3x^2 y^2 + 2y^3 - \sin x \]
   \[ f_y(x, y) = 2x^3 y + 6xy^2 \]

   \[ f_{xx}(x, y) = \frac{\partial}{\partial x} (f_x) = \frac{\partial}{\partial x} (3x^2 y^2 + 2y^3 - \sin x) = 6xy^2 - \cos x \]
   \[ f_{xy}(x, y) = \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} (3x^2 y^2 + 2y^3) = 2x^3 + 12xy \]
   \[ f_{yx}(x, y) = \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} (2x^3 y + 6xy^2) = 6x^2 y + 6y^2 \]

2. \[ f(x, y) = \frac{x^3}{y^2} = x^3 y^{-2} \]
   \[ f_x(x, y) = \frac{3x^2}{y^2} \]
   \[ f_y(x, y) = -\frac{2x^3}{y^3} \]

   \[ f_{xx}(x, y) = \frac{\partial}{\partial x} (f_x) = \frac{\partial}{\partial x} \left( \frac{3x^2}{y^2} \right) = \frac{6x}{y^2} \]
   \[ f_{yy}(x, y) = \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} \left( -\frac{2x^3}{y^3} \right) = \frac{6x^2}{y^4} \]
   \[ f_{xy}(x, y) = \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} \left( \frac{3x^2}{y^2} \right) = -\frac{6x^2}{y^3} \]
   \[ f_{yx}(x, y) = \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} \left( -\frac{2x^3}{y^3} \right) = -\frac{6x^2}{y^3} \]

Notice how in both of the functions in Example 4, \( f_{xy} = f_{yx} \). Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

**Theorem 106  Mixed Partial Derivatives**

Let \( f \) be defined such that \( f_{xy} \) and \( f_{yx} \) are continuous on an open set \( S \). Then for each point \( (x, y) \) in \( S \), \( f_{xy}(x, y) = f_{yx}(x, y) \).

Finding \( f_{xy} \) and \( f_{yx} \) independently and comparing the results provides a convenient way of checking our work.

Notes:
Understanding Second Partial Derivatives

Now that we know how to find second partials, we investigate what they tell us.

Again we refer back to a function \( y = f(x) \) of a single variable. The second derivative of \( f \) is “the derivative of the derivative,” or “the rate of change of the rate of change.” The second derivative measures how much the derivative is changing. If \( f''(x) < 0 \), then the derivative is getting smaller (so the graph of \( f \) is concave down); if \( f''(x) > 0 \), then the derivative is growing, making the graph of \( f \) concave up.

Now consider \( z = f(x, y) \). Similar statements can be made about \( f_{xx} \) and \( f_{yy} \) as could be made about \( f''(x) \) above. When taking derivatives with respect to \( x \) twice, we measure how much \( f_x \) changes with respect to \( x \). If \( f_{xx}(x, y) < 0 \), it means that as \( x \) increases, \( f_x \) decreases, and the graph of \( f \) will be concave down in the \( x \)-direction. Using the analogy of standing in the rolling meadow used earlier in this section, \( f_{xx} \) measures whether one’s path is concave up/down when walking due east.

Similarly, \( f_{yy} \) measures the concavity in the \( y \)-direction. If \( f_{yy}(x, y) > 0 \), then \( f_y \) is increasing with respect to \( y \) and the graph of \( f \) will be concave up in the \( y \)-direction. Appealing to the rolling meadow analogy again, \( f_{yy} \) measures whether one’s path is concave up/down when walking due north.

We now consider the mixed partials \( f_{xy} \) and \( f_{yx} \). The mixed partial \( f_{xy} \) measures how much \( f_x \) changes with respect to \( y \). Once again using the rolling meadow analogy, \( f_x \) measures the slope if one walks due east. Looking east, begin walking north (side–stepping). Is the path towards the east getting steeper? If so, \( f_{xy} > 0 \). Is the path towards the east not changing in steepness? If so, then \( f_{xy} = 0 \). A similar thing can be said about \( f_{yx} \): consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and graphs.

**Example 5**  
**Understanding second partial derivatives**

Let \( z = x^2 - y^2 + xy \). Evaluate the 6 first and second partial derivatives at \((-1/2, 1/2)\) and interpret what each of these numbers mean.

**Solution**

We find that:

\[
\begin{align*}
    f_x(x, y) &= 2x + y, \\
    f_y(x, y) &= -2y + x, \\
    f_{xx}(x, y) &= 2, \\
    f_{yy}(x, y) &= -2 \\
    f_{xy}(x, y) &= f_{yx}(x, y) = 1. 
\end{align*}
\]

Thus at \((-1/2, 1/2)\) we have

\[
\begin{align*}
    f_x(-1/2, 1/2) &= -1/2, \\
    f_y(-1/2, 1/2) &= -3/2. 
\end{align*}
\]

The slope of the tangent line at \((-1/2, 1/2, -1/4)\) in the direction of \( x \) is \(-1/2\); if one moves from that point parallel to the \( x \)-axis, the instantaneous rate of change will be \(-1/2\). The slope of the tangent line at this point in the direction...
of \( y \) is \(-3/2\): if one moves from this point parallel to the \( y \)-axis, the instantaneous rate of change will be \(-3/2\). These tangents lines are graphed in Figure 13.13(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 13.13(a). Three directed tangent lines are drawn (two are dashed), each in the direction of \( x \); that is, each has a slope determined by \( f_x \). Note how as \( y \) increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the slopes are increasing. The slopes given by \( f_x \) are increasing as \( y \) increases, meaning \( f_{xy} \) must be positive.

Since \( f_{xy} = f_{yx} \), we also expect \( f_y \) to increase as \( x \) increases. Consider Figure 13.13(b) where again three directed tangent lines are drawn, this time each in the direction of \( y \) with slopes determined by \( f_y \). As \( x \) increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of \( f_x, f_y, \) and \( f_{xy} = f_{yx} \). We now interpret \( f_{xx} \) and \( f_{yy} \). In Figure 13.13(a), we see a curve drawn where \( x \) is held constant at \( x = -1/2 \): only \( y \) varies. This curve is clearly concave down, corresponding to the fact that \( f_{yy} < 0 \). In part (b) of the figure, we see a similar curve where \( y \) is constant and only \( x \) varies. This curve is concave up, corresponding to the fact that \( f_{xx} > 0 \).

**Partial Derivatives and Functions of Three Variables**

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

**Definition 88 Partial Derivatives with Three Variables**

Let \( w = f(x, y, z) \) be a continuous function on an open set \( S \) in \( \mathbb{R}^3 \).

The **partial derivative of \( f \) with respect to \( x \)** is:

\[
f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.
\]

Similar definitions hold for \( f_y(x, y, z) \) and \( f_z(x, y, z) \).

By taking partial derivatives of partial derivatives, we can find second partial derivatives of \( f \) with respect to \( z \) then \( y \), for instance, just as before.

**Notes:**
Example 6  Partial derivatives of functions of three variables
For each of the following, find \( f_x, f_y, f_z, f_{xz}, f_{yz}, \) and \( f_{zz}. \)

1. \( f(x, y, z) = x^2 y^3 z^4 + x^3 y^2 + x^2 z^3 + y^4 z^4 \)

2. \( f(x, y, z) = x \sin(yz) \)

SOLUTION

1. \begin{align*}
    f_x &= 2xy^3z^4 + 2x^2 + 3x^2z^3 \\
    f_y &= 3x^2y^2z^4 + 2x^2y + 4y^3z^4 \\
    f_z &= 4x^2y^3z^2 + 3x^3z + 4y^4z^3 \\
    f_{xz} &= 8x^2y^3z^3 + 9x^2z^2 \\
    f_{yz} &= 12x^2y^3z^3 + 16y^3z^3 \\
    f_{zz} &= 12x^2y^3z^3 + 6x^3z + 12y^4z^2
\end{align*}

2. \begin{align*}
    f_x &= \sin(yz) \\
    f_y &= xz \cos(yz) \\
    f_z &= xy \cos(yz) \\
    f_{xz} &= y \cos(yz) \\
    f_{yz} &= x \cos(yz) - xyz \sin(yz) \\
    f_{zz} &= -xy^2 \sin(xy)
\end{align*}

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

\[
    f_{xyz}(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right)
\]
and

\[
    f_{xyz}(x, y, z) = \frac{\partial}{\partial z} \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right).
\]

Example 7  Higher order partial derivatives

1. Let \( f(x, y) = x^2 y^2 + \sin(xy). \) Find \( f_{xxy} \) and \( f_{xxx}. \)

2. Let \( f(x, y, z) = x^3 e^y + \cos(z). \) Find \( f_{xyz}. \)
Chapter 13  Functions of Several Variables

SOLUTION
1. To find \( f_{xxy} \), we first find \( f_x \), then \( f_{xx} \), then \( f_{xxy} \):

\[
f_x = 2xy^2 + y \cos(xy) \quad f_{xx} = 2y^2 - y^2 \sin(xy) \quad f_{xxy} = 4y - 2y \sin(xy) - xy^2 \cos(xy).
\]

To find \( f_{yxx} \), we first find \( f_y \), then \( f_{yx} \), then \( f_{yxx} \):

\[
f_y = 2x^2 y + x \cos(xy) \quad f_{yx} = 4xy + \cos(xy) - xy \sin(xy) \quad f_{yxx} = 4y - y \sin(xy) - (y \sin(xy) + xy^2 \cos(xy)) \quad f_{yxx} = 4y - 2y \sin(xy) - xy^2 \cos(xy).
\]

Note how \( f_{xxy} = f_{yxx} \).

2. To find \( f_{xyz} \), we find \( f_x \), then \( f_{xy} \), then \( f_{xyz} \):

\[
f_x = 3x^2 e^{xy} + x^3 ye^{xy} \quad f_{xy} = 3x^2 e^{xy} + x^3 e^{xy} + x^4 ye^{xy} = 4x^2 e^{xy} + x^4 ye^{xy} \quad f_{xyz} = 0.
\]

In the previous example we saw that \( f_{xxy} = f_{yxx} \); this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, \( f_{xxy} = f_{yxx} = f_{xxy} \).

This can be useful at times. Had we known this, the second part of Example 7 would have been much simpler to compute. Instead of computing \( f_{xyz} \) in the \( x \), \( y \) then \( z \) orders, we could have applied the \( z \), then \( x \) then \( y \) order (as \( f_{xyz} = f_{zyx} \)). It is easy to see that \( f_z = -\sin z \); then \( f_{zx} \) and \( f_{zxy} \) are clearly 0 as \( f_z \) does not contain an \( x \) or \( y \).

We have seen that partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With \( z = f(x, y) \), the partial derivatives \( f_x \) and \( f_y \) measure the instantaneous rate of change of \( z \) when moving parallel to the \( x \)- and \( y \)-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector \( \langle 2, 1 \rangle \)? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 13.6. First, we need to define what it means for a function of two variables to be differentiable.

Notes:
Exercises 13.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?
2. Given a function \( z = f(x, y) \), explain in your own words how to compute \( f_x \).
3. In the expression \( f_{xy} \), which is computed first, \( f_x \) or \( f_y \)?
4. In the expression \( \frac{\partial^2 f}{\partial x \partial y} \), which is computed first, \( f_x \) or \( f_y \)?

Problems

In Exercises 5–8, evaluate \( f_x(x, y) \) and \( f_y(x, y) \) at the indicated point.
5. \( f(x, y) = x^2y - x + 2y + 3 \) at \( (1, 2) \)
6. \( f(x, y) = x^3 - 3x + y^2 - 6y \) at \( (-1, 3) \)
7. \( f(x, y) = \sin y \cos x \) at \( (\pi/3, \pi/3) \)
8. \( f(x, y) = \ln(xy) \) at \( (-2, -3) \)

In Exercises 9–26, find \( f_x, f_y, f_{xx}, f_{yy}, f_{xy} \) and \( f_{yx} \).
9. \( f(x, y) = x^2y + 3x^2 + 4y - 5 \)
10. \( f(x, y) = y^3 + 3xy^2 + 3y^2 + x^3 \)
11. \( f(x, y) = \frac{x}{y} \)
12. \( f(x, y) = \frac{4}{xy} \)
13. \( f(x, y) = e^{x^2+y^2} \)
14. \( f(x, y) = e^{x+2y} \)
15. \( f(x, y) = \sin x \cos y \)
16. \( f(x, y) = (x + y)^3 \)
17. \( f(x, y) = \cos(5xy^2) \)
18. \( f(x, y) = \sin(5x^2 + 2y^3) \)
19. \( f(x, y) = \sqrt{4xy^2 + 1} \)
20. \( f(x, y) = (2x + 5y)\sqrt{y} \)
21. \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \)
22. \( f(x, y) = 5x - 17y \)
23. \( f(x, y) = 3x^2 + 1 \)
24. \( f(x, y) = \ln(x^2 + y) \)
25. \( f(x, y) = \frac{\ln x}{4y} \)
26. \( f(x, y) = 5e^y \sin y + 9 \)

In Exercises 27–30, form a function \( z = f(x, y) \) such that \( f_x \) and \( f_y \) match those given.
27. \( f_x = \sin y + 1, \quad f_y = x \cos y \)
28. \( f_x = x + y, \quad f_y = x + y \)
29. \( f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2 \)
30. \( f_x = \frac{2x}{x^2 + y^2}, \quad f_y = \frac{2y}{x^2 + y^2} \)

In Exercises 31–34, find \( f_x, f_y, f_z, f_{xz} \) and \( f_{yz} \).
31. \( f(x, y, z) = x^2 e^{2y - 3z} \)
32. \( f(x, y, z) = x^3y^2 + x^3z + y^3 \)
33. \( f(x, y, z) = \frac{3x}{7y^2 z} \)
34. \( f(x, y, z) = \ln(xyz) \)
13.4 Differentiability and the Total Differential

We studied differentials in Section 4.3, where Definition 20 states that if \( y = f(x) \) and \( f \) is differentiable, then \( dy = f'(x) dx \). One important use of this differential is in Integration by Substitution. Another important application is approximation. Let \( \Delta x = dx \) represent a change in \( x \). When \( dx \) is small, \( dy \approx \Delta y \), the change in \( y \) resulting from the change in \( x \). Fundamental in this understanding is this: as \( dx \) gets small, the difference between \( \Delta y \) and \( dy \) goes to 0. Another way of stating this: as \( dx \) goes to 0, the error in approximating \( \Delta y \) with \( dy \) goes to 0.

We extend this idea to functions of two variables. Let \( z = f(x, y) \), and let \( \Delta x = dx \) and \( \Delta y = dy \) represent changes in \( x \) and \( y \), respectively. Let \( \Delta z = f(x + dx, y + dy) - f(x, y) \) be the change in \( z \) over the change in \( x \) and \( y \). Recalling that \( f_x \) and \( f_y \) give the instantaneous rates of \( z \)-change in the \( x \)- and \( y \)-directions, respectively, we can approximate \( \Delta z \) with \( dz = f_x dx + f_y dy \); in words, the total change in \( z \) is approximately the change caused by changing \( x \) plus the change caused by changing \( y \). In a moment we give an indication of whether or not this approximation is any good. First we give a name to \( dz \).

**Definition 89 Total Differential**

Let \( z = f(x, y) \) be continuous on an open set \( S \). Let \( dx \) and \( dy \) represent changes in \( x \) and \( y \), respectively. Where the partial derivatives \( f_x \) and \( f_y \) exist, the total differential of \( z \) is

\[
dz = f_x(x, y) dx + f_y(x, y) dy.
\]

Watch the video: Differentials of Functions of Two Variables at https://youtu.be/C1XcJ5XmngC

**Example 1 Finding the total differential**

Let \( z = x^4 e^{3y} \). Find \( dz \).

**SOLUTION** We compute the partial derivatives: \( f_x = 4x^3 e^{3y} \) and \( f_y = 3x^4 e^{3y} \). Following Definition 89, we have

\[
dz = 4x^3 e^{3y} dx + 3x^4 e^{3y} dy.
\]

Notes:
We can approximate $\Delta z$ with $dz$, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point $(x_0, y_0)$, let $E_x$ and $E_y$ be functions of $dx$ and $dy$ such that $E_x dx + E_y dy$ describes this error. Then

$$
\Delta z = dz + E_x dx + E_y dy
$$

If the approximation of $\Delta z$ by $dz$ is good, then as $dx$ and $dy$ get small, so does $E_x dx + E_y dy$. The approximation of $\Delta z$ by $dz$ is even better if, as $dx$ and $dy$ go to 0, so do $E_x$ and $E_y$. This leads us to our definition of differentiability.

**Definition 90**  **Multivariable Differentiability**

Let $z = f(x, y)$ be defined on an open set $S$ containing $(x_0, y_0)$ where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let $dz$ be the total differential of $z$ at $(x_0, y_0)$, let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let $E_x$ and $E_y$ be functions of $dx$ and $dy$ such that

$$
\Delta z = dz + E_x dx + E_y dy.
$$

1. $f$ is differentiable at $(x_0, y_0)$ if

$$
\lim_{{(dx, dy) \to (0, 0)}} \| (E_x, E_y) \| = 0.
$$

2. $f$ is differentiable on $S$ if $f$ is differentiable at every point in $S$. If $f$ is differentiable on $\mathbb{R}^2$, we say that $f$ is differentiable everywhere.

**Example 2**  **Showing a function is differentiable**

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 90.

**Solution**  
We begin by finding $f(x + dx, y + dy)$, $\Delta z$, $f_x$, and $f_y$.

$$
f(x + dx, y + dy) = (x + dx)(y + dy) + 3(y + dy)^2
$$

$$
= xy + xdy + ydx + dx dy + 3y^2 + 6ydy + 3dy^2.
$$

Then

$$
\Delta z = f(x + dx, y + dy) - f(x, y),
$$

so

$$
\Delta z = xdy + ydx + dx dy + 6ydy + 3dy^2.
$$
It is straightforward to compute \( f_x = y \) and \( f_y = x + 6y \). Consider once more \( \Delta z \):

\[
\Delta z = ydx + xdy + 6ydx + 6ydy + 3dy^2
\]

\[
= ydx + xdy + 6ydy + dxdy + 3dy^2
\]

\[
= (y) \, dx + (x + 6y) \, dy + (dy) \, dx + (3dy) \, dy
\]

\[
= f_x dx + f_y dy + E_x dx + E_y dy.
\]

With \( E_x = dy \) and \( E_y = 3dy \), it is clear that as \( dx \) and \( dy \) go to 0, \( E_x \) and \( E_y \) also go to 0. Since this did not depend on a specific point \((x_0, y_0)\), we can say that \( f(x, y) \) is differentiable for all pairs \((x, y)\) in \( \mathbb{R}^2 \), or, equivalently, that \( f \) is differentiable everywhere.

Our intuitive understanding of differentiability of functions \( y = f(x) \) of one variable was that the graph of \( f \) was “smooth.” A similar intuitive understanding of functions \( z = f(x, y) \) of two variables is that the surface defined by \( f \) is also “smooth,” not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

**Theorem 107 Continuity and Differentiability of Multivariable Functions**

Let \( z = f(x, y) \) be defined on an open set \( S \) containing \((x_0, y_0)\). If \( f \) is differentiable at \((x_0, y_0)\), then \( f \) is continuous at \((x_0, y_0)\).

**Theorem 108 Differentiability of Multivariable Functions**

Let \( z = f(x, y) \) be defined on an open set \( S \) containing \((x_0, y_0)\). If \( f_x \) and \( f_y \) are both continuous on \( S \), then \( f \) is differentiable on \( S \).

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 90 and Theorem 108, though: it is possible for a function \( f \) to be differentiable yet \( f_x \) or \( f_y \) is not continuous. Such strange behavior of functions is a source of delight for many mathematicians. When this happens, we need to use other methods to determine whether or not \( f \) is differentiable at that point.
Approximating with the Total Differential

By the definition, when \( f \) is differentiable \( dz \) is a good approximation for \( \Delta z \) when \( dx \) and \( dy \) are small. We give some simple examples of how this is used here.

Example 3 Approximating with the total differential

Let \( f(x, y) = \sqrt{x} \sin y \). Approximate \( f(4.1, 0.2) \).

**Solution** We can approximate \( f(4.1, 0.2) \) using \( f(4, 0) = 0 \). Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let \( \Delta z = f(4.1, 0.2) - f(4, 0) \). The total differential \( dz \) is approximately equal to \( \Delta z \), so

\[
\begin{align*}
f(4.1, 0.2) - f(4, 0) & \approx dz \\
\Rightarrow f(4.1, 0.2) & \approx dz + f(4, 0) \quad (13.1)
\end{align*}
\]

To find \( dz \), we need \( f_x \) and \( f_y \).

\[
\begin{align*}
f_x(x, y) &= \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \quad f_x(4, 0) = \frac{\sin 0}{2\sqrt{4}} = 0 \\
f_y(x, y) &= \sqrt{x} \cos y \quad \Rightarrow \quad f_y(4, 0) = \sqrt{4} \cos 0 = 2
\end{align*}
\]

Approximating 4.1 with 4 gives \( dx = 0.1 \); approximating 0.2 with 0 gives \( dy = 0.2 \). Thus

\[
dz(4, 0) = f_x(4, 0)(0.1) + f_y(4, 0)(0.2) = 0(0.1) + 2(0.2) = 0.4.
\]

Returning to Equation (13.1), we have

\[
f(4.1, 0.2) \approx 0.4 + 0 = .4.
\]

We, of course, can compute the actual value of \( f(4.1, 0.2) \) with a calculator; to 5 places after the decimal, this is 0.40228. Obviously our approximation is quite good.

The point of the previous example was not to develop an approximation method for known functions. After all, we can very easily compute \( f(4.1, 0.2) \) using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

"New position = old position + amount of change," so

"New position \( \approx \) old position + approximate amount of change."

---

Notes:
In the previous example, we could easily compute $f(4, 0)$ and could approximate the amount of $z$-change when computing $f(4.1, 0.2)$, letting us approximate the new $z$-value.

It may be surprising to learn that it is not uncommon to know the values of $f$, $f_x$ and $f_y$ at a particular point without actually knowing the function $f$. The total differential gives a good method of approximating $f$ at nearby points.

**Example 4** Approximating an unknown function

Given that $f(2, -3) = 6$, $f_x(2, -3) = 1.3$ and $f_y(2, -3) = -0.6$, approximate $f(2.1, -3.03)$.

**Solution** The total differential approximates how much $f$ changes from the point $(2, -3)$ to the point $(2.1, -3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$dz = f_x(2, -3)dx + f_y(2, -3)dy$$

$$= 1.3(0.1) + (-0.6)(-0.03)$$

$$= 0.148.$$

The change in $z$ is approximately 0.148, so we approximate $f(2.1, -3.03) \approx 6.148$.

**Error/Sensitivity Analysis**

The total differential gives an approximation of the change in $z$ given small changes in $x$ and $y$. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

**Example 5** Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

**Solution** A cylindrical solid with height $h$ and radius $r$ has volume $V = \pi r^2 h$. We can view $V$ as a function of two variables, $r$ and $h$. We can compute partial derivatives of $V$:

$$\frac{\partial V}{\partial r} = V_r(r, h) = 2\pi rh \quad \text{and} \quad \frac{\partial V}{\partial h} = V_h(r, h) = \pi r^2.$$
The total differential is \( dV = (2\pi rh)dr + (\pi r^2)dh \). When \( h = 10 \) and \( r = 2 \), we have \( dV = 40\pi dr + 4\pi dh \). Note that the coefficient of \( dr \) is \( 40\pi \); the coefficient of \( dh \) is a tenth of that. A small change in radius will be multiplied by \( 40\pi \), whereas a small change in height will be multiplied by \( 4\pi \). Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 10 ft and radius of 5 ft would be more sensitive to changes in height than in radius.

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

**Differentiability of Functions of Three Variables**

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

**Definition 91  Total Differential**

Let \( w = f(x, y, z) \) be continuous on an open set \( S \). Let \( dx, dy \) and \( dz \) represent changes in \( x, y \) and \( z \), respectively. Where the partial derivatives \( f_x, f_y \) and \( f_z \) exist, the total differential of \( w \) is

\[
dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.
\]

This differential can be a good approximation of the change in \( w \) when \( w = f(x, y, z) \) is differentiable.
Definition 92  Multivariable Differentiability
Let $w = f(x, y, z)$ be defined on an open set $S$ containing $(x_0, y_0, z_0)$ where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let $dw$ be the total differential of $w$ at $(x_0, y_0, z_0)$, let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let $E_x$, $E_y$ and $E_z$ be functions of $dx$, $dy$ and $dz$ such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$ 

1. $f$ is differentiable at $(x_0, y_0, z_0)$ if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|\langle dx, dy, dz \rangle\| < \delta$, then $\|\langle E_x, E_y, E_z \rangle\| < \varepsilon$.

2. $f$ is differentiable on $S$ if $f$ is differentiable at every point in $S$. If $f$ is differentiable on $\mathbb{R}^3$, we say that $f$ is differentiable everywhere.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 108.

Theorem 109  Continuity and Differentiability of Functions of Three Variables
Let $w = f(x, y, z)$ be defined on an open set $S$ containing $(x_0, y_0, z_0)$.

1. If $f$ is differentiable at $(x_0, y_0, z_0)$, then $f$ is continuous at $(x_0, y_0, z_0)$.

2. If $f_x$, $f_y$ and $f_z$ are continuous on $S$, then $f$ is differentiable on $S$.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we encounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a function to be “differentiable,” along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:
Exercises 13.4

Terms and Concepts

1. T/F: If \(f(x, y)\) is differentiable on \(S\), the \(f\) is continuous on \(S\).
2. T/F: If \(f_r\) and \(f_y\) are continuous on \(S\), then \(f\) is differentiable on \(S\).
3. T/F: If \(z = f(x, y)\) is differentiable, then the change in \(z\) over small changes \(dx\) and \(dy\) in \(x\) and \(y\) is approximately \(dz\).
4. Finish the sentence: “The new \(z\)-value is approximately the old \(z\)-value plus the approximate ________.”

Problems

In Exercises 5–8, find the total differential \(dz\).

5. \(z = x \sin y + x^3\)
6. \(z = (2x^2 + 3y)^2\)
7. \(z = 5x - 7y\)
8. \(z = xe^{x+y}\)

In Exercises 9–12, a function \(z = f(x, y)\) is given. Give the indicated approximation using the total differential.

9. \(f(x, y) = \sqrt{x^2 + y}\). Approximate \(f(2.95, 7.1)\) knowing \(f(3, 7) = 4\).
10. \(f(x, y) = \sin x \cos y\). Approximate \(f(0.1, -0.1)\) knowing \(f(0, 0) = 0\).
11. \(f(x, y) = x^2y - xy^2\). Approximate \(f(2.04, 3.06)\) knowing \(f(2, 3) = -6\).
12. \(f(x, y) = \ln(x - y)\). Approximate \(f(5.1, 3.98)\) knowing \(f(5, 4) = 0\).

Exercises 13–16 ask a variety of questions dealing with approximating error and sensitivity analysis.

13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

14. Projectile Motion: The \(x\)-value of an object moving under the principles of projectile motion is \(x(\theta, v_0, t) = (v_0 \cos \theta)t\). A particular projectile is fired with an initial velocity of \(v_0 = 250\)ft/s and an angle of elevation of \(\theta = 60^\circ\). It travels a distance of 375ft in 3 seconds.

Is the projectile more sensitive to errors in initial speed or angle of elevation?

15. The length \(\ell\) of a long wall is to be approximated. The angle \(\theta\), as shown in the diagram (not to scale), is measured to be \(85^\circ\), and the distance \(x\) is measured to be \(30'\). Assume that the triangle formed is a right triangle.

Is the measurement of the length of \(\ell\) more sensitive to errors in the measurement of \(x\) or in \(\theta\)?

16. It is “common sense” that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance \(D\) can be viewed as the product of the length \(\ell\) of a measuring tape times the number \(n\) of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (i.e., \(30 = 12 \times 2.5\)). Thus \(D = n\ell\).

Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (i.e., \(d\ell = 1/16'' \approx 0.005\)ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

In Exercises 17–18, find the total differential \(dw\).

17. \(w = x^2y^2\)
18. \(w = e^x \sin y \ln z\)

In Exercises 19–22, use the information provided and the total differential to make the given approximation.

19. \(f(3, 1) = 7, f_r(3, 1) = 9, f_y(3, 1) = -2\). Approximate \(f(3.05, 0.9)\).

20. \(f(-4, 2) = 13, f_r(-4, 2) = 2.6, f_y(-4, 2) = 5.1\). Approximate \(f(-4.12, 2.07)\).

21. \(f(2, 4, 5) = -1, f_r(2, 4, 5) = 2, f_y(2, 4, 5) = -3, f_z(2, 4, 5) = 3.7\). Approximate \(f(2.5, 4.1, 4.8)\).

22. \(f(3, 3, 3) = 5, f_r(3, 3, 3) = 2, f_y(3, 3, 3) = 0, f_z(3, 3, 3) = -2\). Approximate \(f(3.1, 3.1, 3.1)\).

23. Find where the function \(z = \sqrt{x^2 + y^2}\) is differentiable.
13.5 The Multivariable Chain Rule

The Chain Rule, as learned in Section 2.5, states that \( \frac{d}{dx} \left( f(g(x)) \right) = f'(g(x))g'(x) \).

If \( t = g(x) \), we can express the Chain Rule as
\[
\frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx}.
\]

In this section we extend the Chain Rule to functions of more than one variable.

**Theorem 110** Multivariable Chain Rule, Part I

Let \( z = f(x, y) \), \( x = g(t) \) and \( y = h(t) \), where \( f \), \( g \) and \( h \) are differentiable functions. Then \( z = f(g(t), h(t)) \) is a function of \( t \), and
\[
\frac{dz}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}.
\]

**Proof**

By definition,
\[
\frac{df}{dt}(x, y) = \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x, y)}{h}.
\]

Let
\[
\Delta f = f(x(t+h), y(t+h)) - f(x, y),
\]
\[d x = x(t+h) - x(t), \quad \text{and}\]
\[d y = y(t+h) - y(t).
\]

Because \( f \) is differentiable, Definition 90 gives us functions \( E_x \) and \( E_y \) so that
\[
E_x dx + E_y dy = \Delta f - f_x(x, y) dx - f_y(x, y) dy
\]
and \( \lim_{(dx, dy) \to 0} (E_x, E_y) = 0 \).

This means that
\[
\frac{df}{dt}(x, y) = \lim_{h \to 0} \frac{f_x(x, y) dx + f_y(x, y) dy + E_x dx + E_y dy}{h}
\]
\[= f_x(x, y) \lim_{h \to 0} \frac{dx}{h} + f_y(x, y) \lim_{h \to 0} \frac{dy}{h}
\]
\[+ \lim_{h \to 0} E_x \lim_{h \to 0} \frac{dx}{h} + \lim_{h \to 0} E_y \lim_{h \to 0} \frac{dy}{h}
\]
\[= f_x(x, y)x'(t) + f_y(x, y)y'(t) + 0x'(t) + 0y'(t).
\]

\[\square\]

Notes:
It is good to understand what the situation of \( z = f(x, y) \), \( x = g(t) \) and \( y = h(t) \) describes. We know that \( z = f(x, y) \) describes a surface; we also recognize that \( x = g(t) \) and \( y = h(t) \) are parametric equations for a curve in the \( x-y \) plane. Combining these together, we are describing a curve that lies on the surface described by \( f \). The parametric equations for this curve are \( x = g(t) \), \( y = h(t) \) and \( z = f(g(t), h(t)) \).

Consider Figure 13.14 in which a surface is drawn, along with a dashed curve in the \( x-y \) plane. Restricting \( f \) to just the points on this circle gives the curve shown on the surface. The derivative \( \frac{df}{dt} \) gives the instantaneous rate of change of \( f \) with respect to \( t \). If we consider an object traveling along this path, \( \frac{df}{dt} \) gives the rate at which the object rises/falls.

We now practice applying the Multivariable Chain Rule.

**Example 1**  
**Using the Multivariable Chain Rule**

Let \( z = x^2y + x \), where \( x = \sin t \) and \( y = e^{5t} \). Find \( \frac{dz}{dt} \) using the Chain Rule.

**Solution**  
Following Theorem 110, we find

\[
\begin{align*}
 f_x(x, y) &= 2xy + 1, \\
 f_y(x, y) &= x^2, \\
 \frac{dx}{dt} &= \cos t, \\
 \frac{dy}{dt} &= 5e^{5t}.
\end{align*}
\]

Applying the theorem, we have

\[
\frac{dz}{dt} = (2xy + 1) \cos t + 5x^2e^{5t}.
\]

This may look odd, as it seems that \( \frac{dz}{dt} \) is a function of \( x, y \) and \( t \). Since \( x \) and \( y \) are functions of \( t \), \( \frac{dx}{dt} \) is really just a function of \( t \), and we can replace \( x \) with \( \sin t \) and \( y \) with \( e^{5t} \): 

\[
\frac{dz}{dt} = (2\sin t \cos t + 1) \cos t + 5\sin^2 t e^{5t}.
\]

The previous example can make us wonder: if we substituted for \( x \) and \( y \) at the end to show that \( \frac{dz}{dt} \) is really just a function of \( t \), why not substitute before differentiating, showing clearly that \( z \) is a function of \( t \)?

That is, \( z = x^2y + x = (\sin t)^2e^{5t} + \sin t \). Applying the Chain and Product Rules, we have

\[
\frac{dz}{dt} = 2 \sin t \cos t e^{5t} + 5 \sin^2 t e^{5t} + \cos t,
\]

which matches the result from the example.

Notes:
This may now make one wonder “What’s the point? If we could already find the derivative, why learn another way of finding it?” In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where \( z = f(x, y) \), \( x = g(t) \) and \( y = h(t) \), the Chain Rule is extremely powerful when we do not know what \( f \), \( g \) and/or \( h \) are. It may be hard to believe, but often in “the real world” we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we’ll see in the next section).

We demonstrate this in the next example.

**Example 2 Applying the Multivariable Chain Rule**

An object travels along a path on a surface. The exact path and surface are not known, but at time \( t = t_0 \) it is known that:

\[
\frac{\partial z}{\partial x} = 5, \quad \frac{\partial z}{\partial y} = -2, \quad \frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = 7.
\]

Find \( \frac{dz}{dt} \) at time \( t_0 \).

**SOLUTION**

The Multivariable Chain Rule states that

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
= 5(3) + (-2)(7)
\]

\[
= 1.
\]

By knowing certain rates–of–change information about the surface and about the path of the particle in the \( x\)-\( y \) plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

**Example 3 Applying the Multivariable Chain Rule**

Consider the surface \( z = x^2 + y^2 - xy \), a paraboloid, on which a particle moves with \( x \) and \( y \) coordinates given by \( x = \cos t \) and \( y = \sin t \). Find \( \frac{dz}{dt} \) when \( t = 0 \), and find where the particle reaches its maximum/minimum \( z \)-values.

**SOLUTION**

It is straightforward to compute

\[
f_x(x, y) = 2x - y, \quad f_y(x, y) = 2y - x, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.
\]

Notes:
Combining these according to the Chain Rule gives:

\[
\frac{dz}{dt} = -(2x - y) \sin t + (2y - x) \cos t.
\]

When \( t = 0, x = 1 \) and \( y = 0 \). Thus \( \frac{dz}{dt} = -(2)(0) + (-1)(1) = -1 \). When \( t = 0 \), the particle is moving down, as shown in Figure 13.15.

To find where \( z \)-value is maximized/minimized on the particle’s path, we set \( \frac{dz}{dt} = 0 \) and solve for \( t \):

\[
\frac{dz}{dt} = 0 = -(2x - y) \sin t + (2y - x) \cos t
\]

\[
0 = -(2 \cos t - \sin t) \sin t + (2 \sin t - \cos t) \cos t
\]

\[
0 = \sin^2 t - \cos^2 t
\]

\[
\cos^2 t = \sin^2 t
\]

\[
t = n \frac{\pi}{4} \quad \text{(for odd } n\text{)}
\]

We can use the First Derivative Test to find that on \([0, 2\pi]\), \( z \) has reaches its absolute minimum at \( t = \frac{\pi}{4} \) and \( 5\frac{\pi}{4} \); it reaches its absolute maximum at \( t = 3\frac{\pi}{4} \) and \( 7\frac{\pi}{4} \), as shown in Figure 13.15.

We can extend the Chain Rule to include the situation where \( z \) is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where \( z = f(x, y) \), and \( x \) and \( y \) are functions of two variables, say \( s \) and \( t \).
Theorem 111  Multivariable Chain Rule, Part II

1. Let \( z = f(x, y), x = g(s, t) \) and \( y = h(s, t) \), where \( f, g \) and \( h \) are differentiable functions. Then \( z \) is a function of \( s \) and \( t \), and

\[
\begin{align*}
\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \\
\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.
\end{align*}
\]

2. Let \( z = f(x_1, x_2, \ldots, x_m) \) be a differentiable function of \( m \) variables, where each of the \( x_i \) is a differentiable function of the variables \( t_1, t_2, \ldots, t_n \). Then \( z \) is a function of the \( t_i \), and

\[
\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t_i}.
\]

The proof of Part II follows quickly from Part I, because \( \frac{\partial}{\partial t_i} \) means that we hold the other variables constant and we are back to the one variable case already proved. A helpful way to remember the derivatives is to examine the following chart

Each possible path from \( f \) to the variable \( t_i \) contributes a term to the sum, and each line segment in a path contributes a factor to that term.

Notes:
Example 4 Using the Multivariable Chain Rule, Part II
Let \( z = x^2 y + x, x = s^2 + 3t \) and \( y = 2s - t \). Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \), and evaluate each when \( s = 1 \) and \( t = 2 \).

**Solution** Following Theorem 111, we compute the following partial derivatives:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2xy + 1, & \frac{\partial f}{\partial y} &= x^2, \\
\frac{\partial x}{\partial s} &= 2s, & \frac{\partial x}{\partial t} &= 3, & \frac{\partial y}{\partial s} &= 2, & \frac{\partial y}{\partial t} &= -1.
\end{align*}
\]

Thus

\[
\begin{align*}
\frac{\partial z}{\partial s} &= (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \\
\frac{\partial z}{\partial t} &= (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.
\end{align*}
\]

When \( s = 1 \) and \( t = 2 \), \( x = 7 \) and \( y = 0 \), so

\[
\begin{align*}
\frac{\partial z}{\partial s} &= 100 \quad \text{and} \quad \frac{\partial z}{\partial t} = -46.
\end{align*}
\]

Example 5 Using the Multivariable Chain Rule, Part II
Let \( w = xy + z^2 \), where \( x = t^2 e^t \), \( y = t \cos s \), and \( z = s \sin t \). Find \( \frac{\partial w}{\partial t} \) when \( s = 0 \) and \( t = \pi \).

**Solution** Following Theorem 111, we compute the following partial derivatives:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= y, & \frac{\partial f}{\partial y} &= x, & \frac{\partial f}{\partial z} &= 2z, \\
\frac{\partial x}{\partial t} &= 2te^t, & \frac{\partial y}{\partial t} &= \cos s, & \frac{\partial z}{\partial t} &= s \cos t.
\end{align*}
\]

Notes:
Thus
\[ \frac{\partial w}{\partial t} = y(2te^x) + x(\cos s) + 2z(s \cos t). \]
When \( s = 0 \) and \( t = \pi \), we have \( x = \pi^2 \), \( y = \pi \) and \( z = 0 \). Thus
\[ \frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2. \]

**Implicit Differentiation**

We studied finding \( \frac{dy}{dx} \) when \( y \) is given as an implicit function of \( x \) in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding \( \frac{dy}{dx} \).

For instance, consider the implicit function \( x^2y - xy^3 = 3 \). We learned to use the following steps to find \( \frac{dy}{dx} \):

\[
\frac{d}{dx}(x^2y - xy^3) = \frac{d}{dx}(3)
\]
\[2xy + x^2 \frac{dy}{dx} - y^3 - 3xy^2 \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{2xy - y^3}{x^2 - 3xy^2}. \]  \hspace{1cm} (13.2)

Instead of using this method, consider \( z = x^2y - xy^3 \). The implicit function above describes the level curve \( z = 3 \). Considering \( x \) and \( y \) as functions of \( x \), the Multivariable Chain Rule states that
\[ \frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx}. \]  \hspace{1cm} (13.3)

Since \( z \) is constant (in our example, \( z = 3 \)), \( \frac{dz}{dx} = 0 \). We also know \( \frac{dx}{dx} = 1 \). Equation (13.3) becomes
\[ 0 = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y} \frac{dy}{dx} \]
\[ \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{f_x}{f_y}. \]

Note how our solution for \( \frac{dy}{dx} \) in Equation (13.2) is just the partial derivative of \( z \) with respect to \( x \), divided by the partial derivative of \( z \) with respect to \( y \).

We state the above as a theorem for two and three variables.

---

Notes:
Theorem 112  Implicit Differentiation
If \( f \) is a differentiable function of \( x \) and \( y \), where \( f(x, y) = c \) defines \( y \) as an implicit function of \( x \) for some constant \( c \), then
\[
\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.
\]

If \( f \) is a differentiable function of \( x, y, \) and \( z \), where \( f(x, y, z) = c \) defines \( z \) as an implicit function of \( x \) and \( y \) for some constant \( c \), then
\[
\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)}.
\]

We practice using Theorem 112 by applying it to a problem from Section 2.6.

Example 6  Implicit Differentiation
Given the implicitly defined function \( \sin(x^2y^2) + y^3 = x + y \), find \( y' \). Note: this is the same problem as given in Example 2.6.4 of Section 2.6.

SOLUTION  Let \( f(x, y) = \sin(x^2y^2) + y^3 - x - y \); the implicitly defined function above is equivalent to \( f(x, y) = 0 \). We find \( \frac{dy}{dx} \) by applying Theorem 112. We find
\[
f_x(x, y) = 2xy^2 \cos(x^2y^2) - 1 \quad \text{and} \quad f_y(x, y) = 2x^2y \cos(x^2y^2) + 3y^2 - 1,
\]
so
\[
\frac{dy}{dx} = -\frac{2xy^2 \cos(x^2y^2) + 3y^2 - 1}{2x^2y \cos(x^2y^2) - 1},
\]
which matches our solution from Example 2.6.4.

Notes:
Terms and Concepts

1. Let a level curve of \( z = f(x, y) \) be described by \( x = g(t), \ y = h(t) \). Explain why \( \frac{dz}{dt} = 0 \).
2. Fill in the blank: The single variable Chain Rule states \( \frac{d}{dx} \left( f(g(x)) \right) = f'(g(x)) \cdot \ldots \).
3. Fill in the blank: The Multivariable Chain Rule states \( \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \ldots + \frac{\partial f}{\partial y} \cdot \ldots \cdot \frac{dy}{dt} \).
4. If \( z = f(x, y) \), where \( x = g(t) \) and \( y = h(t) \), we can substitute and write \( z \) as an explicit function of \( t \).
5. T/F: Using the Multivariable Chain Rule to find \( \frac{dz}{dt} \) is sometimes easier than first substituting and then taking the derivative.
6. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.
7. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two \ldots \) derivatives.

Problems

In Exercises 7–12, functions \( z = f(x, y) \), \( x = g(t) \) and \( y = h(t) \) are given.

(a) Use the Multivariable Chain Rule to compute \( \frac{dz}{dt} \).

(b) Evaluate \( \frac{dz}{dt} \) at the indicated \( t \)-value.

7. \( z = 3x + 4y \), \( x = t^2 \), \( y = 2t \); \( t = 1 \)
8. \( z = x^2 - y^2 \), \( x = t \), \( y = t^2 - 1 \); \( t = 1 \)
9. \( z = 5x + 2y \), \( x = 2 \cos t + 1 \), \( y = \sin t - 3 \); \( t = \pi/4 \)
10. \( z = \frac{x}{y^2 + 1} \), \( x = \cos t \), \( y = \sin t \); \( t = \pi/2 \)
11. \( z = \sqrt{x} + 2y^2 \), \( x = \sin t \), \( y = 3 \sin t \); \( t = \pi/4 \)
12. \( z = \cos x \sin y \), \( x = \pi t \), \( y = 2\pi t + \pi/2 \); \( t = 3 \)

In Exercises 13–18, functions \( z = f(x, y) \), \( x = g(t) \) and \( y = h(t) \) are given. Find the values of \( t \) where \( \frac{dz}{dt} = 0 \). Note: these are the same surfaces/curves as found in Exercises 7–12.

13. \( z = 3x + 4y \), \( x = t^2 \), \( y = 2t \)
14. \( z = x^2 - y^2 \), \( x = t \), \( y = t^2 - 1 \)
15. \( z = 5x + 2y \), \( x = 2 \cos t + 1 \), \( y = \sin t - 3 \)
16. \( z = \frac{x}{y^2 + 1} \), \( x = \cos t \), \( y = \sin t \)
17. \( z = x^2 + 2y^2 \), \( x = \sin t \), \( y = 3 \sin t \)
18. \( z = \cos x \sin y \), \( x = \pi t \), \( y = 2\pi t + \pi/2 \)

In Exercises 19–22, functions \( z = f(x, y) \), \( x = g(s, t) \) and \( y = h(s, t) \) are given.

(a) Use the Multivariable Chain Rule to compute \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

(b) Evaluate \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) at the indicated \( s \) and \( t \) values.

19. \( z = x^2 + y \), \( x = s - t \), \( y = 2s + 4t \); \( s = 1, t = 0 \)
20. \( z = \cos (\pi x + \pi y) \), \( x = st^2 \), \( y = s^2 t \); \( s = 1, t = 1 \)
21. \( z = x^2 + y^2 \), \( x = s \cos t \), \( y = s \sin t \); \( s = 2, t = \pi/4 \)
22. \( z = e^{-\sqrt{x^2 + y^2}} \), \( x = t \), \( y = st^2 \); \( s = 1, t = 1 \)

In Exercises 23–26, find \( \frac{dy}{dx} \) using Implicit Differentiation and Theorem 112.

23. \( x \tan y = 50 \)
24. \( (3x^2 + 2y^3)^4 = 2 \)
25. \( \frac{x^2 + y^2}{x + y^2} = 17 \)
26. \( \ln(x^2 + xy + y^2) = 1 \)

In Exercises 27–30, find \( \frac{dz}{dt} \) or \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \), using the supplied information.

27. \( \frac{\partial z}{\partial x} = 2, \frac{\partial z}{\partial y} = 1 \), \( \frac{dx}{dt} = 4 \), \( \frac{dy}{dt} = 5 \)
28. \( \frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = -3 \), \( \frac{dx}{dt} = 6 \), \( \frac{dy}{dt} = 2 \)
29. \( \frac{\partial z}{\partial x} = -4, \frac{\partial z}{\partial y} = 9 \), \( \frac{dx}{dt} = 5 \), \( \frac{dy}{dt} = 7 \), \( \frac{dz}{dt} = -2 \), \( \frac{dy}{ds} = 6 \)
30. \( \frac{\partial z}{\partial x} = 2, \frac{\partial z}{\partial y} = 1 \), \( \frac{dz}{dt} = 2 \), \( \frac{\partial z}{\partial s} = 3 \), \( \frac{dy}{ds} = 2 \), \( \frac{dy}{dt} = -1 \)
13.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the $x$ and $y$ directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to $f_x$. Likewise, the rise/fall in moving due north is comparable to $f_y$. The steeper the slope, the greater in magnitude $f_y$.

But what if we didn’t move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates directional derivatives, which do measure this rate of change.

We begin with a definition.

**Definition 93 Directional Derivatives**

Let $z = f(x, y)$ be continuous on an open set $S$ and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points $(x, y)$, the directional derivative of $f$ at $(x, y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives $f_x$ and $f_y$ are defined with similar limits, but only $x$ or $y$ varies with $h$, not both. Here both $x$ and $y$ vary with a weighted $h$, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

**Theorem 113 Directional Derivatives**

Let $z = f(x, y)$ be differentiable at $(x_0, y_0)$, and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of $f$ at $(x_0, y_0)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

**Proof**

This is really a quick application of Definition 90. Because $f$ is differentiable at

Notes:
Chapter 13  Functions of Several Variables

\[(x_0, y_0),\]

\[D_uf(x_0, y_0) = \lim_{h \to 0} \frac{f_x(x_0, y_0)hu_1 + f_y(x_0, y_0)hu_2 + E_x(x_0, y_0)hu_1 + E_y(x_0, y_0)hu_2}{h}\]

\[= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 + \lim_{h \to 0} E_x(x_0, y_0)u_1 + E_y(x_0, y_0)u_2\]

\[= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.\]

Watch the video:
Finding the Directional Derivative — Ex 1 at
https://youtu.be/uCY0XYXgYQo

Example 1  Computing directional derivatives
Let \(z = 14 - x^2 - y^2\) and let \(P = (1, 2)\). Find the directional derivative of \(f\), at \(P\), in the following directions:

1. toward the point \(Q = (3, 4)\),
2. in the direction of \((2, -1)\), and
3. toward the origin.

**Solution**  The surface is plotted in Figure 13.16, where the point \(P = (1, 2)\) is indicated in the \(x, y\)-plane as well as the point \((1, 2, 9)\) which lies on the surface of \(f\). We find that \(f_x(x, y) = -2x\) and \(f_x(1, 2) = -2; f_y(x, y) = -2y\) and \(f_y(1, 2) = -4\).

1. Let \(\vec{u}_1\) be the unit vector that points from the point \((1, 2)\) to the point \(Q = (3, 4)\), as shown in the figure. The vector \(PQ = (2, 2)\); the unit vector in this direction is \(\vec{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\). Thus the directional derivative of \(f\) at \((1, 2)\) in the direction of \(\vec{u}_1\) is

\[D_{\vec{u}_1}f(1, 2) = -2\left(\frac{1}{\sqrt{2}}\right) + (-4)\left(\frac{1}{\sqrt{2}}\right) = -6/\sqrt{2}.\]

Thus the instantaneous rate of change in moving from the point \((1, 2, 9)\) on the surface in the direction of \(\vec{u}_1\) (which points toward the point \(Q\)) is \(-3\sqrt{2}.\) Moving in this direction moves one steeply downward.

Notes:
2. We seek the directional derivative in the direction of \((2, -1)\). The unit vector in this direction is \(\mathbf{u}_2 = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle\). Thus the directional derivative of \(f\) at \((1, 2)\) in the direction of \(\mathbf{u}_2\) is

\[
D_{\mathbf{u}_2}f(1, 2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.
\]

Starting on the surface of \(f\) at \((1, 2)\) and moving in the direction of \((2, -1)\) (or \(\mathbf{u}_2\)) results in no instantaneous change in \(z\)-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just “along the side” of the hill.

Finding these directions of “no elevation change” is important.

3. At \(P = (1, 2)\), the direction towards the origin is given by the vector \(\langle -1, -2 \rangle\); the unit vector in this direction is \(\mathbf{u}_3 = \langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle\).

The directional derivative of \(f\) at \(P\) in the direction of the origin is

\[
D_{\mathbf{u}_3}f(1, 2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = \frac{10}{\sqrt{5}}.
\]

Moving towards the origin means “walking uphill” quite steeply, with an initial slope of \(\frac{2}{\sqrt{5}}\).

As we study directional derivatives, it will help to make an important connection between the unit vector \(\mathbf{u} = \langle u_1, u_2 \rangle\) that describes the direction and the partial derivatives \(f_x\) and \(f_y\). We start with a definition and follow this with a Key Idea.

---

**Definition 94 Gradient**

Let \(z = f(x, y)\) be differentiable on an open set \(S\) that contains the point \((x_0, y_0)\).

1. The **gradient of \(f\)** is \(\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle\).

2. The **gradient of \(f\) at \((x_0, y_0)\)** is \(\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle\).

To simplify notation, we often express the gradient as \(\nabla f = \langle f_x, f_y \rangle\). It is often useful to think of the gradient \(\nabla\) as an operator:

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).
\]

The operator \(\nabla\) only has any meaning when it operates on a function; it doesn’t mean anything by itself. But this notation does help us to apply it correctly to find the gradient. The gradient allows us to compute directional derivatives in terms of a dot product.

---

**Note:** The symbol “\(\nabla\)” is named “nabla,” derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression \(\nabla f\) is pronounced “del \(f\).”

---

Notes:
The directional derivative of \( z = f(x, y) \) in the direction of the unit vector \( \vec{u} \) is

\[ D_{\vec{u}} f = \nabla f \cdot \vec{u}. \]

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of \( z \) when moving in the direction of \( \vec{u} \), three questions naturally arise:

1. In what direction(s) is the change in \( z \) the greatest (i.e., the “steepest uphill”)?
2. In what direction(s) is the change in \( z \) the least (i.e., the “steepest downhill”)?
3. In what direction(s) is there no change in \( z \)?

Using the key property of the dot product, we have

\[ \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos \theta = \|\nabla f\| \cos \theta, \quad (13.4) \]

where \( \theta \) is the angle between the gradient and \( \vec{u} \). (Since \( \vec{u} \) is a unit vector, \( \|\vec{u}\| = 1 \).) This equation allows us to answer the three questions stated previously.

1. Equation (13.4) is maximized when \( \cos \theta = 1 \), i.e., when the gradient and \( \vec{u} \) have the same direction. We conclude the gradient points in the direction of greatest \( z \) change.
2. Equation (13.4) is minimized when \( \cos \theta = -1 \), i.e., when the gradient and \( \vec{u} \) have opposite directions. We conclude the gradient points in the opposite direction of the least \( z \) change.
3. Equation (13.4) is 0 when \( \cos \theta = 0 \), i.e., when the gradient and \( \vec{u} \) are orthogonal to each other. We conclude the gradient is orthogonal to directions of no \( z \) change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side-stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the \( x-y \) plane along which the \( z \)-values of a function do not change. Let a surface \( z = f(x, y) \) be given, and let’s
13.6 Directional Derivatives

represent one such level curve as a vector–valued function, \( \mathbf{r}(t) = (x(t), y(t)) \).
As the output of \( f \) does not change along this curve, \( f(x(t), y(t)) = c \) for all \( t \), for some constant \( c \).

Since \( f \) is constant for all \( t \), \( \frac{df}{dt} = 0 \). By the Multivariable Chain Rule, we also know

\[
\frac{df}{dt} = f_x(x, y)x'(t) + f_y(x, y)y'(t)
= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle x'(t), y'(t) \rangle
= \nabla f \cdot \mathbf{r}'(t)
= 0.
\]

This last equality states \( \nabla f \cdot \mathbf{r}'(t) = 0 \): the gradient is orthogonal to the derivative of \( \mathbf{r} \). Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

**Theorem 114 The Gradient and Directional Derivatives**

Let \( z = f(x, y) \) be differentiable on an open set \( S \) with gradient \( \nabla f \), let \( P = (x_0, y_0) \) be a point in \( S \) and let \( \mathbf{u} \) be a unit vector.

1. The maximum value of \( D_uf(x_0, y_0) \) is \( \|\nabla f(x_0, y_0)\| \); the direction of maximal \( z \) increase is \( \nabla f(x_0, y_0) \).

2. The minimum value of \( D_uf(x_0, y_0) \) is \( -\|\nabla f(x_0, y_0)\| \); the direction of minimal \( z \) increase is \( -\nabla f(x_0, y_0) \).

3. At \( P \), \( \nabla f(x_0, y_0) \) is orthogonal to the level curve passing through \( (x_0, y_0, f(x_0, y_0)) \).

**Example 2 Finding directions of maximal and minimal increase**

Let \( f(x, y) = \sin x \cos y \) and let \( P = (\pi/3, \pi/3) \). Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of \( z \) change is 0.

**SOLUTION** We begin by finding the gradient. We see that \( f_x = \cos x \cos y \) and \( f_y = -\sin x \sin y \), thus

\[
\nabla f = (\cos x \cos y, -\sin x \sin y) \quad \text{and, at } P, \quad \nabla f \left( \frac{\pi}{3}, \frac{\pi}{3} \right) = \left( \frac{1}{4}, \frac{3}{4} \right).
\]

**Notes:**
Thus the direction of maximal increase is \((1/4, -3/4)\). In this direction, the instantaneous rate of \(z\) change is \(\|\langle 1/4, -3/4 \rangle\| = \sqrt{10}/4\).

Figure 13.17 shows the surface plotted from two different perspectives. In each, the gradient is drawn at \(P\) with a dashed line (because of the nature of this surface, the gradient points “into” the surface). Let \(\vec{u} = \langle u_1, u_2 \rangle\) be the unit vector in the direction of \(\nabla f\) at \(P\). Each graph of the figure also contains the vector \(\langle u_1, u_2, \|\nabla f\| \rangle\). This vector has a “run” of 1 (because in the \(x\)-\(y\) plane it moves 1 unit) and a “rise” of \(\|\nabla f\|\), hence we can think of it as a vector with slope of \(\|\nabla f\|\) in the direction of \(\nabla f\), helping us visualize how “steep” the surface is in its steepest direction.

The direction of minimal increase is \((-1/4, 3/4)\); in this direction the instantaneous rate of \(z\) change is \(-\sqrt{10}/4\).

Any direction orthogonal to \(\nabla f\) is a direction of no \(z\) change. We have two choices: the direction of \(\langle 3, 1 \rangle\) and the direction of \(\langle -3, -1 \rangle\). The unit vector in the direction of \(\langle 3, 1 \rangle\) is shown in each graph of the figure as well. The level curve at \(z = \sqrt{3}/4\) is drawn: recall that along this curve the \(z\)-values do not change. Since \(\langle 3, 1 \rangle\) is a direction of no \(z\)-change, this vector is tangent to the level curve at \(P\).

**Example 3** Understanding when \(\nabla f = \vec{0}\)

Let \(f(x, y) = -x^2 + 2x - y^2 + 2y + 1\). Find the directional derivative of \(f\) in any direction at \(P = (1, 1)\).

**Solution** We find \(\nabla f = \langle -2x + 2, -2y + 2 \rangle\). At \(P\), we have \(\nabla f(1, 1) = \langle 0, 0 \rangle\). According to Theorem 114, this is the direction of maximal increase. However, \(\langle 0, 0 \rangle\) is directionless; it has no displacement. And regardless of the unit vector \(\vec{u}\) chosen, \(D_{\vec{u}} f = 0\).

Figure 13.18 helps us understand what this means. We can see that \(P\) lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of \(\vec{0} = \langle 0, 0 \rangle\), as this indicates that all directional derivatives are 0.

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

**Example 4** The flow of water downhill

Consider the surface given by \(f(x, y) = 20 - x^2 - 2y^2\). Water is poured on the surface at \((1, 1/4)\). What path does it take as it flows downhill?

**Solution** Let \(\vec{r}(t) = \langle x(t), y(t) \rangle\) be the vector–valued function describing the path of the water in the \(x\)-\(y\) plane; we seek \(x(t)\) and \(y(t)\). We know
that water will always flow downhill in the steepest direction; therefore, at any
point on its path, it will be moving in the direction of \(-\nabla f\). (We ignore the phys-
ical effects of momentum on the water.) Thus \(\vec{r}'(t)\) will be parallel to \(\nabla f\), and
there is some constant \(c\) such that \(c\nabla f = \vec{r}'(t) = (x'(t), y'(t))\).

We find \(\nabla f = (-2x, -4y)\) and write \(x'(t)\) as \(\frac{dx}{dt}\) and \(y'(t)\) as \(\frac{dy}{dt}\). Then
\[
\vec{r}'(t) = (\nabla f) = \langle x', y' \rangle = \langle -2cx, -4cy \rangle = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle.
\]

This implies
\[
-2cx = \frac{dx}{dt} \quad \text{and} \quad -4cy = \frac{dy}{dt}, \quad \text{i.e.,}
\]
\[
c = -\frac{1}{2x} \frac{dx}{dt} \quad \text{and} \quad c = -\frac{1}{4y} \frac{dy}{dt}.
\]

As \(c\) equals both expressions, we have
\[
\frac{1}{2x} \frac{dx}{dt} = \frac{1}{4y} \frac{dy}{dt}.
\]

To find an explicit relationship between \(x\) and \(y\), we can integrate both sides with
respect to \(t\). Recall from our study of differentials that \(\frac{dx}{dt} = dx\). Thus:

\[
\int \frac{1}{2x} \frac{dx}{dt} dt = \int \frac{1}{4y} \frac{dy}{dt} dt
\]
\[
\int \frac{1}{2x} dx = \int \frac{1}{4y} dy
\]
\[
\frac{1}{2} \ln |x| = \frac{1}{4} \ln |y| + C_1
\]
\[
2 \ln |x| = \ln |y| + C_1
\]
\[
\ln |x^2| = \ln |y| + C_1
\]

Now raise both sides as a power of \(e\):
\[
x^2 = e^{\ln |y| + C_1}
\]
\[
x^2 = e^{\ln |y|} e^{C_1} \quad \text{(Note that } e^{C_1} \text{ is just a constant.)}
\]
\[
x^2 = y C_2
\]
\[
\frac{1}{C_2} x^2 = y \quad \text{(Note that } 1/C_2 \text{ is just a constant.)}
\]
\[
CX^2 = y.
\]

Notes:
As the water started at the point \((1, 1/4)\), we can solve for \(C\):

\[
C(1)^2 = \frac{1}{4} \implies C = \frac{1}{4}.
\]

Thus the water follows the curve \(y = x^2/4\) in the \(x\)-\(y\) plane. The surface and the path of the water is graphed in Figure 13.19(a). In part (b) of the figure, the level curves of the surface are plotted in the \(x\)-\(y\) plane, along with the curve \(y = x^2/4\). Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 93 and 94 and Theorem 113 into one set of definitions.

**Definition 95** Directional Derivatives and Gradient with Three Variables

Let \(w = F(x, y, z)\) be differentiable on an open ball \(B\) and let \(\vec{u}\) be a unit vector in \(\mathbb{R}^3\).

1. The **gradient** of \(F\) is \(\nabla F = (F_x, F_y, F_z)\).
2. The **gradient of \(F\) at \((x_0, y_0, z_0)\)** is

\[
\nabla F(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)).
\]

3. The **directional derivative of \(F\) in the direction of \(\vec{u}\)** is

\[
D_\vec{u} F = \nabla F \cdot \vec{u}.
\]

The same properties of the gradient given in Theorem 114, when \(f\) is a function of two variables, hold for \(F\), a function of three variables.

Notes:
Theorem 115  The Gradient and Directional Derivatives with Three Variables
Let \( w = F(x, y, z) \) be differentiable on an open set \( S \) with gradient \( \nabla F \), let \( P = (x_0, y_0, z_0) \) be a point in \( S \), and let \( \vec{u} \) be a unit vector.

1. The maximum value of \( D_{\vec{u}} F(x_0, y_0, z_0) \) is \( \| \nabla F(x_0, y_0, z_0) \| \); the direction of maximal increase is \( \nabla F(x_0, y_0, z_0) \).

2. The minimum value of \( D_{\vec{u}} F(x_0, y_0, z_0) \) is \( -\| \nabla F(x_0, y_0, z_0) \| \); the direction of minimal increase is \( -\nabla F(x_0, y_0, z_0) \).

3. At \( P \), \( D_{\vec{u}} F(x_0, y_0, z_0) = 0 \) when \( \nabla F(x_0, y_0, z_0) \) and \( \vec{u} \) are orthogonal.

We interpret the third statement of the theorem as “the gradient is orthogonal to level surfaces,” the three–variable analogue to level curves.

Example 5  Finding directional derivatives with functions of three variables
If a point source \( S \) is radiating energy, the intensity \( I \) at a given point \( P \) in space is inversely proportional to the square of the distance between \( S \) and \( P \). That is, when \( S = (0, 0, 0) \), \( I(x, y, z) = \frac{k}{x^2 + y^2 + z^2} \) for some constant \( k \).

Let \( k = 1 \), let \( \vec{u} = (2/3, 2/3, 1/3) \) be a unit vector, and let \( P = (2, 3, 3) \). Measure distances in inches. Find the directional derivative of \( I \) at \( P \) in the direction of \( \vec{u} \), and find the direction of greatest intensity increase at \( P \).

SOLUTION  We need the gradient \( \nabla I \), meaning we need \( I_x, I_y \) and \( I_z \). Each partial derivative requires a simple application of the Quotient Rule, giving

\[
\nabla I = \begin{pmatrix} -2x \over (x^2 + y^2 + z^2)^2 & -2y \over (x^2 + y^2 + z^2)^2 & -2z \over (x^2 + y^2 + z^2)^2 \\
\end{pmatrix} 
\]

\[
\nabla I(2, 5, 3) = \begin{pmatrix} -4/444 \over 444 & -10/444 \over 444 & -6/444 \over 444 \\
\end{pmatrix} 
\]

\[
D_{\vec{u}} I = \nabla I(2, 5, 3) \cdot \vec{u} 
\]

\[
= \begin{pmatrix} -17/2166 \\
\end{pmatrix} 
\]

The directional derivative tells us that moving in the direction of \( \vec{u} \) from \( P \) results in a slight decrease in intensity. (The intensity is decreasing as \( \vec{u} \) moves one farther from the origin than \( P \).)
Chapter 13  Functions of Several Variables

The gradient gives the direction of greatest intensity increase. Notice that

\[
\nabla I(2, 5, 3) = \begin{pmatrix} -4 \\ 1444 \\ -10 \\ 1444 \\ -6 \\ 1444 \end{pmatrix}
\]

\[
= \frac{2}{1444} \begin{pmatrix} -2 \\ 1444 \\ -5 \\ 1444 \\ -3 \end{pmatrix}
\]

That is, the gradient at (2, 5, 3) is pointing in the direction of \((-2, -5, -3)\), that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards the source of the energy.

The directional derivative allows us to find the instantaneous rate of \(z\) change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are tangent to a surface at a point, which is the topic of the next section.
Exercises 13.6

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?
2. For what \( \vec{u} \) is \( D_\vec{u} f = f_x \)?
3. For what \( \vec{u} \) is \( D_\vec{u} f = f_y \)?
4. The gradient is _________ to level curves.
5. The gradient points in the direction of _________ increase.
6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a _________ product.

Problems

In Exercises 7–12, a function \( z = f(x, y) \) is given. Find \( \nabla f \).

7. \( f(x, y) = -x^2y + xy^2 + xy \)
8. \( f(x, y) = \sin x \cos y \)
9. \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \)
10. \( f(x, y) = -4x + 3y \)
11. \( f(x, y) = x^2 + 2y^2 - xy - 7x \)
12. \( f(x, y) = x^4y^3 - 2x \)

In Exercises 13–18, a function \( z = f(x, y) \) and a point \( P \) are given. Find the directional derivative of \( f \) in the indicated directions. Note: these are the same functions as in Exercises 7–12.

13. \( f(x, y) = -x^2y + xy^2 + xy, P = (2, 1) \)
   (a) In the direction of \( \vec{v} = (3, 4) \)
   (b) In the direction toward the point \( Q = (1, -1) \).
14. \( f(x, y) = \sin x \cos y, P = \left( \frac{\pi}{4}, \frac{\pi}{3} \right) \)
   (a) In the direction of \( \vec{v} = (1, 1) \)
   (b) In the direction toward the point \( Q = (0, 0) \).
15. \( f(x, y) = \frac{1}{x^2 + y^2 + 1}, P = (1, 1) \)
   (a) In the direction of \( \vec{v} = (1, -1) \)
   (b) In the direction toward the point \( Q = (-2, -2) \).

16. \( f(x, y) = -4x + 3y, P = (5, 2) \)
   (a) In the direction of \( \vec{v} = (3, 1) \)
   (b) In the direction toward the point \( Q = (2, 7) \).
17. \( f(x, y) = x^2 + 2y^2 - xy - 7x, P = (4, 1) \)
   (a) In the direction of \( \vec{v} = (-2, 5) \)
   (b) In the direction toward the point \( Q = (4, 0) \).
18. \( f(x, y) = x^4y^3 - 2x, P = (1, 1) \)
   (a) In the direction of \( \vec{v} = (3, 3) \)
   (b) In the direction toward the point \( Q = (1, 2) \).

In Exercises 19–24, a function \( z = f(x, y) \) and a point \( P \) are given.

(a) Find the direction of maximal increase of \( f \) at \( P \).
(b) What is the maximal value of \( D_\vec{u} f \) at \( P \)?
(c) Find the direction of minimal increase of \( f \) at \( P \).
(d) Give a direction \( \vec{u} \) such that \( D_\vec{u} f = 0 \) at \( P \).

Note: these are the same functions and points as in Exercises 13–18.

19. \( f(x, y) = -x^2y + xy^2 + xy, P = (2, 1) \)
20. \( f(x, y) = \sin x \cos y, P = \left( \frac{\pi}{4}, \frac{\pi}{3} \right) \)
21. \( f(x, y) = \frac{1}{x^2 + y^2 + 1}, P = (1, 1) \)
22. \( f(x, y) = -4x + 3y, P = (5, 4) \)
23. \( f(x, y) = x^2 + 2y^2 - xy - 7x, P = (4, 1) \)
24. \( f(x, y) = x^4y^3 - 2x, P = (1, 1) \)

In Exercises 25–28, a function \( w = F(x, y, z) \), a vector \( \vec{v} \) and a point \( P \) are given.

(a) Find \( \nabla F(x, y, z) \).
(b) Find \( D_\vec{v} F \) at \( P \).
25. \( F(x, y, z) = 3x^2z^2 + 4xy - 3z^2, \vec{v} = (1, 1, 1) \), \( P = (3, 2, 1) \)
26. \( F(x, y, z) = \sin(x) \cos(y) e^z, \vec{v} = (2, 2, 1), P = (0, 0, 0) \)
27. \( F(x, y, z) = x^3y^2 - y^3z^2, \vec{v} = (-1, 7, 3) \), \( P = (1, 0, -1) \)
28. \( F(x, y, z) = \frac{2}{x^2 + y^2 + z^2}, \vec{v} = (1, 1, -2) \), \( P = (1, 1, 1) \)
13.7 Tangent Lines, Normal Lines, and Tangent Planes

Derivatives and tangent lines go hand-in-hand. Given \( y = f(x) \), the line tangent to the graph of \( f \) at \( x = x_0 \) is the line through \((x_0, f(x_0))\) with slope \( f'(x_0) \); that is, the slope of the tangent line is the instantaneous rate of change of \( f \) at \( x_0 \).

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being “tangent” to the surface.

In Figure 13.20 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be “tangent to a surface.”

**Definition 96 Directional Tangent Line**

Let \( z = f(x, y) \) be differentiable on an open set \( S \) containing \((x_0, y_0)\) and let \( \vec{u} = \langle u_1, u_2 \rangle \) be a unit vector. The line \( \ell_{\vec{u}} \) through \((x_0, y_0, f(x_0, y_0))\) parallel to \( \langle u_1, u_2, D_{\vec{u}}f(x_0, y_0) \rangle \) is the tangent line to \( f \) in the direction of \( \vec{u} \) at \((x_0, y_0)\).

We will also follow the convention that

\[
\ell_{(1,0)} = \ell_x \quad \text{and} \quad \ell_{(0,1)} = \ell_y.
\]

It is instructive to consider each of three directions given in the definition in terms of “slope.” The direction of \( \ell_x \) is \( \langle 1, 0, f_x(x_0, y_0) \rangle \); that is, the “run” is one unit in the \( x \)-direction and the “rise” is \( f_x(x_0, y_0) \) units in the \( z \)-direction. Note how the slope is just the partial derivative with respect to \( x \). A similar statement can be made for \( \ell_y \). The direction of \( \ell_{\vec{u}} \) is \( \langle u_1, u_2, D_{\vec{u}}f(x_0, y_0) \rangle \); the “run” is one unit in the \( \vec{u} \) direction (where \( \vec{u} \) is a unit vector) and the “rise” is the directional derivative of \( z \) in that direction.

Definition 96 leads to the following parametric equations of directional tangent lines:

\[
\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1t \\ y = y_0 + u_2t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}
\]

Notes:
Example 1  

Finding directional tangent lines

Find the lines tangent to the surface \( z = \sin x \cos y \) at \( (\pi/2, \pi/2) \) in the \( x \) and \( y \) directions and also in the direction of \( \vec{v} = (1, 1) \).

**Solution**  
The partial derivatives with respect to \( x \) and \( y \) are:

\[
\begin{align*}
  f_x(x, y) &= \cos x \cos y, \\
  f_y(x, y) &= -\sin x \sin y.
\end{align*}
\]

At \( (\pi/2, \pi/2) \), the \( z \)-value is 0.

Thus the parametric equations of the line tangent to \( f \) at \( (\pi/2, \pi/2) \) in the directions of \( \vec{v} \) are:

\[
\ell_v(t) = \begin{cases}
  x = \pi/2 + t \\
  y = \pi/2 \\
  z = 0
\end{cases}
\]

The two lines are shown with the surface in Figure 13.21(a). To find the equation of the tangent line in the direction of \( \vec{v} \), we first find the unit vector in the direction of \( \vec{v} \): \( \vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle \). The directional derivative at \( (\pi/2, \pi/2) \) in the direction of \( \vec{u} \) is

\[
D_{\vec{u}}f(\pi/2, \pi/2) = \langle 0, -1 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = -1/\sqrt{2}.
\]

Thus the directional tangent line is

\[
\ell_{\vec{u}}(t) = \begin{cases}
  x = \pi/2 - t/\sqrt{2} \\
  y = \pi/2 + t/\sqrt{2} \\
  z = -t/\sqrt{2}
\end{cases}
\]

The curve through \( (\pi/2, \pi/2, 0) \) in the direction of \( \vec{v} \) is shown in Figure 13.21(b) along with \( \ell_{\vec{u}}(t) \).

Notes:

Figure 13.21: A surface and directional tangent lines in Example 1.
Example 2  Finding directional tangent lines
Let \( f(x, y) = 4xy - x^4 - y^4 \). Find the equations of all directional tangent lines to \( f \) at \((1, 1)\).

**Solution**  First note that \( f(1, 1) = 2 \). We need to compute directional derivatives, so we need \( \nabla f \). We begin by computing partial derivatives.

\[
\begin{align*}
  f_x &= 4y - 4x^3 \Rightarrow f_x(1, 1) = 0; \\
  f_y &= 4x - 4y^3 \Rightarrow f_y(1, 1) = 0.
\end{align*}
\]

Thus \( \nabla f(1, 1) = (0, 0) \). Let \( \vec{u} = (u_1, u_2) \) be any unit vector. The directional derivative of \( f \) at \((1, 1)\) will be \( D_{\vec{u}} f(1, 1) = (0, 0) \cdot (u_1, u_2) = 0 \). It does not matter what direction we choose; the directional derivative is always 0. Therefore

\[
\ell_{\vec{u}}(t) = \begin{cases} 
  x = 1 + u_1 t \\
  y = 1 + u_2 t \\
  z = 2
\end{cases}
\]

Figure 13.22 shows a graph of \( f \) and the point \((1, 1, 2)\). Note that this point comes at the top of a “hill,” and therefore every tangent line through this point will have a “slope” of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

**Normal Lines**

When dealing with a function \( y = f(x) \) of one variable, we stated that a line through \((c, f(c))\) was tangent to \( f \) if the line had a slope of \( f'(c) \) and was normal (or, perpendicular, orthogonal) to \( f \) if it had a slope of \(-1/f'(c)\). We extend the concept of normal, or orthogonal, to functions of two variables.

Let \( z = f(x, y) \) be a differentiable function of two variables. By Definition 96, at \((x_0, y_0)\), \( \ell_x(t) \) is a line parallel to the vector \( \vec{d}_x = \langle 1, 0, f_x(x_0, y_0) \rangle \) and \( \ell_y(t) \) is a line parallel to \( \vec{d}_y = \langle 0, 1, f_y(x_0, y_0) \rangle \). Since lines in these directions through \((x_0, y_0, f(x_0, y_0))\) are tangent to the surface, a line through this point and orthogonal to these directions would be orthogonal, or normal, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to \( \vec{d}_x \) and \( \vec{d}_y \), hence the direction is parallel to \( \vec{d}_n = \vec{d}_x \times \vec{d}_y \). It turns out this cross product has a very simple form:

\[
\vec{d}_n = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle.
\]

It is often more convenient to refer to the opposite of this direction, namely \( \langle f_x, f_y, -1 \rangle \). This leads to a definition.

Notes:
13.7 Tangent Lines, Normal Lines, and Tangent Planes

**Definition 97 Normal Line**
Let \( z = f(x, y) \) be differentiable on an open set \( S \) containing \((x_0, y_0)\).

1. A nonzero vector parallel to \( \vec{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1) \) is **orthogonal to** \( f \) at \( P = (x_0, y_0, f(x_0, y_0)) \).

2. The line \( \ell_n \) through \( P \) with direction parallel to \( \vec{n} \) is the **normal line** to \( f \) at \( P \).

Thus the parametric equations of the normal line to a surface \( f \) at \((x_0, y_0, f(x_0, y_0))\) is:

\[
\ell_n(t) = \begin{cases} 
    x = x_0 + f_x(x_0, y_0)t \\
    y = y_0 + f_y(x_0, y_0)t \\
    z = f(x_0, y_0) - t
\end{cases}
\]

**Example 3 Finding a normal line**
Find the equation of the normal line to \( z = -x^2 - y^2 + 2 \) at \((0, 1)\).

**Solution**
We find \( z_x(x, y) = -2x \) and \( z_y(x, y) = -2y \); at \((0, 1)\), we have \( z_x = 0 \) and \( z_y = -2 \). We take the direction of the normal line, following Definition 97, to be \( \vec{n} = (0, -2, -1) \). The line with this direction going through the point \((0, 1, 1)\) is

\[
\ell_n(t) = \begin{cases} 
    x = 0 \\
    y = -2t + 1 \\
    z = -t + 1
\end{cases}
\]

The surface \( z = -x^2 - y^2 + 2 \), along with the found normal line, is graphed in Figure 13.23.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point \( Q \) in space, it is a general geometric concept to define the distance from \( Q \) to the surface as being the length of the shortest line segment \( PQ \) over all points \( P \) on the surface. This, in turn, implies that \( PQ \) will be orthogonal to the surface at \( P \). Therefore we can measure the distance from \( Q \) to the surface \( f \) by finding a point \( P \) on the surface such that \( PQ \) is parallel to the normal line to \( f \) at \( P \).

**Example 4 Finding the distance from a point to a surface**
Let \( f(x, y) = 2 - x^2 - y^2 \) and let \( Q = (2, 2, 2) \). Find the distance from \( Q \) to the surface defined by \( f \).
Chapter 13  Functions of Several Variables

This surface is used in Example 2, so we know that at \((x, y)\), the direction of the normal line will be \(d_n = (-2x, -2y, -1)\). A point \(P\) on the surface will have coordinates \((x, y, 2 - x^2 - y^2)\), so \(\vec{PQ} = (2 - x, 2 - y, x^2 + y^2)\).

To find where \(\vec{PQ}\) is parallel to \(d_n\), we need to find \(x\), \(y\) and \(c\) such that \(c\vec{PQ} = d_n\).

\[
\begin{align*}
\vec{PQ} &= \vec{d_n} \\
\langle 2 - x, 2 - y, x^2 + y^2 \rangle &= \langle -2x, -2y, -1 \rangle.
\end{align*}
\]

This implies

\[
\begin{align*}
c(2 - x) &= -2x \\
c(2 - y) &= -2y \\
c(x^2 + y^2) &= -1
\end{align*}
\]

In each equation, we can solve for \(c\):

\[
c = \frac{-2x}{2 - x} = \frac{-2y}{2 - y} = \frac{-1}{x^2 + y^2}.
\]

The first two fractions imply \(x = y\), and so the last fraction can be rewritten as \(c = -1/(2x^2)\). Then

\[
\begin{align*}
\frac{-2x}{2 - x} &= \frac{-1}{2x^2} \\
-2x(2x^2) &= -1(2 - x) \\
4x^3 &= 2 - x \\
4x^3 + x - 2 &= 0.
\end{align*}
\]

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that \(x = 0.689\), hence \(P = (0.689, 0.689, 1.051)\). We find the distance from \(Q\) to the surface of \(f\) is

\[
||\vec{PQ}|| = \sqrt{(2 - 0.689)^2 + (2 - 0.689)^2 + (2 - 1.051)^2} = 2.083.
\]

We can take the concept of measuring the distance from a point to a surface to find a point \(Q\) a particular distance from a surface at a given point \(P\) on the surface.

**Example 5**  Finding a point a set distance from a surface

Let \(f(x, y) = x - y^2 + 3\). Let \(P = (2, 1, f(2, 1)) = (2, 1, 4)\). Find points \(Q\) in space that are 4 units from the surface of \(f\) at \(P\). That is, find \(Q\) such that \(||\vec{PQ}|| = 4\) and \(\vec{PQ}\) is orthogonal to \(f\) at \(P\).
We begin by finding partial derivatives:

\[ f_x(x, y) = 1 \quad \Rightarrow \quad f_x(2, 1) = 1 \]
\[ f_y(x, y) = -2y \quad \Rightarrow \quad f_y(2, 1) = -2 \]

The vector \( \vec{n} = (1, -2, -1) \) is orthogonal to \( f \) at \( P \). For reasons that will become more clear in a moment, we find the unit vector in the direction of \( \vec{n} \):

\[ \vec{u} = \frac{\vec{n}}{||\vec{n}||} = \frac{(1, -2, -1)}{\sqrt{6}}. \]

Thus a the normal line to \( f \) at \( P \) can be written as

\[ \ell_n(t) = \langle 2, 1, 4 \rangle + \frac{t}{\sqrt{6}} (1, -2, -1). \]

An advantage of this parametrization of the line is that letting \( t = t_0 \) gives a point on the line that is \(|t_0|\) units from \( P \). (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from \( P \):

\[ Q_1 = \ell_n(4) = \left\langle 2 + \frac{4}{\sqrt{6}}, 1 - \frac{8}{\sqrt{6}}, 4 - \frac{4}{\sqrt{6}} \right\rangle \]
\[ Q_2 = \ell_n(-4) = \left\langle 2 - \frac{4}{\sqrt{6}}, 1 + \frac{8}{\sqrt{6}}, 4 + \frac{4}{\sqrt{6}} \right\rangle \]

The surface is graphed along with points \( P, Q_1, Q_2 \) and a portion of the normal line to \( f \) at \( P \).

**Tangent Planes**

We can use the direction of the normal line to define a plane. With \( a = f_x(x_0, y_0), b = f_y(x_0, y_0) \) and \( P = (x_0, y_0, f(x_0, y_0)) \), the vector \( \vec{n} = \langle a, b, -1 \rangle \) is orthogonal to \( f \) at \( P \). The plane through \( P \) with normal vector \( \vec{n} \) is therefore **tangent** to \( f \) at \( P \).

**Definition 98**  
**Tangent Plane**

Let \( z = f(x, y) \) be differentiable on an open set \( S \) containing \( (x_0, y_0) \), where \( a = f_x(x_0, y_0), b = f_y(x_0, y_0), \vec{n} = \langle a, b, -1 \rangle \) and \( P = (x_0, y_0, f(x_0, y_0)) \).

The plane through \( P \) with normal vector \( \vec{n} \) is the **tangent plane to** \( f \) at \( P \). The standard form of this plane is

\[ a(x - x_0) + b(y - y_0) - (z - f(x_0, y_0)) = 0. \]
Chapter 13  Functions of Several Variables

Example 6  Finding tangent planes
Find the equation of the tangent plane to \( z = -x^2 - y^2 + 2 \) at \((0, 1)\).

**SOLUTION**  Note that this is the same surface and point used in Example 3. There we found \( \mathbf{n} = \langle 0, -2, -1 \rangle \) and \( P = (0, 1, 1) \). Therefore the equation of the tangent plane is

\[
-2(y - 1) - (z - 1) = 0.
\]

The surface \( z = -x^2 - y^2 + 2 \) and tangent plane are graphed in Figure 13.25.

Example 7  Using the tangent plane to approximate function values
The point \((3, -1, 4)\) lies on the surface of an unknown differentiable function \( f \) where \( f_x(3, -1) = 2 \) and \( f_y(3, -1) = -1/2 \). Find the equation of the tangent plane to \( f \) at \( P \), and use this to approximate the value of \( f(2.9, -0.8) \).

**SOLUTION**  Knowing the partial derivatives at \((3, -1)\) allows us to form the normal vector to the tangent plane, \( \mathbf{n} = \langle 2, -1/2, -1 \rangle \). Thus the equation of the tangent line to \( f \) at \( P \) is:

\[
2(x - 3) - 1/2(y + 1) - (z - 4) = 0 \quad \Rightarrow \quad z = 2(x - 3) - 1/2(y + 1) + 4. \tag{13.5}
\]

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So \( f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7 \).

This is not a new method of approximation. Compare the right hand expression for \( z \) in Equation (13.5) to the total differential:

\[
dz = f_x dx + f_y dy \quad \text{and} \quad z = \frac{2}{f_x} (x - 3) + \frac{-1/2}{f_y} (y + 1) + 4.
\]

Thus the “new \( z \)-value” is the sum of the change in \( z \) (i.e., \( dz \)) and the old \( z \)-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

The Gradient and Normal Lines, Tangent Planes
The methods developed in this section so far give a straightforward method of finding equations of normal lines and tangent planes for surfaces with explicit...
equations of the form \( z = f(x, y) \). However, they do not handle implicit equations well, such as \( x^2 + y^2 + z^2 = 1 \). There is a technique that allows us to find vectors orthogonal to these surfaces based on the gradient.

Recall that when \( z = f(x, y) \), the gradient \( \nabla f = \langle f_x, f_y \rangle \) is orthogonal to level curves of \( f \). Theorem 115 part 3 made an analogous statement about the gradient \( \nabla F \), where \( w = F(x, y, z) \). Given a point \( (x_0, y_0, z_0) \), let \( c = F(x_0, y_0, z_0) \). Then \( F(x, y, z) = c \) is a level surface that contains the point \( (x_0, y_0, z_0) \) and \( \nabla F(x_0, y_0, z_0) \) is orthogonal to this level surface. This direction can be used to find tangent planes and normal lines.

**Example 8** Using the gradient to find a tangent plane

Find the equation of the plane tangent to the ellipsoid \( \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} = 1 \) at \( P = (1, 2, 1) \).

**Solution** We consider the equation of the ellipsoid as a level surface of a function \( F \) of three variables, where \( F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} \). The gradient is:

\[
\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle = \langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2} \rangle.
\]

At \( P \), the gradient is \( \nabla F(1, 2, 1) = \langle 1/6, 2/3, 1/2 \rangle \). Thus the equation of the plane tangent to the ellipsoid at \( P \) is

\[
\frac{1}{6}(x - 1) + \frac{2}{3}(y - 2) + \frac{1}{2}(z - 1) = 0.
\]

The ellipsoid and tangent plane are graphed in Figure 13.26.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

The next section investigates another use of partial derivatives: determining relative extrema. When dealing with functions of the form \( y = f(x) \), we found relative extrema by finding \( x \) where \( f'(x) = 0 \). We can start finding relative extrema of \( z = f(x, y) \) by setting \( f_x \) and \( f_y \) to 0, but it turns out that there is more to consider.

Notes:
Terms and Concepts

1. Explain how the vector \( \mathbf{v} = (1, 0, 3) \) can be thought of as having a “slope” of 3.
2. Explain how the vector \( \mathbf{v} = (0.6, 0.8, -2) \) can be thought of as having a “slope” of \(-2\).
3. T/F: Let \( z = f(x, y) \) be differentiable at \( P \). If \( \mathbf{n} \) is a normal vector to the tangent plane of \( f \) at \( P \), then \( \mathbf{n} \) is orthogonal to \( f_x \) and \( f_y \) at \( P \).
4. Explain in your own words why we do not refer to the tangent line to a surface at a point, but rather to directional tangent lines to a surface at a point.

Problems

In Exercises 5–8, a function \( z = f(x, y) \), a vector \( \mathbf{v} \) and a point \( P \) are given. Give the parametric equations of the following directional tangent lines to \( f \) at \( P \):

(a) \( f_x(t) \)
(b) \( f_y(t) \)
(c) \( f_z(t) \), where \( \mathbf{u} \) is the unit vector in the direction of \( \mathbf{v} \).

5. \( f(x, y) = 2x^2y - 4xy^2, \mathbf{v} = (1, 3), P = (2, 3) \).
6. \( f(x, y) = 3 \cos x \sin y, \mathbf{v} = (1, 2), P = (\pi/3, \pi/6) \).
7. \( f(x, y) = 3x - 5y, \mathbf{v} = (1, 1), P = (4, 2) \).
8. \( f(x, y) = x^2 - 2x - y^2 + 4y, \mathbf{v} = (1, 1), P = (1, 2) \).

In Exercises 9–12, a function \( z = f(x, y) \) and a point \( P \) are given. Find the equation of the normal line to \( f \) at \( P \). Note: these are the same functions as in Exercises 5–8.

9. \( f(x, y) = 2x^2y - 4xy^2, P = (2, 3) \).
10. \( f(x, y) = 3 \cos x \sin y, P = (\pi/3, \pi/6) \).
11. \( f(x, y) = 3x - 5y, P = (4, 2) \).
12. \( f(x, y) = x^2 - 2x - y^2 + 4y, P = (1, 2) \).

In Exercises 13–16, a function \( z = f(x, y) \) and a point \( P \) are given. Find the two points that are 2 units from the surface \( f \) at \( P \). Note: these are the same functions as in Exercises 5–8.

13. \( f(x, y) = 2x^2y - 4xy^2, P = (2, 3) \).
14. \( f(x, y) = 3 \cos x \sin y, P = (\pi/3, \pi/6) \).
15. \( f(x, y) = 3x - 5y, P = (4, 2) \).
16. \( f(x, y) = x^2 - 2x - y^2 + 4y, P = (1, 2) \).

In Exercises 17–20, a function \( z = f(x, y) \) and a point \( P \) are given. Find the equation of the tangent plane to \( f \) at \( P \). Note: these are the same functions as in Exercises 5–8.

17. \( f(x, y) = 2x^2y - 4xy^2, P = (2, 3) \).
18. \( f(x, y) = 3 \cos x \sin y, P = (\pi/3, \pi/6) \).
19. \( f(x, y) = 3x - 5y, P = (4, 2) \).
20. \( f(x, y) = x^2 - 2x - y^2 + 4y, P = (1, 2) \).

In Exercises 21–24, an implicitly defined function of \( x, y \) and \( z \) is given along with a point \( P \) that lies on the surface. Use the gradient \( \nabla F \) to:

(a) find the equation of the normal line to the surface at \( P \), and

(b) find the equation of the plane tangent to the surface at \( P \).

21. \( \frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{16} = 1 \), at \( P = (1, \sqrt{2}, \sqrt{6}) \)
22. \( x^2 - \frac{x^2}{4} - \frac{y^2}{3} = 0 \), at \( P = (4, -3, \sqrt{5}) \)
23. \( xy^2 - xz^2 = 0 \), at \( P = (2, 1, -1) \)
24. \( \sin(xy) + \cos(yz) = 0 \), at \( P = (2, \pi/12, 4) \)
13.8 Extreme Values

Given a function \( z = f(x, y) \), we are often interested in points where \( z \) takes on the largest or smallest values. For instance, if \( z \) represents a cost function, we would likely want to know what \((x, y)\) values minimize the cost. If \( z \) represents the ratio of a volume to surface area, we would likely want to know where \( z \) is greatest. This leads to the following definition.

**Definition 99 Relative and Absolute Extrema**

Let \( z = f(x, y) \) be defined on a set \( S \) containing the point \( P = (x_0, y_0) \).

1. If there is an open disk \( D \) containing \( P \) such that \( f(x_0, y_0) \geq f(x, y) \) for all \((x, y)\) in \( D \), then \( f \) has a **relative maximum** at \( P \); if \( f(x_0, y_0) \leq f(x, y) \) for all \((x, y)\) in \( D \), then \( f \) has a **relative minimum** at \( P \).

2. If \( f(x_0, y_0) \geq f(x, y) \) for all \((x, y)\) in \( S \), then \( f \) has an **absolute maximum** at \( P \); if \( f(x_0, y_0) \leq f(x, y) \) for all \((x, y)\) in \( S \), then \( f \) has an **absolute minimum** at \( P \).

3. If \( f \) has a relative maximum or minimum at \( P \), then \( f \) has a **relative extrema** at \( P \); if \( f \) has an absolute maximum or minimum at \( P \), then \( f \) has an **absolute extrema** at \( P \).

If \( f \) has a relative or absolute maximum at \( P = (x_0, y_0) \), it means every curve on the surface of \( f \) through \( P \) will also have a relative or absolute maximum at \( P \). Recalling what we learned in Section 3.1, the slopes of the tangent lines to these curves at \( P \) must be 0 or undefined. Since directional derivatives are computed using \( f_x \) and \( f_y \), we are led to the following definition and theorem.

**Definition 100 Critical Point**

Let \( z = f(x, y) \) be continuous on an open set \( S \). A **critical point** \( P = (x_0, y_0) \) of \( f \) is a point in \( S \) such that

- \( f_x(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) = 0 \), or
- \( f_x(x_0, y_0) \) or \( f_y(x_0, y_0) \) is undefined.

Notes:
Chapter 13  Functions of Several Variables

**Theorem 116  Critical Points and Relative Extrema**

Let \( z = f(x, y) \) be defined on an open set \( S \) containing \( P = (x_0, y_0) \). If \( f \) has a relative extrema at \( P \), then \( P \) is a critical point of \( f \).

Therefore, to find relative extrema, we find the critical points of \( f \) and determine which correspond to relative maxima, relative minima, or neither.

Watch the video:
Local Maximum and Minimum Values / Function of Two Variables at [https://youtu.be/Hm5QnuDjNmY](https://youtu.be/Hm5QnuDjNmY)

The following examples demonstrate this process.

**Example 1  Finding critical points and relative extrema**

Let \( f(x, y) = x^2 + y^2 - xy - x - 2 \). Find the relative extrema of \( f \).

**Solution**

We start by computing the partial derivatives of \( f \):

\[
f_x(x, y) = 2x - y - 1 \quad \text{and} \quad f_y(x, y) = 2y - x.
\]

Each is never undefined. A critical point occurs when \( f_x \) and \( f_y \) are simultaneously 0, leading us to solve the following system of linear equations:

\[
2x - y - 1 = 0 \quad \text{and} \quad -x + 2y = 0.
\]

This solution to this system is \( x = 2/3, y = 1/3 \). (Check that at \((2/3, 1/3)\), both \( f_x \) and \( f_y \) are 0.)

The graph in Figure 13.27 shows \( f \) along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

**Example 2  Finding critical points and relative extrema**

Let \( f(x, y) = -\sqrt{x^2 + y^2} + 2 \). Find the relative extrema of \( f \).
We start by computing the partial derivatives of $f$:

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that $f_x = 0$ when $x = 0$ & $y \neq 0$, and that $f_y = 0$ when $y = 0$ & $x \neq 0$. At $(0, 0)$, both $f_x$ and $f_y$ are not 0, but rather undefined. The point $(0, 0)$ is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of $f$.

The surface of $f$ is graphed in Figure 13.28 along with the point $(0, 0, 2)$. The graph shows that this point is the absolute maximum of $f$.

In each of the previous two examples, we found a critical point of $f$ and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

**Example 3 Finding critical points and relative extrema**

Let $f(x, y) = x^3 - 3x - y^2 + 4y$. Find the relative extrema of $f$.

Once again we start by finding the partial derivatives of $f$:

$$f_x(x, y) = 3x^2 - 3 \quad \text{and} \quad f_y(x, y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for $x$ and $y$, we find

$$f_x(x, y) = 0 \quad \Rightarrow \quad x = \pm 1$$
$$f_y(x, y) = 0 \quad \Rightarrow \quad y = 2.$$

We have two critical points: $(-1, 2)$ and $(1, 2)$. To determine if they correspond to a relative maximum or minimum, we consider the graph of $f$ in Figure 13.29.

The critical point $(-1, 2)$ clearly corresponds to a relative maximum. However, the critical point at $(1, 2)$ is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the $y$-axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the $x$-axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

Notes:
**Definition 101 Saddle Point**

Let \( P = (x_0, y_0) \) be in the domain of \( f \) where \( f_x = 0 \) and \( f_y = 0 \) at \( P \). We say \( P \) is a **saddle point** of \( f \) if, for every open disk \( D \) containing \( P \), there are points \( (x_1, y_1) \) and \( (x_2, y_2) \) in \( D \) such that \( f(x_0, y_0) > f(x_1, y_1) \) and \( f(x_0, y_0) < f(x_2, y_2) \).

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with \( z \)-values both less than and greater than the \( z \)-value of the saddle point.

Before Example 3 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of \( f \).

Recall that with single variable functions, such as \( y = f(x) \), if \( f''(c) > 0 \), then \( f \) is concave up at \( c \), and if \( f'(c) = 0 \), then \( f \) has a relative minimum at \( x = c \). (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is “both” concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

\[
\begin{align*}
    f_{xx} \text{ and } f_{yy} > 0 & \implies \text{relative minimum} \\
    f_{xx} \text{ and } f_{yy} < 0 & \implies \text{relative maximum} \\
    f_{xx} \text{ and } f_{yy} \text{ have opposite signs} & \implies \text{saddle point}.
\end{align*}
\]

However, this is not the case. Functions \( f \) exist where \( f_{xx} \) and \( f_{yy} \) are both positive but a saddle point still exists. In such a case, while the concavity in the \( x \)-direction is up (i.e., \( f_{xx} > 0 \)) and the concavity in the \( y \)-direction is also up (i.e., \( f_{yy} > 0 \)), the concavity switches somewhere in between the \( x \)- and \( y \)-directions.

To account for this, consider \( D = f_{xx}f_{yy} - f_{xy}^2 \). Since \( f_{xy} \) and \( f_{yx} \) are equal when continuous (refer back to Theorem 106), we can rewrite this as \( D = f_{xx}f_{yy} - f_{xy}^2 \). Then \( D \) can be used to test whether the concavity at a point changes depending on direction. If \( D > 0 \), the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If \( D < 0 \), the concavity does switch. If \( D = 0 \), our test fails to determine whether concavity switches or not.

We state the use of \( D \) in the following theorem.

Notes:
Theorem 117  Second Derivative Test
Let \( z = f(x, y) \) be defined on an open set containing a critical point \( P = (x_0, y_0) \) where all second order derivatives of \( f \) are continuous at \( P \). Define
\[
D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).
\]
1. If \( D > 0 \) and \( f_{xx}(x_0, y_0) > 0 \), then \( P \) is a relative minimum of \( f \).
2. If \( D > 0 \) and \( f_{xx}(x_0, y_0) < 0 \), then \( P \) is a relative maximum of \( f \).
3. If \( D < 0 \), then \( P \) is a saddle point of \( f \).
4. If \( D = 0 \), the test is inconclusive.

Proof
Let \( \vec{u} = (h, k) \) be a unit vector. Then at the critical point \( P \), \( D\vec{u}f = 0 \). This means that along the line \( h(y - y_0) = k(x - x_0) \), \( P \) is a critical point that is a maximum or minimum according to the sign of \( D\vec{u}f \). Now,
\[
D\vec{u}f = D\vec{u}(f_x + f_y)
= (f_xh)h + (f_xh)k + (f_yk)h + (f_yk)
= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2
\]
because \( f_{xy} \) and \( f_{yx} \) are continuous and therefore equal.

Suppose now that \( D > 0 \). Then we must have \( f_{xx} \neq 0 \), and we can complete the square to see that
\[
D^2\vec{u}f = f_{xx}
\left( h + \frac{f_{xy}k}{f_{xx}} \right)^2 + f_{yy}k^2 - \frac{f_{xy}^2k^2}{f_{xx}} = f_{xx}
\left[ \left( h + \frac{f_{xy}k}{f_{xx}} \right)^2 + \frac{Dk^2}{f_{xx}} \right].
\]
Because we assumed \( D > 0 \), everything in the brackets is positive, and \( D^2\vec{u}f \) always has the same sign as \( f_{xx} \). This shows parts 1 and 2.

If \( D < 0 \), our task is easier because we only need to find two different \( \vec{u} \) that give \( D^2\vec{u}f \) opposite signs. If \( f_{xx} \neq 0 \), let \( \vec{v} = (-f_{xy}, f_{xx}) \) and we can choose
\[
\vec{u}_1 = \frac{1}{\|\vec{v}\|}\vec{v} \quad \Rightarrow \quad D^2\vec{u}_1f = \frac{1}{\|\vec{v}\|}f_{xx}D
\]
\[
\vec{u}_2 = (1, 0) \quad \Rightarrow \quad D^2\vec{u}_2f = f_{xx}.
\]

Notes:
Similarly, if \( f_{yy} \neq 0 \), let \( \vec{v} = (f_{yy}, -f_{yy}) \) and we can choose

\[
\vec{u}_1 = \frac{1}{||\vec{v}||} \vec{v} \quad \Rightarrow \quad D_{\vec{u}_1}^2 f = \frac{1}{||\vec{v}||^2} f_{yy} D
\]

\[
\vec{u}_2 = (0, 1) \quad D_{\vec{u}_2}^2 f = f_{yy}.
\]

Finally, if \( f_{xx} = f_{yy} = 0 \), then \( D_{\vec{u}_1}^2 f = 2f_{xy}h^2 \) has opposite signs for the vectors \( \vec{u}_1 = \frac{1}{\sqrt{2}} (1, 1) \) and \( \vec{u}_2 = \frac{1}{\sqrt{2}} (1, -1) \).

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

**Example 4 Using the Second Derivative Test**

Let \( f(x, y) = x^3 - 3x - y^3 + 4y \) as in Example 3. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

**SOLUTION**

We determined previously that the critical points of \( f \) are \((-1, 2)\) and \((1, 2)\). To use the Second Derivative Test, we must find the second partial derivatives of \( f \):

\[
f_{xx} = 6x; \quad f_{yy} = -2; \quad f_{xy} = 0.
\]

Thus \( D(x, y) = -12x \).

At \((-1, 2)\): \( D(-1, 2) = 12 > 0 \), and \( f_{xx}(-1, 2) = -6 \). By the Second Derivative Test, \( f \) has a relative maximum at \((-1, 2)\).

At \((1, 2)\): \( D(1, 2) = -12 < 0 \). The Second Derivative Test states that \( f \) has a saddle point at \((1, 2)\).

The Second Derivative Test confirmed what we determined visually.

**Example 5 Using the Second Derivative Test**

Find the relative extrema of \( f(x, y) = x^2y + y^2 + xy \).

**SOLUTION**

We start by finding the first and second partial derivatives of \( f \):

\[
f_x = 2xy + y \quad f_y = x^2 + 2y + x
\]

\[
f_{xx} = 2y \quad f_{yy} = 2
\]

\[
f_{xy} = 2x + 1 \quad f_{yx} = 2x + 1.
\]

We find the critical points by finding where \( f_x \) and \( f_y \) are simultaneously 0 (they are both never undefined). Setting \( f_x = 0 \), we have:

\[
f_x = 0 \Rightarrow 2xy + y = 0 \Rightarrow y(2x + 1) = 0.
\]
This implies that for \( f_x = 0 \), either \( y = 0 \) or \( 2x + 1 = 0 \).

Assume \( y = 0 \) then consider \( f_y = 0 \):

\[
\begin{align*}
f_y &= 0 \\
x^2 + 2y + x &= 0, \quad \text{and since } y = 0, \text{ we have } \\
x^2 + x &= 0 \\
x(x + 1) &= 0.
\end{align*}
\]

Thus if \( y = 0 \), we have either \( x = 0 \) or \( x = -1 \), giving two critical points: \((-1, 0)\) and \((0, 0)\).

Going back to \( f_x \), now assume \( 2x + 1 = 0 \), i.e., that \( x = -1/2 \), then consider \( f_y = 0 \):

\[
\begin{align*}
f_y &= 0 \\
x^2 + 2y + x &= 0, \quad \text{and since } x = -1/2, \text{ we have } \\
1/4 + 2y - 1/2 &= 0 \\
y &= 1/8.
\end{align*}
\]

Thus if \( x = -1/2, y = 1/8 \) giving the critical point \((-1/2, 1/8)\).

With \( D = 4y - (2x+1)^2 \), we apply the Second Derivative Test to each critical point.

- At \((-1, 0)\), \( D < 0 \), so \((-1, 0)\) is a saddle point.
- At \((0, 0)\), \( D < 0 \), so \((0, 0)\) is also a saddle point.
- At \((-1/2, 1/8)\), \( D > 0 \) and \( f_{xx} > 0 \), so \((-1/2, 1/8)\) is a relative minimum.

Figure 13.30 shows a graph of \( f \) and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

**Constrained Optimization**

When optimizing functions of one variable such as \( y = f(x) \), we made use of Theorem 22, the Extreme Value Theorem, that said that over a closed interval \( I \), a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated \( f \) at all critical points in the interval, as well as at the endpoints (the “boundary”) of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.
Chapter 13  Functions of Several Variables

**Theorem 118  Extreme Value Theorem**

Let \( z = f(x,y) \) be a continuous function on a closed, bounded set \( S \). Then \( f \) has a maximum and minimum value on \( S \).

**Example 6  Finding extrema on a closed set**

Let \( f(x,y) = x^2 - y^2 + 5 \) and let \( S \) be the triangle with vertices \((-1, -2), (0, 1)\) and \((2, -2)\). Find the maximum and minimum values of \( f \) on \( S \).

**SOLUTION**  

It can help to see a graph of \( f \) along with the set \( S \). In Figure 13.31(a) the triangle defining \( S \) is shown in the \( x-y \) plane in a dashed line. Above it is the surface of \( f \); we are only concerned with the portion of \( f \) enclosed by the “triangle” on its surface.

We begin by finding the critical points of \( f \). With \( f_x = 2x \) and \( f_y = -2y \), we find only one critical point, at \((0, 0)\).

We now find the maximum and minimum values that \( f \) attains along the boundary of \( S \), that is, along the edges of the triangle. In Figure 13.31(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line \( y = -2 \). If \( y = -2 \), then on the surface, we are considering points \( f(x, -2) \); that is, our function reduces to \( f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x) \). We want to maximize/minimize \( f_1(x) = x^2 + 1 \) on the interval \([-1, 2]\). To do so, we evaluate \( f_1(x) \) at its critical points and at the endpoints. The critical points of \( f_1 \) are found by setting its derivative equal to 0:

\[
f_1'(x) = 0 \quad \Rightarrow \quad x = 0,
\]

so that we will need to evaluate \( f \) at the points \((-1, -2), (0, -2), \) and \((2, -2)\).

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line \( y = 3x + 1 \), we substitute \( 3x + 1 \) in for \( y \) in \( f(x, y) \):

\[
f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).
\]

We want the maximum and minimum values of \( f_2 \) on the interval \([-1, 0]\), so we evaluate \( f_2 \) at its critical points and the endpoints of the interval. We find the critical points:

\[
f_2'(x) = -16x - 6 = 0 \quad \Rightarrow \quad x = -3/8,
\]

Notes:
so that we will need to evaluate \( f \) at the points \((-1, -2), (-\frac{3}{8}, -\frac{1}{8})\), and \((0, 1)\).

Finally, we evaluate \( f \) along the right edge of the triangle, where \( y = -3/2x + 1 \).

\[
f(x, y) = f(x, -3/2x + 1) = x^2 - (-3/2x + 1)^2 + 5 = \frac{5}{4}x^2 + 3x + 4 = f_3(x).
\]

The critical points of \( f_3(x) \) are:

\[
f_3'(x) = 0 \implies x = 6/5,
\]

so that we will need to evaluate \( f \) at the points \((0, 1), (\frac{6}{5}, -\frac{4}{5}), \) and \((2, -2)\).

We now evaluate \( f \) at a total of 7 different places, all shown in Figure 13.32.

\[
f(-1, -2) = 2, \quad f(0, -2) = 1, \quad f(2, -2) = 5, \quad f\left(-\frac{3}{8}, -\frac{1}{8}\right) = \frac{41}{8}, \quad f(0, 1) = 4, \quad \text{and} \quad f\left(\frac{6}{5}, -\frac{4}{5}\right) = \frac{29}{5}.
\]

Of all the \( z \)-values found, the maximum is \( \frac{29}{5} \), found at \((\frac{6}{5}, -\frac{4}{5})\); the minimum is 1, found at \((0, -2)\).

This portion of the text is entitled “Constrained Optimization” because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a constraint – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real “meaning” behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

**Example 7 Constrained Optimization**

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130”. Given a rectangular box, the “length” is the longest side, and the “girth” is twice the width+height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

**Solution** Let \( w, h, \) and \( \ell \) denote the width, height, and length of a rectangular box; we assume here that \( w = h \). The girth is then \( 2(w + h) = 4w \).

The volume of the box is \( V(w, \ell) = wh\ell = w^2\ell \). We wish to maximize this
volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0$, $w > 0$, so that we don’t need to check the boundary where either is zero.)

We begin by finding the critical values of $V$. We find that $V_w = 2w\ell$ and $V_\ell = w^2$; these are simultaneously 0 only at $(0, 0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w, \ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the $w$-interval $[0, 32.5]$ as indicated in the figure. Thus we want to maximize $V_1$ on $[0, 32.5]$.

Finding the critical values of $V_1$, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \Rightarrow w(260 - 12w) = 0 \Rightarrow w = 0, \frac{260}{12} \approx 21.67.$$ 

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution; the maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a volume of $V(21.67, 43.33) \approx 19,408\text{in}^3$.

The volume function $V(w, \ell)$ is shown in Figure 13.33 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the $x$-$y$ plane and also along the surface of the function. The point where the volume is maximized is indicated.

It is hard to overemphasize the importance of optimization. In “the real world,” we routinely seek to make something better. By expressing the something as a mathematical function, “making something better” means “optimize some function.”

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of “find the gradient and set it equal to $\vec{0}$” highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.
**Terms and Concepts**

1. T/F: Theorem 116 states that if \( f \) has a critical point at \( P \), then \( f \) has a relative extrema at \( P \).
2. T/F: A point \( P \) is a critical point of \( f \) if \( f_x \) and \( f_y \) are both 0 at \( P \).
3. T/F: A point \( P \) is a critical point of \( f \) if \( f_x \) or \( f_y \) are undefined at \( P \).
4. Explain what it means to “solve a constrained optimization” problem.

**Problems**

In Exercises 5–14, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

5. \( f(x, y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x \)
6. \( f(x, y) = x^2 + 4x + y^2 - 9y + 3xy \)
7. \( f(x, y) = x^2 + 3y^2 - 6y + 4xy \)
8. \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \)
9. \( f(x, y) = x^2 + y^3 - 3y + 1 \)
10. \( f(x, y) = \frac{1}{3}x^3 - x + \frac{1}{3}y^3 - 4y \)
11. \( f(x, y) = x^2y^2 \)
12. \( f(x, y) = x^4 - 2x^2 + y^2 - 27y - 15 \)
13. \( f(x, y) = \sqrt{16 - (x - 3)^2 - y^2} \)
14. \( f(x, y) = \sqrt{x^2 + y^2} \)

In Exercises 15–18, find the absolute maximum and minimum of the function subject to the given constraint.

15. \( f(x, y) = x^2 + y^2 + y + 1 \), constrained to the triangle with vertices \((0, 1), (-1, -1)\) and \((1, -1)\).
16. \( f(x, y) = 5x - 7y \), constrained to the region bounded by \( y = x^2 \) and \( y = 1 \).
17. \( f(x, y) = x^2 + 2x + y^2 + 2y \), constrained to the region bounded by the circle \( x^2 + y^2 = 4 \).
18. \( f(x, y) = 3y - 2x^2 \), constrained to the region bounded by the parabola \( y = x^2 + x - 1 \) and the line \( y = x \).
13.9 Lagrange Multipliers

In the previous section, we were concerned with finding maxima and minima of functions without any constraints on the variables (other than being in the domain of the function). We ended by discussing what we would do if there were constraints on the variables. The following example illustrates a simple case of this type of problem.

Example 1 Maximizing an Area
For a rectangle whose perimeter is 20 m, find the dimensions that will maximize the area.

**SOLUTION** The area $A$ of a rectangle with width $x$ and height $y$ is $A = xy$. The perimeter $P$ of the rectangle is then given by the formula $P = 2x + 2y$. Since we are given that the perimeter $P = 20$, this problem can be stated as:

Maximize $f(x, y) = xy$ subject to $2x + 2y = 20$

The reader is probably familiar with a simple method, using single-variable calculus, for solving this problem. Since we must have $2x + 2y = 20$, then we can solve for, say, $y$ in terms of $x$ using that equation. This gives $y = 10 - x$, which we then substitute into $f$ to get $f(x, y) = xy = x(10 - x) = 10x - x^2$. This is now a function of $x$ alone, so we now just have to maximize the function $f(x) = 10x - x^2$ on the interval $[0, 10]$. Since $f'(x) = 10 - 2x = 0 \Rightarrow x = 5$ and $f''(5) = -2 < 0$, then the Second Derivative Test tells us that $x = 5$ is a local maximum for $f$, and hence $x = 5$ must be the global maximum on the interval $[0, 10]$ (since $f = 0$ at the endpoints of the interval). So since $y = 10 - x = 5$, then the maximum area occurs for a rectangle whose width and height both are 5 m.

**Note:** Joseph Louis Lagrange (1736–1813) was a French mathematician and astronomer.

Notice in the above example that the ease of the solution depended on being able to solve for one variable in terms of the other in the equation $2x + 2y = 20$. But what if that were not possible (which is often the case)? In this section we will use a general method, called the Lagrange multiplier method, for solving constrained optimization problems:

Maximize (or minimize) $f(x, y)$ subject to $g(x, y) = c$

for some constant $c$. The equation $g(x, y) = c$ is called the constraint equation, and we say that $x$ and $y$ are constrained by $g(x, y) = c$. Points $(x, y)$ which are maxima or minima of $f(x, y)$ with the condition that they satisfy the constraint equation $g(x, y) = c$ are called constrained maximum or constrained minimum points, respectively. Similar definitions hold for functions of three variables.
The previous section optimized a function on a set $S$. In this section, “subject to $g(x, y) = c$” is the same as saying that the set $S$ is given by $\{ (x, y) | g(x, y) = c \}$. The Lagrange multiplier method for solving such problems can now be stated:

**Theorem 119   Lagrange Multipliers**

Let $f(x, y)$ and $g(x, y)$ be functions with continuous partial derivatives of all orders, and suppose that $c$ is a scalar constant such that $\nabla g(x, y) \neq \vec{0}$ for all $(x, y)$ that satisfy the equation $g(x, y) = c$. Then to solve the constrained optimization problem

Maximize (or minimize) $f(x, y)$ subject to $g(x, y) = c$,

find the points $(x, y)$ that solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some constant $\lambda$ (the number $\lambda$ is called the *Lagrange multiplier*). If there is a constrained maximum or minimum, then it must be at such a point.

A rigorous proof of the above theorem is well beyond the scope of this text. Note that the theorem only gives a *necessary* condition for a point to be a constrained maximum or minimum. Whether a point $(x, y)$ that satisfies $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda$ actually is a constrained maximum or minimum can sometimes be determined by the nature of the problem itself. For instance, in Example 1 it was clear that there had to be a global maximum.

So how can you tell when a point that satisfies the condition in Theorem 119 really is a constrained maximum or minimum? The answer is that it depends on the constraint function $g(x, y)$, together with any implicit constraints. It can be shown that if the constraint equation $g(x, y) = c$ (plus any hidden constraints) describes a bounded set $B$ in $\mathbb{R}^2$, then the constrained maximum or minimum of $f(x, y)$ will occur either at a point $(x, y)$ satisfying $\nabla f(x, y) = \lambda \nabla g(x, y)$ or at a “boundary” point of the set $B$.

In Example 1 the constraint equation $2x + 2y = 20$ describes a line in $\mathbb{R}^2$, which by itself is not bounded. However, there are “hidden” constraints, due to

---

**Notes:**
the nature of the problem, namely $0 \leq x, y \leq 10$, which cause that line to be restricted to a line segment in $\mathbb{R}^2$ (including the endpoints of that line segment), which is bounded.

**Example 2  Maximizing an Area**  
For a rectangle whose perimeter is 20 m, use the Lagrange multiplier method to find the dimensions that will maximize the area.

**Solution** As we saw in Example 1, with $x$ and $y$ representing the width and height, respectively, of the rectangle, this problem can be stated as:

Maximize $f(x, y) = xy$ subject to $g(x, y) = 2x + 2y = 20$.

Then solving the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda$ means solving the equations $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$, namely:

\[
\begin{align*}
y &= 2\lambda, \\
x &= 2\lambda
\end{align*}
\]

The general idea is to solve for $\lambda$ in both equations, then set those expressions equal (since they both equal $\lambda$) to solve for $x$ and $y$. Doing this we get

\[
\frac{y}{2} = \lambda = \frac{x}{2} \implies x = y,
\]

so now substitute either of the expressions for $x$ or $y$ into the constraint equation to solve for $x$ and $y$:

\[
20 = g(x, y) = 2x + 2y = 2x + 2x = 4x \implies x = 5 \implies y = 5
\]

There must be a maximum area, since the minimum area is 0 and $f(5, 5) = 25 > 0$, so the point $(5, 5)$ that we found (called a constrained critical point) must be the constrained maximum. Therefore, the maximum area occurs for a rectangle whose width and height both are 5 m.

**Example 3  Extreme Values on a Circle**  
Find the points on the circle $x^2 + y^2 = 80$ which are closest to and farthest from the point $(1, 2)$.

**Solution** The distance $d$ from any point $(x, y)$ to the point $(1, 2)$ is

\[
d = \sqrt{(x - 1)^2 + (y - 2)^2},
\]

Notes:
and minimizing the distance is equivalent to minimizing the square of the distance. Thus the problem can be stated as:

Maximize (and minimize) \( f(x, y) = (x-1)^2 + (y-2)^2 \) subject to \( g(x, y) = x^2 + y^2 = 80 \).

Solving \( \nabla f(x, y) = \lambda \nabla g(x, y) \) means solving the following equations:

\[
\begin{align*}
2(x - 1) &= 2 \lambda x, \\
2(y - 2) &= 2 \lambda y
\end{align*}
\]

Note that \( x \neq 0 \) since otherwise we would get \(-2 = 0\) in the first equation. Similarly, \( y \neq 0 \). So we can solve both equations for \( \lambda \) as follows:

\[
\frac{x - 1}{x} = \lambda = \frac{y - 2}{y} \quad \Rightarrow \quad xy - y = xy - 2x \quad \Rightarrow \quad y = 2x
\]

Substituting this into \( g(x, y) = x^2 + y^2 = 80 \) yields \( 5x^2 = 80 \), so \( x = \pm 4 \). So the two constrained critical points are \((4, 8)\) and \((-4, -8)\). Since \( f(4, 8) = 45 \) and \( f(-4, -8) = 125 \), and since there must be points on the circle closest to and farthest from \((1, 2)\), then it must be the case that \((4, 8)\) is the point on the circle closest to \((1, 2)\) and \((-4, -8)\) is the farthest from \((1, 2)\) (see Figure 13.34).

Notice that since the constraint equation \( x^2 + y^2 = 80 \) describes a circle, which is a bounded set in \( \mathbb{R}^2 \), then we were guaranteed that the constrained critical points we found were indeed the constrained maximum and minimum.

The Lagrange multiplier method can be extended to functions of three variables.

**Example 4 Maximizing a Function of Three Variables**
Maximize (and minimize) \( f(x, y, z) = x + z \) subject to \( g(x, y, z) = x^2 + y^2 + z^2 = 1 \).

**SOLUTION** Solve the equation \( \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \):

\[
\begin{align*}
1 &= 2 \lambda x, \\
0 &= 2 \lambda y, \\
1 &= 2 \lambda z
\end{align*}
\]

The first equation implies \( \lambda \neq 0 \) (otherwise we would have \( 1 = 0 \)), so we can divide by \( \lambda \) in the second equation to get \( y = 0 \) and we can divide by \( \lambda \) in the first and third equations to get \( x = \frac{1}{2\lambda} = z \). Substituting these expressions into the constraint equation \( g(x, y, z) = x^2 + y^2 + z^2 = 1 \) yields the constrained critical points \( \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \) and \( \left( -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \). Since \( f \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) > f \left( -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \),

Notes:
and since the constraint equation \( x^2 + y^2 + z^2 = 1 \) describes a sphere (which is bounded) in \( \mathbb{R}^3 \), then \( \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \) is the constrained maximum point and \( \left( -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \) is the constrained minimum point.

### Two Constraints

When we have two constraints, we can still use Lagrange multipliers once we’ve made a slight modification. The optimization problem

Maximize (or minimize) \( f(x, y, z) \) subject to \( g(x, y, z) = c_1 \) and \( h(x, y, z) = c_2 \)

is satisfied when \( \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \).

**Example 5 Optimizing with Two Constraints**

The plane \( x - y + z = 2 \) intersects the cylinder \( x^2 + y^2 = 4 \) in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

**Solution**

We can optimize the distance \( \sqrt{x^2 + y^2 + z^2} \) by optimizing the function \( f(x, y, z) = x^2 + y^2 + z^2 \), which has a simpler derivative. Let \( g(x, y, z) = x - y + z \) be the plane constraint, and \( h(x, y, z) = x^2 + y^2 \) be the cylinder constraint. We see that

\[
\begin{align*}
\nabla f(x, y, z) &= \langle 2x, 2y, 2z \rangle \\
\nabla g(x, y, z) &= \langle 1, -1, 1 \rangle \\
\nabla h(x, y, z) &= \langle 2x, 2y, 0 \rangle .
\end{align*}
\]

The equation \( \nabla f = \lambda \nabla g + \mu \nabla h \) means that

\[
\begin{align*}
2x &= \lambda + 2\mu x \\
2y &= -\lambda + 2\mu y \\
2z &= \lambda .
\end{align*}
\]

Adding the first two equations tells us that \( x + y = \mu(x + y) \), so that \( \mu = 1 \) or \( x = -y \). If \( \mu = 1 \), then \( \lambda = z = 0 \), and the constraint equations become

\[
\begin{align*}
x - y &= 2 \\
x^2 + y^2 &= 4 .
\end{align*}
\]

Substituting \( x = y + 2 \) into \( x^2 + y^2 = 4 \) tells us that \((y + 2)^2 + y^2 = 4\), which simplifies to \( 2y(y + 2) = 0 \). This means that we need to look at the points...
\[(2, 0, 0)\) and \((0, -2, 0)\), which are both distance 2 from the origin. If \(x = -y\), then the constraint equations become
\[
2x + z = 2 \\
2x^2 = 4
\]
and we need to look at the points \((\pm \sqrt{2}, \mp \sqrt{2}, 2 \mp 2\sqrt{2})\). These have distance
\[
\sqrt{2 + 2 + (2 \mp 2\sqrt{2})^2} = \sqrt{16 \mp 8\sqrt{2}},
\]
which are both greater than 2. Therefore, the closest points are \((2, 0, 0)\) and \((0, -2, 0)\), while the furthest point is \((-\sqrt{2}, \sqrt{2}, 2 + \sqrt{2})\).

Finally, note that solving the equation \(\nabla f(x, y) = \lambda \nabla g(x, y)\) means having to solve a system of two (possibly nonlinear) equations in three unknowns, which as we have seen before, may not be possible to do. And the 3-variable case can get even more complicated. All of this somewhat restricts the usefulness of Lagrange's method to relatively simple functions. Luckily there are many numerical methods for solving constrained optimization problems, though we will not discuss them here.

Notes:
Exercises 13.9

Problems

1. Find the constrained maxima and minima of \( f(x, y) = 2x + y \) given that \( x^2 + y^2 = 4 \).
2. Find the constrained maxima and minima of \( f(x, y) = xy \) given that \( x^2 + 3y^2 = 6 \).
3. Find the points on the circle \( x^2 + y^2 = 100 \) which are closest to and farthest from the point \((2, 3)\).
4. Find the constrained maxima and minima of \( f(x, y, z) = x + y^2 + 2z \) given that \( 4x^2 + 9y^2 - 36z^2 = 36 \).
5. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).
6. Find the minimum surface area of a box that holds 2 cubic meters.
7. The girth of a box is the perimeter of a cross section perpendicular to its length. The US post office will accept packages whose combined length and girth are at most 130 inches. Find the dimensions of the largest volume box that will be accepted.
8. Using Lagrange multipliers, find the shortest distance from the point \((x_0, y_0, z_0)\) to the plane \( ax + by + cz = d \). (See also Key Idea 54.)
9. Find all points on the surface \( xz - y^2 + 1 = 0 \) that are closest to the origin.
10. Find the three positive numbers whose sum is 60 and whose product is as large as possible.
11. Find all points on the plane \( x + y + z = 5 \) in the first octant at which \( f(x, y, z) = x^2yz^2 \) has a maximum value.
12. Find the points on the surface \( x^2 - xy = 5 \) that are closest to the origin.
13. Find the maximum and minimum points of \( f(x, y) = xy + \sqrt{9 - x^2 - y^2} \) when \( x^2 + y^2 \leq 9 \).
14. Find three real numbers whose sum is 12 and the sum of whose squares is a small as possible.
15. Find the maximum volume of a rectangular box inscribed in the unit sphere.
16. The plane \( x + y - z = 1 \) intersects the cylinder \( x^2 + y^2 = 1 \) in an ellipse. Find the points on the ellipse closest to and farthest from the origin.
14: **Multiple Integration**

The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter 5 we learned how the definite integral of a single variable function gave us “area under the curve.” In this chapter we will see that integration applied to a multivariable function gives us “volume under a surface.” And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

### 14.1 Iterated Integrals and Area

In Chapter 13 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that \( f_x(x, y) = 2xy \), we can treat \( y \) as staying constant and integrate to obtain \( f(x, y) \):

\[
\begin{align*}
  f(x, y) &= \int f_x(x, y) \, dx \\
  &= \int 2xy \, dx \\
  &= x^2y + C.
\end{align*}
\]

Make a careful note about the constant of integration, \( C \). This “constant” is something with a derivative of 0 with respect to \( x \), so it could be any expression that contains only constants and functions of \( y \). For instance, if \( f(x, y) = x^2y + \sin y + y^3 + 17 \), then \( f_x(x, y) = 2xy \). To signify that \( C \) is actually a function of \( y \), we write:

\[
  f(x, y) = \int f_x(x, y) \, dx = x^2y + C(y).
\]

Using this process we can even evaluate definite integrals.

**Example 1** Integrating functions of more than one variable

Evaluate the integral \( \int_1^{2y} 2xy \, dx \).
We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

\[
\int_1^{2y} 2xy \, dx = x^2y \bigg|_1^{2y} = (2y)^2y - (1)^2y = 4y^3 - y.
\]

We can also integrate with respect to \(y\). In general,

\[
\int_{h_1(y)}^{h_2(y)} f_x(x, y) \, dx = f(x, y) \bigg|_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y),
\]

and

\[
\int_{g_1(x)}^{g_2(x)} f_y(x, y) \, dy = f(x, y) \bigg|_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)).
\]

Note that when integrating by \(x\), the bounds do not depend on \(x\), and the result is no longer a function of \(x\). When integrating by \(y\), the bounds do not depend on \(y\), and the result is no longer a function of \(y\). Another example will help us understand this.

**Example 2**

**Integrating functions of more than one variable**

Evaluate \(\int_1^x (5x^3y^{-3} + 6y^2) \, dy\).

**Solution**

We consider \(x\) as staying constant and integrate with respect to \(y\):

\[
\int_1^x (5x^3y^{-3} + 6y^2) \, dy = \left( \frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3} \right) \bigg|_1^x = \left( \frac{-5}{2}x^3 + 2x^2 \right) - \left( \frac{-5}{2} + 2 \right) = \frac{9}{2}x^3 - \frac{5}{2}x - 2.
\]

Note how the bounds of the integral are from \(y = 1\) to \(y = x\) and that the final answer is a function of \(x\).

In the previous example, we integrated a function with respect to \(y\) and ended up with a function of \(x\). We can integrate this as well. This process is known as *iterated integration*, or *multiple integration*.

Notes:
Example 3  Integrating an integral

Evaluate \( \int_1^2 \left( \int_1^x (5x^3y^{-3} + 6y^2) \, dy \right) \, dx \).

SOLUTION  We follow a standard “order of operations” and perform the operations inside parentheses first (which is the integral evaluated in Example 2.)

\[
\begin{align*}
\int_1^2 \left( \int_1^x (5x^3y^{-3} + 6y^2) \, dy \right) \, dx &= \int_1^2 \left( \int_1^x \left( \frac{5x^3y^{-2}}{2} + \frac{6y^3}{3} \right) \, dy \right) \, dx \\
&= \int_1^2 \left( \frac{9}{2}x^3 - \frac{5}{2}x - 2 \right) \, dx \\
&= \left( \frac{9}{8} - \frac{5}{4}x^2 - \frac{2x}{1} \right) \bigg|_1^2 \\
&= \frac{89}{8}.
\end{align*}
\]

Note how the bounds of \( x \) were \( x = 1 \) to \( x = 2 \) and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know why we would be interested in doing so nor what the result, such as the number \( \frac{89}{8} \), means. Before we investigate these questions, we offer some definitions.
Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let \( a, b, c \) and \( d \) be numbers and let \( g_1(x), g_2(x), h_1(y) \) and \( h_2(y) \) be functions of \( x \) and \( y \), respectively. Then:

1. \[
   \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \right) \, dy.
   \]

2. \[
   \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right) \, dx.
   \]

Again make note of the bounds of these iterated integrals.

With \( \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \), \( x \) varies from \( h_1(y) \) to \( h_2(y) \), whereas \( y \) varies from \( c \) to \( d \). That is, the bounds of \( x \) are curves, the curves \( x = h_1(y) \) and \( x = h_2(y) \), whereas the bounds of \( y \) are constants, \( y = c \) and \( y = d \). It is useful to remember that after integrating with respect to a variable, that variable is no longer present.

We now begin to investigate why we are interested in iterated integrals and what they mean.

Area of a plane region

Consider the plane region \( R \) bounded by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), shown in Figure 14.1. We learned in Section 6.1 that the area of \( R \) is given by

\[
   \text{area of } R = \int_a^b (g_2(x) - g_1(x)) \, dx.
\]

We can view the expression \( (g_2(x) - g_1(x)) \) as

\[
   (g_2(x) - g_1(x)) = \int_{g_1(x)}^{g_2(x)} 1 \, dy = \int_{g_1(x)}^{g_2(x)} dy,
\]

meaning we can express the area of \( R \) as an iterated integral:

\[
   \text{area of } R = \int_a^b (g_2(x) - g_1(x)) \, dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} dy \right) \, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx.
\]

Figure 14.1: Calculating the area of a plane region \( R \) with an iterated integral.
In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region $R$ could also be defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, as shown in Figure 14.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy.$$ 

We state this formally in a theorem.

**Theorem 120  Area of a plane region**

1. Let $R$ be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where $g_1$ and $g_2$ are continuous functions on $[a, b]$. The area $A$ of $R$ is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx.$$ 

2. Let $R$ be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where $h_1$ and $h_2$ are continuous functions on $[c, d]$. The area $A$ of $R$ is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy.$$ 

The following examples should help us understand this theorem.

**Example 4  Area of a rectangle**

Find the area $A$ of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 14.3.

**Solution**  Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region $R$ is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to $y$ first, we have

$$A = \int_{-1}^3 \int_{1}^3 1 \, dy \, dx = \int_{-1}^3 \left( y \Big|_1^3 \right) dx = \int_{-1}^3 2 \, dx = 2 \Big|_{-1}^3 = 8.$$ 

We could also integrate with respect to $x$ first, giving:

$$A = \int_{-1}^3 \int_{-1}^3 1 \, dx \, dy = \int_{-1}^3 \left( x \Big|_{-1}^3 \right) dy = \int_{-1}^3 4 \, dy = 4 \Big|_{-1}^3 = 8.$$ 

**Notes:**
Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

Example 5  

**Area of a triangle**  

Find the area of the triangle with vertices at (1, 1), (3, 1) and (5, 5), as shown in Figure 14.4.

**Solution** The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$ to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since $x$-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y + 5)/2$, is the upper bound.) The area is

\[
A = \int_{1}^{5} \int_{y}^{\frac{y+5}{2}} dx 
\]

\[
= \int_{1}^{5} \left( x \left|_{y}^{\frac{y+5}{2}} \right. \right) dy
\]

\[
= \int_{1}^{5} \left( \frac{1}{2}y + \frac{5}{2} \right) dy
\]

\[
= \left( \frac{1}{4}y^2 + \frac{5}{2}y \right)_{1}^{5}
\]

\[
= 4.
\]

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region $R$, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

\[
A = \int_{1}^{3} \int_{1}^{x} 1 dy 
\]

\[
\left. \frac{1}{2}y^2 \right|_{x}^{1} dx
\]

\[
= \int_{1}^{3} (x - 1) dx
\]

\[
= 2
\]

\[
A = \int_{3}^{5} \int_{2x-5}^{x} 1 dy 
\]

\[
\left. \frac{1}{2}y^2 \right|_{2x-5}^{x} dx
\]

\[
= \int_{3}^{5} (-x + 5) dx
\]

\[
= 2
\]

\[
= 4.
\]

As expected, we get the same answer both ways. This equality will also be justified by Theorem 122 in the next section.

Notes:
Example 6  Area of a plane region
Find the area of the region enclosed by \( y = 2x \) and \( y = x^2 \), as shown in Figure 14.5.

**SOLUTION** Once again we'll find the area of the region using both orders of integration.

Using \( dy \ dx \):

\[
\int_0^2 \int_{x^2}^{2x} 1 \, dy \, dx = \int_0^2 (2x - x^2) \, dx = \left( x^2 - \frac{1}{3}x^3 \right) \bigg|_0^2 = \frac{4}{3}.
\]

Using \( dx \ dy \):

\[
\int_0^4 \int_{\sqrt{y}}^{\sqrt{y} - 2} 1 \, dx \, dy = \int_0^4 \left( \sqrt{y} - y/2 \right) \, dy = \left( \frac{2}{3}y^{3/2} - \frac{1}{4}y^2 \right) \bigg|_0^4 = \frac{4}{3}.
\]

Changing Order of Integration
In each of the previous examples, we have been given a region \( R \) and found the bounds needed to find the area of \( R \) using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 4), and so:

\[
\int_a^b \int_c^d 1 \, dy \, dx = \int_c^d \int_a^b 1 \, dx \, dy.
\]

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 7  Changing the order of integration
Rewrite the iterated integral \( \int_0^b \int_0^{x^3/3} 1 \, dy \, dx \) with the order of integration \( dx \ dy \).
We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that \( y \) is bounded by \( 0 \) and \( x/3 \); \( x \) is bounded by \( 0 \) and \( 6 \). We plot these four curves: \( y = 0 \), \( y = x/3 \), \( x = 0 \) and \( x = 6 \) to find the region described by the bounds. Figure 14.6 shows these curves, indicating that \( R \) is a triangle.

To change the order of integration, we need to consider the curves that bound the \( x \)-values. We see that the lower bound is \( x = 3y \) and the upper bound is \( x = 6 \). The bounds on \( y \) are \( 0 \) to \( 2 \). Thus we can rewrite the integral as

\[
\int_0^6 \int_{3y}^{6} 1 \, dx \, dy.
\]

**Example 8** Changing the order of integration

Change the order of integration of

\[
\int_0^4 \int_{y^2/4}^{(y+4)/2} 1 \, dx \, dy.
\]

**Solution** We sketch the region described by the bounds to help us change the integration order. We see \( x \) is bounded below and above (i.e., to the left and right) by \( x = y^2/4 \) and \( x = (y + 4)/2 \) respectively, and \( y \) is bounded between \( 0 \) and \( 4 \). Graphing the previous curves, we find the region \( R \) to be that shown in Figure 14.7.

To change the order of integration, we need to establish curves that bound \( y \). The figure makes it clear that there are two lower bounds for \( y \): \( y = 0 \) on \( 0 \leq x \leq 2 \), and \( y = 2x - 4 \) on \( 2 \leq x \leq 4 \). Thus we need two double integrals. The upper bound for each is \( y = 2\sqrt{x} \). Thus we have

\[
\int_0^4 \int_{y^2/4}^{(y+4)/2} 1 \, dx \, dy = \int_0^2 \int_{0}^{2\sqrt{x}} 1 \, dy \, dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1 \, dy \, dx.
\]

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The “real” goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

---

**Notes:**

---

858


Terms and Concepts

1. When integrating \( f(x, y) \) with respect to \( x \), the constant of integration \( C \) is really which: \( C(x) \) or \( C(y) \)? What does this mean?

2. Integrating an integral is called ________ ________.

3. When evaluating an iterated integral, we integrate from ________ to ________, then from ________ to ________.

4. One understanding of an iterated integral is that

\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx
\]

gives the ________ of a plane region.

Problems

In Exercises 5–10, evaluate the integral and subsequent iterated integral.

5. (a) \( \int_2^5 \left( 6x^2 + 4xy - 3y^2 \right) \, dy \)

(b) \( \int_3^5 \int_2^7 \left( 6x^2 + 4xy - 3y^2 \right) \, dy \, dx \)

6. (a) \( \int_0^\pi \left( 2x \cos y + \sin x \right) \, dx \)

(b) \( \int_0^{\pi/2} \int_0^\pi \left( 2x \cos y + \sin x \right) \, dx \, dy \)

7. (a) \( \int_1^2 \left( x^2y - y + 2 \right) \, dy \)

(b) \( \int_0^2 \int_1^2 \left( x^2y - y + 2 \right) \, dy \, dx \)

8. (a) \( \int_y^2 \left( x - y \right) \, dx \)

(b) \( \int_{-1}^1 \int_y^2 \left( x - y \right) \, dx \, dy \)

9. (a) \( \int_0^\pi \left( \cos x \sin y \right) \, dx \)

(b) \( \int_0^\pi \int_0^\pi \left( \cos x \sin y \right) \, dx \, dy \)

10. (a) \( \int_0^2 \left( \frac{1}{1 + x^2} \right) \, dy \)

(b) \( \int_1^2 \int_0^2 \left( \frac{1}{1 + x^2} \right) \, dy \, dx \)

In Exercises 11–16, a graph of a planar region \( R \) is given. Give the iterated integrals, with both orders of integration \( dy \, dx \) and \( dx \, dy \), that give the area of \( R \). Evaluate one of the iterated integrals to find the area.

11. \( R \)

12. \( R \)

13. \( R \)

14. \( R \)
In Exercises 17–22, iterated integrals are given that compute the area of a region $R$ in the $x$-$y$ plane. Sketch the region $R$, and give the iterated integral(s) that give the area of $R$ with the opposite order of integration.

17. $\int_{-2}^{2} \int_{0}^{4-x^2} dy \, dx$

18. $\int_{0}^{5} \int_{5}^{-5x^2} dy \, dx$

19. $\int_{-2}^{2} \int_{0}^{2\sqrt{4-y^2}} dx \, dy$

20. $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy \, dx$

21. $\int_{0}^{\sqrt{7}} dx \, dy + \int_{\sqrt{7}}^{\sqrt{9}} \int_{\sqrt{7}}^{\sqrt{9}} dx \, dy$

22. $\int_{-1}^{1} \int_{(1-x)^2/2}^{(x-1)^2/2} dy \, dx$
14.2 Double Integration and Volume

The definite integral of \( f \) over \([a, b]\), \( \int_a^b f(x) \, dx \), was introduced as “the signed area under the curve.” We approximated the value of this area by first subdividing \([a, b]\) into \(n\) subintervals, where the \(i^{th}\) subinterval has length \(\Delta x_i\), and letting \(c_i\) be any value in the \(i^{th}\) subinterval. We formed rectangles that approximated part of the region under the curve with width \(\Delta x_i\), height \(f(c_i)\), and hence with area \(f(c_i) \Delta x_i\). Summing all the rectangle’s areas gave an approximation of the definite integral, and Theorem 36 stated that

\[
\int_a^b f(x) \, dx = \lim_{\|\Delta x\| \to 0} \sum f(c_i) \Delta x_i,
\]

connecting the area under the curve with sums of the areas of rectangles.

We use a similar approach in this section to find volume under a surface.

Let \(R\) be a closed, bounded region in the \(x-y\) plane and let \(z = f(x, y)\) be a continuous function defined on \(R\). We wish to find the signed volume under the surface of \(f\) over \(R\). (We use the term “signed volume” to denote that space above the \(x-y\) plane, under \(f\), will have a positive volume; space above \(f\) and under the \(x-y\) plane will have a “negative” volume, similar to the notion of signed area used before.)

We start by partitioning \(R\) into \(n\) rectangular subregions as shown in Figure 14.8(a). For simplicity’s sake, we let all widths be \(\Delta x\) and all heights be \(\Delta y\). Note that the sum of the areas of the rectangles is not equal to the area of \(R\), but rather is a close approximation. Arbitrarily number the rectangles 1 through \(n\), and pick a point \((x_i, y_i)\) in the \(i^{th}\) subregion.

The volume of the rectangular solid whose base is the \(i^{th}\) subregion and whose height is \(f(x_i, y_i)\) is \(V_i = f(x_i, y_i) \Delta x \Delta y\). Such a solid is shown in Figure 14.8(b). Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

For each subregion \(R_i\) used to approximate \(R\), create the rectangular solid with base area \(\Delta x \Delta y\) and height \(f(x_i, y_i)\). The sum of all rectangular solids is

\[
\sum_{i=1}^{n} f(x_i, y_i) \Delta x \Delta y.
\]

This approximates the signed volume under \(f\) over \(R\). As we have done before, to get a better approximation we can use more rectangles to approximate the region \(R\).

In general, each rectangle could have a different width \(\Delta x_i\) and height \(\Delta y_k\), giving the \(i^{th}\) rectangle an area \(\Delta A_i = \Delta x_i \Delta y_k\) and the \(i^{th}\) rectangular solid a

Notes:
Chapter 14  Multiple Integration

**Note:** Recall that the integration symbol “∫” is an “elongated S,” representing the word “sum.” We interpreted \( \int_a^b f(x) \, dx \) as “take the sum of the areas of rectangles over the interval \([a, b]\).” The double integral uses two integration symbols to represent a “double sum.” When adding up the volumes of rectangular solids over a partition of a region \(R\), as done in Figure 14.8, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x_i \Delta y_j.
\]

One can rewrite this as

\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{m} f(x_i, y_j) \Delta x_i \right) \Delta y_j.
\]

The summation inside the parenthesis indicates the sum of heights \(\times\) widths, which gives an area; multiplying these areas by the thickness \(\Delta y_j\) gives a volume. The illustration in Figure 14.9 relates to this understanding.

Volume of \(f(x_i, y_j) \Delta A_i\). Let \(\| \Delta A \|\) denote the length of the longest diagonal of all rectangles in the subdivision of \(R\); \(\| \Delta A \| \to 0\) means each rectangle’s width and height are both approaching 0. If \(f\) is a continuous function, as \(\| \Delta A \|\) shrinks (and hence \(n \to \infty\)) the summation \(\sum_{i=1}^{n} f(x_i, y_i) \Delta A_i\) approximates the signed volume better and better. This leads to a definition.

**Definition 103  Double Integral, Signed Volume**

Let \(z = f(x, y)\) be a continuous function defined over a closed region \(R\) in the \(x-y\) plane. The **signed volume** \(V\) under \(f\) over \(R\) is denoted by the **double integral**

\[
V = \iint_R f(x, y) \, dA.
\]

Alternate notations for the double integral are

\[
\iint_R f(x, y) \, dA = \iint_R f(x, y) \, dx \, dy = \iint_R f(x, y) \, dy \, dx.
\]

The definition above does not state how to find the signed volume, though the notation offers a hint. We need the next two theorems to evaluate double integrals to find volume.

**Theorem 121  Double Integrals and Signed Volume**

Let \(z = f(x, y)\) be a continuous function defined over a closed region \(R\) in the \(x-y\) plane. Then the signed volume \(V\) under \(f\) over \(R\) is

\[
V = \iint_R f(x, y) \, dA = \lim_{\| \Delta A \| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i.
\]

This theorem states that we can find the exact signed volume using a limit of sums. The partition of the region \(R\) is not specified, so any partitioning where the diagonal of each rectangle shrinks to 0 results in the same answer.

This does not offer a very satisfying way of computing volume, though. Our experience has shown that evaluating the limits of sums can be tedious. We seek a more direct method.

Recall Theorem 45 in Section 6.2. This stated that if \(A(x)\) gives the cross-sectional area of a solid at \(x\), then \(\int_a^b A(x) \, dx\) gave the volume of that solid over \([a, b]\).

**Notes:**
Consider Figure 14.9, where a surface \( z = f(x, y) \) is drawn over a region \( R \). Fixing a particular \( x \) value, we can consider the area under \( f \) over \( R \) where \( x \) has that fixed value. That area can be found with a definite integral, namely

\[
A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy.
\]

Remember that though the integrand contains \( x \), we are viewing \( x \) as fixed. Also note that the bounds of integration are functions of \( x \): the bounds depend on the value of \( x \).

As \( A(x) \) is a cross-sectional area function, we can find the signed volume \( V \) under \( f \) by integrating it:

\[
V = \int_a^b A(x) \, dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right) \, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
\]

This gives a concrete method for finding signed volume under a surface. We could do a similar procedure where we started with \( y \) fixed, resulting in a iterated integral with the order of integration \( dx \, dy \). The following theorem states that both methods give the same result, which is the value of the double integral. It is such an important theorem it has a name associated with it.

**Theorem 122 Fubini’s Theorem**

Let \( R \) be a closed, bounded region in the \( x-y \) plane and let \( z = f(x, y) \) be a continuous function on \( R \).

1. If \( R \) is bounded by \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \), where \( g_1 \) and \( g_2 \) are continuous functions on \([a, b]\), then

\[
\iint_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
\]

2. If \( R \) is bounded by \( c \leq y \leq d \) and \( h_1(y) \leq x \leq h_2(y) \), where \( h_1 \) and \( h_2 \) are continuous functions on \([c, d]\), then

\[
\iint_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
\]

Note that the bounds of integration follow a “curve to curve, point to point” pattern. In fact, one of the main points of the previous section is developing the skill of describing a region \( R \) with the bounds of an iterated integral. Once this
skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

Example 1  Evaluating a double integral
Let \( f(x, y) = xy + e^y \). Find the signed volume under \( f \) on the region \( R \), which is the rectangle with corners \((3, 1)\) and \((4, 2)\) pictured in Figure 14.10, using Fubini’s Theorem and both orders of integration.

**SOLUTION** We wish to evaluate \( \iint_R (xy + e^y) \, dA \). As \( R \) is a rectangle, the bounds are easily described as \( 3 \leq x \leq 4 \) and \( 1 \leq y \leq 2 \).

Using the order \( dy \, dx \):

\[
\iint_R (xy + e^y) \, dA = \int_3^4 \int_1^2 (xy + e^y) \, dy \, dx
\]
\[
= \int_3^4 \left[ \frac{1}{2}xy^2 + e^y \right]_1^2 \, dx
\]
\[
= \int_3^4 \left( \frac{3}{2}x + e^2 - e \right) \, dx
\]
\[
= \left[ \frac{3}{4}x^2 + (e^2 - e)x \right]_3^4
\]
\[
= \frac{21}{4} + e^2 - e.
\]
Now we check the validity of Fubini’s Theorem by using the order $dx \, dy$:

\[
\iint_R (xy + e^y) \, dA = \int_1^2 \int_3^4 (xy + e^y) \, dx \, dy
\]

\[
= \int_1^2 \left( \frac{1}{2} x^2 y + xe^y \right) \bigg|_3^4 \, dy
\]

\[
= \int_1^2 \left( \frac{7}{2} y + e^y \right) \, dy
\]

\[
= \left( \frac{7}{4} y^2 + e^y \right) \bigg|_1^2
\]

\[
= \frac{21}{4} + e^2 - e.
\]

Both orders of integration return the same result, as expected.

**Example 2** Evaluating a double integral

Evaluate \( \iint_R (3xy - x^2 - y^2 + 6) \, dA \), where \( R \) is the triangle bounded by \( x = 0 \), \( y = 0 \) and \( x/2 + y = 1 \), as shown in Figure 14.11.

**Solution** While it is not specified which order we are to use, we will evaluate the double integral using both orders to help drive home the point that it does not matter which order we use.

Using the order $dy \, dx$: The bounds on $y$ go from “curve to curve,” i.e., $0 \leq y \leq 1 - x/2$, and the bounds on $x$ go from “point to point,” i.e., $0 \leq x \leq 2$.

\[
\iint_R (3xy - x^2 - y^2 + 6) \, dA = \int_0^2 \int_0^{1 - x/2} (3xy - x^2 - y^2 + 6) \, dy \, dx
\]

\[
= \int_0^2 \left( \frac{3}{2} xy^2 - x^2 y - \frac{1}{3} y^3 + 6y \right) \bigg|_0^{1 - x/2} \, dx
\]

\[
= \int_0^2 \left( \frac{11}{12} x^3 - \frac{11}{4} x^2 - x + \frac{17}{3} \right) \, dx
\]

\[
= \left( \frac{11}{48} x^4 - \frac{11}{12} x^3 - \frac{1}{2} x^2 + \frac{17}{3} x \right) \bigg|_0^2
\]

\[
= \frac{17}{3} = 5.6.
\]

Now lets consider the order $dx \, dy$. Here $x$ goes from “curve to curve,” $0 \leq
Chapter 14  Multiple Integration

\[ x \leq 2 - 2y, \text{ and } y \text{ goes from “point to point,” } 0 \leq y \leq 1: \]

\[
\iint_R (3xy - x^2 - y^2 + 6) \, dA = \int_0^1 \int_0^{2-2y} (3xy - x^2 - y^2 + 6) \, dx \, dy
\]

\[
= \int_0^1 \left[ \frac{3}{2}x^2y - \frac{1}{3}x^3 - xy^2 + 6x \right]_0^{2-2y} \, dy
\]

\[
= \int_0^1 \left( \frac{32}{3}y^3 - 22y^2 + 2y + \frac{28}{3} \right) \, dy
\]

\[
= \left( \frac{8}{3}y^4 - \frac{22}{3}y^3 + y^2 + \frac{28}{3}y \right)_{0}^{1}
\]

\[
= \frac{17}{3} = 5.6.
\]

We obtained the same result using both orders of integration.

Note how in these two examples that the bounds of integration depend only on \( R \); the bounds of integration have nothing to do with \( f(x, y) \). This is an important concept, so we include it as a Key Idea.

**Key Idea 60  Double Integration Bounds**

When evaluating \( \iint_R f(x, y) \, dA \) using an iterated integral, the bounds of integration depend only on \( R \). The surface \( f \) does not determine the bounds of integration.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

Notes:
14.2 Double Integration and Volume

**Theorem 123** Properties of Double Integrals
Let $f$ and $g$ be continuous functions over a closed, bounded plane region $R$, and let $c$ be a constant.

1. $\int_R c f(x, y) \, dA = c \int_R f(x, y) \, dA$.

2. $\int_R (f(x, y) \pm g(x, y)) \, dA = \int_R f(x, y) \, dA \pm \int_R g(x, y) \, dA$.

3. If $f(x, y) \geq 0$ on $R$, then $\int_R f(x, y) \, dA \geq 0$.

4. If $f(x, y) \geq g(x, y)$ on $R$, then $\int_R f(x, y) \, dA \geq \int_R g(x, y) \, dA$.

5. Let $R$ be the union of two nonoverlapping regions, $R = R_1 \cup R_2$ (see Figure 14.12). Then

$$ \int_R f(x, y) \, dA = \int_{R_1} f(x, y) \, dA + \int_{R_2} f(x, y) \, dA.$$ 

**Example 3** Evaluating a double integral
Let $f(x, y) = \sin x \cos y$ and $R$ be the triangle with vertices $(-1, 0), (1, 0)$ and $(0, 1)$ (see Figure 14.13). Evaluate the double integral $\int_R f(x, y) \, dA$.

**Solution** If we attempt to integrate using an iterated integral with the order $dy \, dx$, note how there are two upper bounds on $R$ meaning we’ll need to use two iterated integrals. We would need to split the triangle into two regions along the $y$-axis, then use Theorem 123, part 5.

Instead, let’s use the order $dx \, dy$. The curves bounding $x$ are $y - 1 \leq x \leq 1 - y$; the bounds on $y$ are $0 \leq y \leq 1$. This gives us:

$$ \int_R f(x, y) \, dA = \int_0^1 \int_{y-1}^{1-y} \sin x \cos y \, dx \, dy $$

$$ = \int_0^1 \left[ -\cos x \cos y \right]_{y-1}^{1-y} \, dy $$

$$ = \int_0^1 \cos y \left( -\cos(1-y) + \cos(y-1) \right) \, dy. $$

Recall that the cosine function is an even function; that is, $\cos x = \cos(-x)$. Therefore, from the last integral above, we have $\cos(y-1) = \cos(1-y)$. Thus
the integrand simplifies to 0, and we have

\[ \int_R f(x, y) \, dA = \int_0^1 0 \, dy = 0. \]

It turns out that over \( R \), there is just as much volume above the \( x-y \) plane as below (look again at Figure 14.13), giving a final signed volume of 0.

**Example 4**  **Evaluating a double integral**

Evaluate \( \iint_R (4 - y) \, dA \), where \( R \) is the region bounded by the parabolas \( y^2 = 4x \) and \( x^2 = 4y \), graphed in Figure 14.14.

**Solution**  
Graphing each curve can help us find their points of intersection. Solving analytically, the second equation tells us that \( y = x^2/4 \). Substituting this value in for \( y \) in the first equation gives us \( x^4/16 = 4x \). Solving for \( x \):

\[
\begin{align*}
\frac{x^4}{16} &= 4x \\
x^4 - 64x &= 0 \\
x(x^3 - 64) &= 0 \\
x &= 0, 4.
\end{align*}
\]

Thus we’ve found analytically what was easy to approximate graphically: the regions intersect at \((0, 0)\) and \((4, 4)\), as shown in Figure 14.14.

We now choose an order of integration: \( dy \, dx \) or \( dx \, dy \)? Either order works; since the integrand does not contain \( x \), choosing \( dx \, dy \) might be simpler — at least, the first integral is very simple.

Thus we have the following "curve to curve, point to point" bounds: \( y^2/4 \leq x \leq 2\sqrt{y} \), and \( 0 \leq y \leq 4 \).

\[
\begin{align*}
\iint_R (4 - y) \, dA &= \int_0^4 \int_{\sqrt{y}/4}^{2\sqrt{y}} (4 - y) \, dx \, dy \\
&= \int_0^4 \left( x(4 - y) \right)_{\sqrt{y}/4}^{2\sqrt{y}} \, dy \\
&= \int_0^4 \left( 2\sqrt{y} - \frac{y^2}{4} \right) (4 - y) \, dy \\
&= \int_0^4 \left( \frac{y^3}{4} - y^2 - 2y^{3/2} + 8y^{1/2} \right) \, dy \\
&= \left( \frac{y^4}{16} - \frac{y^3}{3} - \frac{4y^{5/2}}{5} + \frac{16y^{3/2}}{3} \right) \bigg|_0^4 \\
&= \frac{176}{15} = 11.73.
\end{align*}
\]

Notes:
The signed volume under the surface $f$ is about 11.7 cubic units.

In the previous section we practiced changing the order of integration of a given iterated integral, where the region $R$ was not explicitly given. Changing the bounds of an integral is more than just an test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

**Example 5**  **Changing the order of integration**

Rewrite the iterated integral $\int_0^3 \int_y^3 e^{-x^2} \, dx \, dy$ with the order $dy \, dx$. Comment on the feasibility to evaluate each integral.

**Solution**  Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on $x$ are from $x = y$ to $x = 3$; the bounds on $y$ are from $y = 0$ to $y = 3$. These curves are sketched in Figure 14.15, enclosing the region $R$.

To change the bounds, note that the curves bounding $y$ are $y = 0$ up to $y = x$; the triangle is enclosed between $x = 0$ and $x = 3$. Thus the new bounds of integration are $0 \leq y \leq x$ and $0 \leq x \leq 3$, giving the iterated integral $\int_0^3 \int_0^x e^{-x^2} \, dy \, dx$.

How easy is it to evaluate each iterated integral? Consider the order of integrating $dx \, dy$, as given in the original problem. The first indefinite integral we need to evaluate is $\int e^{-x^2} \, dx$; we have stated before (see Section 8.7) that this integral cannot be evaluated in terms of elementary functions. We are stuck.

Changing the order of integration makes a big difference here. In the second iterated integral, we are faced with $\int e^{-x^2} \, dy$; integrating with respect to $y$ gives us $ye^{-x^2} + C$, and the first definite integral evaluates to

$$\int_0^x e^{-x^2} \, dy = xe^{-x^2}.$$  

Thus

$$\int_0^3 \int_0^x e^{-x^2} \, dy \, dx = \int_0^3 (xe^{-x^2}) \, dx.$$  

This last integral is easy to evaluate with substitution, giving a final answer of $\frac{1}{2}(1 - e^{-9})$. Figure 14.16 shows the surface over $R$.

In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple.

Notes:
Definition 25 defines the average value of a single-variable function \( f(x) \) on the interval \([a, b]\) as

\[
\text{average value of } f(x) \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(x) \, dx;
\]

that is, it is the “area under \( f \) over an interval divided by the length of the interval.” We make an analogous statement here: the average value of \( z = f(x, y) \) over a region \( R \) is the volume under \( f \) over \( R \) divided by the area of \( R \).

**Definition 104** The Average Value of \( f \) on \( R \\
Let \( z = f(x, y) \) be a continuous function defined over a closed region \( R \) in the \( x-y \) plane. The **average value of \( f \) on \( R \)** is

\[
\text{average value of } f \text{ on } R = \frac{\iint_R f(x, y) \, dA}{\iint_R dA}.
\]

**Example 6** Finding average value of a function over a region \( R \\
Find the average value of \( f(x, y) = 4 - y \) over the region \( R \), which is bounded by the parabolas \( y^2 = 4x \) and \( x^2 = 4y \). Note: this is the same function and region as used in Example 4.

**Solution** In Example 4 we found

\[
\iint_R f(x, y) \, dA = \int_0^4 \int_{y/4}^{\sqrt{y}} (4 - y) \, dx \, dy = \frac{176}{15}.
\]

We find the area of \( R \) by computing \( \iint_R dA \):

\[
\iint_R dA = \int_0^4 \int_{y/4}^{\sqrt{y}} dx \, dy = \frac{16}{3}.
\]

Dividing the volume under the surface by the area gives the average value:

\[
\text{average value of } f \text{ on } R = \frac{176/15}{16/3} = \frac{11}{5} = 2.2.
\]

While the surface, as shown in Figure 14.17, covers \( z \)-values from \( z = 0 \) to \( z = 4 \), the “average” \( z \)-value on \( R \) is 2.2.

Notes:
The previous section introduced the iterated integral in the context of finding the area of plane regions. This section has extended our understanding of iterated integrals; now we see they can be used to find the signed volume under a surface.

This new understanding allows us to revisit what we did in the previous section. Given a region \( R \) in the plane, we computed \( \iint_R 1 \, dA \); again, our understanding at the time was that we were finding the area of \( R \). However, we can now view the function \( z = 1 \) as a surface, a flat surface with constant \( z \)-value of 1. The double integral \( \iint_R 1 \, dA \) finds the volume, under \( z = 1 \), over \( R \), as shown in Figure 14.18. Basic geometry tells us that if the base of a general right cylinder has area \( A \), its volume is \( A \cdot h \), where \( h \) is the height. In our case, the height is 1. We were “actually” computing the volume of a solid, though we interpreted the number as an area.

The next section extends our abilities to find “volumes under surfaces.” Currently, some integrals are hard to compute because either the region \( R \) we are integrating over is hard to define with rectangular curves, or the integrand itself is hard to deal with. Some of these problems can be solved by converting everything into polar coordinates.

Figure 14.18: Showing how an iterated integral used to find area also finds a certain volume.
Exercises 14.2

Terms and Concepts

1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed ______ over a region.
2. Explain why the following statement is false: "Fubini's Theorem states that \( \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy."
3. Explain why if \( f(x, y) > 0 \) over a region \( R \), then \( \iint_R f(x, y) \, dA > 0 \).
4. If \( \iint_R f(x, y) \, dA = \iint_R g(x, y) \, dA \), does this imply \( f(x, y) = g(x, y) \)?

Problems

In Exercises 5–10,

(a) Evaluate the given iterated integral, and
(b) rewrite the integral using the other order of integration.

5. \( \int_1^2 \int_{-1}^1 \left( \frac{x}{y} + 3 \right) \, dx \, dy \)
6. \( \int_{-\pi/2}^{\pi/2} \int_0^\pi \left( \sin x \cos y \right) \, dx \, dy \)
7. \( \int_0^3 \int_0^{3x^2 - y + 2} (3x^2 - y + 2) \, dy \, dx \)
8. \( \int_0^3 \int_0^x (x^2 - xy^2) \, dx \, dy \)
9. \( \int_0^1 \int_{\sqrt{x}}^{\sqrt{x+y}} (x + y + 2) \, dx \, dy \)
10. \( \int_0^\pi \int_{\sqrt{3}}^{\sqrt{7}} (xy^2) \, dx \, dy \)

In Exercises 11–18:

(a) Sketch the region \( R \) given by the problem.
(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region \( R \).
(c) Evaluate one of the iterated integrals to find the signed volume under the surface \( z = f(x, y) \) over the region \( R \).
11. \( \iint_R x^2 y \, dA \), where \( R \) is bounded by \( y = \sqrt{x} \) and \( y = x^2 \).
12. \( \iint_R x^2 y \, dA \), where \( R \) is bounded by \( y = \sqrt{x} \) and \( y = x^3 \).
13. \( \iint_R x^2 - y^2 \, dA \), where \( R \) is the rectangle with corners \((-1, -1), (1, -1), (1, 1) \) and \((-1, 1) \).
14. \( \iint_R ye^2 \, dA \), where \( R \) is bounded by \( x = 0, x = y^2 \) and \( y = 1 \).
15. \( \iint_R (6 - 3x - 2y) \, dA \), where \( R \) is bounded by \( x = 0, y = 0 \) and \( 3x + 2y = 6 \).
16. \( \iint_R e^x \, dA \), where \( R \) is bounded by \( y = \ln x \) and \( y = \frac{1}{e+1}(x+1) \).
17. \( \iint_R (x^2 - y - x) \, dA \), where \( R \) is the half disk \( x^2 + y^2 \leq 9 \) in the first and second quadrants.
18. \( \iint_R (4 - 3y) \, dA \), where \( R \) is bounded by \( y = 0, y = x/e \) and \( y = \ln x \).

In Exercises 19–22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.
19. \( \int_0^1 \int_{1/2}^{1/2} e^{x^2} \, dx \, dy \)
20. \( \int_0^{\sqrt{\pi}/2} \int_x^{\sqrt{\pi}/2} \cos (y^2) \, dy \, dx \)
21. \( \int_0^1 \int_0^{2y/x} (x^2 + y^2) \, dx \, dy \)
22. \( \int_0^1 \int_1^{\sqrt{1 + y}} x \tan^2 y \, dy \, dx \)

In Exercises 23–26, find the average value of \( f \) over the region \( R \). Notice how these functions and regions are related to the iterated integrals given in Exercises 5–8.
23. \( f(x, y) = \frac{x}{y} + 3; \quad R \) is the rectangle with opposite corners \((-1, 1) \) and \((1, 2) \).
24. \( f(x, y) = \sin x \cos y; \quad R \) is bounded by \( x = 0, x = \pi, y = -\pi/2 \) and \( y = \pi/2 \).
25. \( f(x, y) = 3x^2 - y + 2; \quad R \) is bounded by the lines \( y = 0, y = 2 - x/2 \) and \( x = 0 \).
26. \( f(x, y) = x^2 y - xy^2; \quad R \) is bounded by \( y = x, y = 1 \) and \( x = 3 \).
We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, \( z = f(x, y) \), over a region \( R \) of the \( x\)-\( y \) plane. The integrand is simply \( f(x, y) \), and the bounds of the integrals are determined by the region \( R \).

Some regions \( R \) are easy to describe using rectangular coordinates – that is, with equations of the form \( y = f(x) \), \( x = a \), etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form \( r = f(\theta) \), \( \theta = \alpha \), etc.

The basic form of the double integral is

\[
\int_R f(x, y) \, dA.
\]

We interpret this integral as follows: over the region \( R \), sum up lots of products of heights (given by \( f(x_i, y_i) \)) and areas (given by \( \Delta A_i \)). That is, \( dA \) represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area \( dx \, dy \) or \( dy \, dx \) – the area of a rectangle is simply length \( \times \) width – a small change in \( x \) times a small change in \( y \). Thus we replace \( dA \) in the double integral with \( dx \, dy \) or \( dy \, dx \).

Now consider representing a region \( R \) with polar coordinates. Consider Figure 14.19(a). Let \( R \) be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius \( r \), subtended by an angle \( \theta \), is \( A = \frac{1}{2} r^2 \theta \), we can find the area of the shaded region. The whole sector has area \( \frac{1}{2} r_2^2 \Delta \theta \), whereas the smaller, unshaded sector has area \( \frac{1}{2} r_1^2 \Delta \theta \). The area of the shaded region is the difference of these areas:

\[
\Delta A_i = \frac{1}{2} r_2^2 \Delta \theta - \frac{1}{2} r_1^2 \Delta \theta = \frac{1}{2} (r_2^2 - r_1^2) (\Delta \theta) = \frac{r_2 + r_1}{2} (r_2 - r_1) \Delta \theta.
\]

Note that \( (r_2 + r_1)/2 \) is just the average of the two radii.

To approximate the region \( R \), we use many such subregions; doing so shrinks the difference \( r_2 - r_1 \) between radii to 0 and shrinks the change in angle \( \Delta \theta \) also to 0. We represent these infinitesimal changes in radius and angle as \( dr \) and \( d\theta \), respectively. Finally, as \( dr \) is small, \( r_2 \approx r_1 \), and so \( (r_2 + r_1)/2 \approx r_1 \). Thus, when \( dr \) and \( d\theta \) are small,

\[
\Delta A_i \approx r_1 \, dr \, d\theta.
\]

Taking a limit, where the number of subregions goes to infinity and both \( r_2 - r_1 \) and \( \Delta \theta \) go to 0, we get

\[
dA = r \, dr \, d\theta.
\]

So to evaluate \( \iint_R f(x, y) \, dA \), replace \( dA \) with \( r \, dr \, d\theta \). Convert the function \( z = f(x, y) \) to a function with polar coordinates with the substitutions \( x = r \cos \theta \), \( y = r \sin \theta \).
$y = r \sin \theta$. Finally, find bounds $g_1(\theta) \leq r \leq g_2(\theta)$ and $\alpha \leq \theta \leq \beta$ that describe $R$. This is the key principle of this section, so we restate it here as a Key Idea.

**Key Idea 61** Evaluating Double Integrals with Polar Coordinates
Let $R$ be a plane region bounded by the polar equations $\alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)$. Then

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$ 


Examples will help us understand this Key Idea.

**Example 1** Evaluating a double integral with polar coordinates
Find the signed volume under the plane $z = 4 - x - 2y$ over the disk with equation $x^2 + y^2 \leq 1$.

**SOLUTION** The bounds of the integral are determined solely by the region $R$ over which we are integrating. The surface and region are shown in Figure 14.20. In this case, it is a circle with equation $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review Section 10.4; bounds for this circle are $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

We replace $f(x, y)$ with $f(r \cos \theta, r \sin \theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad \Rightarrow \quad 4 - r \cos \theta - 2r \sin \theta.$$ 

Finally, we replace $dA$ in the double integral with $r \, dr \, d\theta$. This gives the final
iterated integral, which we evaluate:

\[
\iint_R f(x, y) \, dA = \int_0^{2\pi} \int_0^1 (4 - r \cos \theta - 2r \sin \theta) \, r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 (4r - r^2 (\cos \theta - 2 \sin \theta)) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3 (\cos \theta - 2 \sin \theta)\right) \bigg|_0^1 \, d\theta
\]

\[
= \int_0^{2\pi} \left(2 - \frac{1}{3}(\cos \theta - 2 \sin \theta)\right) \, d\theta
\]

\[
= \left(2\theta - \frac{1}{3}(\sin \theta + 2 \cos \theta)\right) \bigg|_0^{2\pi}
\]

\[
= 4\pi.
\]

**Example 2  Evaluating a double integral with polar coordinates**

Find the volume under the paraboloid \(z = 4 - (x - 2)^2 - y^2\) over the region bounded by the circles \((x - 1)^2 + y^2 = 1\) and \((x - 2)^2 + y^2 = 4\).

**SOLUTION**  At first glance, this seems like a very hard volume to compute as the region \(R\) (shown in Figure 14.21(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing \(R\) in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle \((x - 1)^2 + y^2 = 1\) has polar equation \(r = 2 \cos \theta\), and that the circle \((x - 2)^2 + y^2 = 4\) has polar equation \(r = 4 \cos \theta\). Each of these circles is traced out on the interval \(0 \leq \theta \leq \pi\). The bounds on \(r\) are \(2 \cos \theta \leq r \leq 4 \cos \theta\).

Replacing \(x\) with \(r \cos \theta\) in the integrand, along with replacing \(y\) with \(r \sin \theta\), prepares us to evaluate the double integral \(\iint_R f(x, y) \, dA\):

Notes:
\[
\iint_{R} f(x, y) \, dA = \int_{0}^{\pi} \int_{2 \cos \theta}^{4 \cos \theta} \left( 4 - (r \cos \theta - 2)^2 - (r \sin \theta)^2 \right) r \, dr \, d\theta \\
= \int_{0}^{\pi} \int_{2 \cos \theta}^{4 \cos \theta} \left( - r^3 + 4r^2 \cos \theta \right) \, dr \, d\theta \\
= \int_{0}^{\pi} \left( \left. \left( - \frac{1}{4} r^4 + \frac{4}{3} r^3 \cos \theta \right) \right|_{2 \cos \theta}^{4 \cos \theta} \right) \, d\theta \\
= \int_{0}^{\pi} \left( \left[ - \frac{1}{4} \left( 256 \cos^4 \theta \right) + \frac{4}{3} \left( 64 \cos^4 \theta \right) \right] - \left[ - \frac{1}{4} \left( 16 \cos^4 \theta \right) + \frac{4}{3} \left( 8 \cos^4 \theta \right) \right] \right) \, d\theta \\
= \int_{0}^{\pi} \frac{44}{3} \cos^4 \theta \, d\theta.
\]

To integrate \( \cos^4 \theta \), rewrite it as \( \cos^2 \theta \cos^2 \theta \) and employ the half-angle formula twice:

\[
\cos^4 \theta = \cos^2 \theta \cos^2 \theta
\]

\[
= \frac{1}{2} \left( 1 + \cos(2\theta) \right) \frac{1}{2} \left( 1 + \cos(2\theta) \right)
\]

\[
= \frac{1}{4} \left( 1 + 2 \cos(2\theta) + \cos^2(2\theta) \right)
\]

\[
= \frac{1}{4} \left( 1 + 2 \cos(2\theta) + \frac{1}{2} \left( 1 + \cos(4\theta) \right) \right)
\]

\[
= \frac{3}{8} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta).
\]

Picking up from where we left off above, we have

\[
\iint_{R} f(x, y) \, dA = \int_{0}^{\pi} \frac{44}{3} \cos^4 \theta \, d\theta
\]

\[
= \int_{0}^{\pi} \frac{44}{3} \left( \frac{3}{8} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta) \right) \, d\theta
\]

\[
= \frac{44}{3} \left( \left. \left( \frac{3}{8} \theta + \frac{1}{4} \sin(2\theta) + \frac{1}{32} \sin(4\theta) \right) \right|_{0}^{\pi} \right)
\]

\[
= \frac{11}{2} \pi.
\]

While this example was not trivial, the double integral would have been much harder to evaluate had we used rectangular coordinates.

Notes:
Example 3  

**Evaluating a double integral with polar coordinates**

Find the volume under the surface \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \) over the sector of the circle with radius \( a \) centered at the origin in the first quadrant, as shown in Figure 14.22.

**Solution**

The region \( R \) we are integrating over is a circle with radius \( a \), restricted to the first quadrant. Thus, in polar, the bounds on \( R \) are \( 0 \leq r \leq a \), \( 0 \leq \theta \leq \pi/2 \). The integrand is rewritten in polar as

\[
\frac{1}{x^2 + y^2 + 1} = \frac{1}{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 1} = \frac{1}{r^2 + 1}.
\]

We find the volume as follows:

\[
\iiint_R f(x, y) \, dA = \int_0^{\pi/2} \int_0^a \frac{r}{r^2 + 1} \, dr \, d\theta
\]

\[
= \int_0^{\pi/2} \left[ \frac{1}{2} \ln |r^2 + 1| \right]_0^a \, d\theta
\]

\[
= \int_0^{\pi/2} \frac{1}{2} \ln(a^2 + 1) \, d\theta
\]

\[
= \left( \frac{1}{2} \ln(a^2 + 1) \right) \left[ \frac{\pi}{2} \right]
\]

\[
= \frac{\pi}{4} \ln(a^2 + 1).
\]

Figure 14.22 shows that \( f \) shrinks to near 0 very quickly. Regardless, as \( a \) grows, so does the volume, without bound.

Example 4  

**Finding the volume of a sphere**

Find the volume of a sphere with radius \( a \).

**Solution**

The sphere of radius \( a \), centered at the origin, has equation \( x^2 + y^2 + z^2 = a^2 \); solving for \( z \), we have \( z = \sqrt{a^2 - x^2 - y^2} \). This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius \( a \), centered at the origin. Polar bounds for this equation are \( 0 \leq r \leq a \), \( 0 \leq \theta \leq 2\pi \).

All together, the volume of a sphere with radius \( a \) is:

\[
2 \iiint_R \sqrt{a^2 - x^2 - y^2} \, dA = 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - (r \cos \theta)^2 - (r \sin \theta)^2} \, r \, dr \, d\theta
\]

\[
= 2 \int_0^{2\pi} \int_0^a r \sqrt{a^2 - r^2} \, dr \, d\theta.
\]

Notes:

**Note:** Previous work has shown that there is finite area under \( \frac{1}{x^2 + 1} \) over the entire x-axis. However, Example 3 shows that there is infinite volume under \( \frac{1}{x^2 + y^2 + 1} \) over the entire x-y plane.
We can evaluate this inner integral with substitution. With \( u = a^2 - r^2, \) \( du = -2r \, dr. \) The new bounds of integration are \( u(0) = a^2 \) to \( u(a) = 0. \) Thus we have:

\[
= \int_0^{2\pi} \int_{a^2}^0 \left( -u^{1/2} \right) \, du \, d\theta \\
= \int_0^{2\pi} \left( \left[ \frac{2}{3} u^{3/2} \right]_{a^2}^0 \right) \, d\theta \\
= \int_0^{2\pi} \frac{2}{3} a^3 \, d\theta \\
= \left( \frac{2}{3} a^3 \theta \right) \bigg|_0^{2\pi} \\
= \frac{4}{3} \pi a^3.
\]

Generally, the formula for the volume of a sphere with radius \( r \) is given as \( \frac{4}{3} \pi r^3; \) we have justified this formula with our calculation.

**Example 5** Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 14.23, where the base of the solid has boundary, in polar coordinates, \( r = \cos(3\theta), \) and the top is defined by the plane \( z = 1 - x + 0.1y. \) Find the volume of the solid.

**Solution** From the outset, we should recognize that knowing how to set up this problem is probably more important than knowing how to compute the integrals. The iterated integral to come is not “hard” to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily–available technology to help compute the final answer.

The region \( R \) that we are integrating over is bound by \( 0 \leq r \leq \cos(3\theta), \) for \( 0 \leq \theta \leq \pi \) (note that this rose curve is traced out on the interval \([0, \pi]\), not \([0, 2\pi]\)). This gives us our bounds of integration. The integrand is \( z = 1 - x + 0.1y; \) converting to polar, we have that the volume \( V \) is:

\[
V = \iint_R f(x, y) \, dA = \int_0^{\pi} \int_0^{\cos(3\theta)} (1 - r \cos \theta + 0.1r \sin \theta) \, r \, dr \, d\theta. \\
\]

Distributing the \( r, \) the inner integral is easy to evaluate, leading to

\[
\int_0^{\pi} \left( \frac{1}{2} \cos^2(3\theta) - \frac{1}{3} \cos^3(3\theta) \cos \theta + \frac{0.1}{3} \cos^3(3\theta) \sin \theta \right) \, d\theta.
\]

Notes:
This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like \( \cos(3\theta) \cos \theta \) need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite \( \frac{1}{2} \cos^2(3\theta) \) as \( \frac{1}{4}(1+\cos(6\theta)) \). We can also rewrite \( \frac{1}{3} \cos^3(3\theta) \cos \theta \) as:

\[
\frac{1}{3} \cos^3(3\theta) \cos \theta = \frac{1}{3} \cos^2(3\theta) \cos(3\theta) \cos \theta = \frac{1}{3} \left( \frac{1 + \cos(6\theta)}{2} \right) (\cos(4\theta) + \cos(2\theta)).
\]

This last expression still needs simplification, but eventually all terms can be reduced to the form \( a \cos(m\theta) \) or \( a \sin(m\theta) \) for various values of \( a \) and \( m \).

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

\[
\int_0^\pi \int_0^{\cos(3\theta)} \left( 1 - r \cos \theta + 0.1r \sin \theta \right) r \, dr \, d\theta = \frac{\pi}{4} \text{ units}^3.
\]

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.).

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than “area under the curve,” iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.
Exercises 14.3

Terms and Concepts

1. When evaluating \( \iint_R f(x,y) \, dA \) using polar coordinates, \( f(x,y) \) is replaced with \( f(r, \theta) \) and \( dA \) is replaced with \( r \, dr \, d\theta \).

2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

In Exercises 3–10, a function \( f(x,y) \) is given and a region \( R \) of the \( x-y \) plane is described. Set up and evaluate \( \iint_R f(x,y) \, dA \) using polar coordinates.

3. \( f(x,y) = 3x - y + 4; \) \( R \) is the region enclosed by the circle \( x^2 + y^2 = 1 \).
4. \( f(x,y) = 4x + 4y; \) \( R \) is the region enclosed by the circle \( x^2 + y^2 = 4 \).
5. \( f(x,y) = 8 - y; \) \( R \) is the region enclosed by the circles with polar equations \( r = \cos \theta \) and \( r = 3 \cos \theta \).
6. \( f(x,y) = 4; \) \( R \) is the region enclosed by the petal of the rose curve \( r = \sin(2\theta) \) in the first quadrant.
7. \( f(x,y) = \ln(x^2 + y^2); \) \( R \) is the annulus enclosed by the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).
8. \( f(x,y) = 1 - x^2 - y^2; \) \( R \) is the region enclosed by the circle \( x^2 + y^2 = 1 \).
9. \( f(x,y) = x^2 - y^2; \) \( R \) is the region enclosed by the circle \( x^2 + y^2 = 36 \) in the first and fourth quadrants.
10. \( f(x,y) = (x - y)/(x + y); \) \( R \) is the region enclosed by the lines \( y = x, y = 0 \) and the circle \( x^2 + y^2 = 1 \) in the first quadrant.

In Exercises 11–14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

11. \( \int_0^5 \int_{-\sqrt{25 - x^2}}^{\sqrt{25 - x^2}} \sqrt{x^2 + y^2} \, dy \, dx \)

12. \( \int_{-8}^{0} \int_{-\sqrt{49 - x^2}}^{\sqrt{49 - x^2}} (2y - x) \, dx \, dy \)

13. \( \int_0^2 \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x + y) \, dx \, dy \)

14. \( \int_{-2}^{-1} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x + 5) \, dy \, dx + \int_{-1}^1 \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x + 5) \, dy \, dx \)

15. Consider \( \int_{R} e^{-(x^2 + y^2)} \, dA \).

(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region \( R \)?

(b) Let \( R \) be the region bounded by the circle of radius \( a \) centered at the origin. Evaluate the double integral using polar coordinates.

(c) Take the limit of your answer from (b), as \( a \to \infty \). What does this imply about the volume under the surface of \( e^{-(x^2 + y^2)} \) over the entire \( x-y \) plane?

16. The surface of a right circular cone with height \( h \) and base radius \( a \) can be described by the equation \( f(x,y) = h - h \sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}} \), where the tip of the cone lies at \((0, 0, h)\) and the circular base lies in the \( x-y \) plane, centered at the origin.

Confirm that the volume of a right circular cone with height \( h \) and base radius \( a \) is \( V = \frac{1}{3} \pi a^2 h \) by evaluating \( \iint_R f(x,y) \, dA \) in polar coordinates.
14.4 Center of Mass

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the mass and center of mass of planar regions.

To find the area of a planar region, we evaluated the double integral \( \iint_R dA \). That is, summing up the areas of lots of little subregions of \( R \) gave us the total area. Informally, we think of \( \iint_R dA \) as meaning “sum up lots of little areas over \( R \).”

To find the signed volume under a surface, we evaluated the double integral \( \iint_R f(x, y) \, dA \). Recall that the “\( dA \)” is not just a “bookend” at the end of an integral; rather, it is multiplied by \( f(x, y) \). We regard \( f(x, y) \) as giving a height, and \( dA \) still giving an area: \( f(x, y) \, dA \) gives a volume. Thus, informally, \( \iint_R f(x, y) \, dA \) means “sum up lots of little volumes over \( R \).”

We now extend these ideas to other contexts.

**Mass and Weight**

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a lamina. So consider a lamina, as shown in Figure 14.24(a), with the shape of some planar region \( R \), as shown in part (b).

We can write a simple double integral that represents the mass of the lamina: \( \iint_R dm \), where “\( dm \)” means “a little mass.” That is, the double integral states the total mass of the lamina can be found by “summing up lots of little masses over \( R \).”

To evaluate this double integral, partition \( R \) into \( n \) subregions as we have done in the past. The \( i^{th} \) subregion has area \( \Delta A_i \). A fundamental property of mass is that “mass=density\times area.” If the lamina has a constant density \( \delta \), then the mass of this \( i^{th} \) subregion is \( \Delta m_i = \delta \Delta A_i \). That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

If density is variable, with density function \( \delta = \delta(x, y) \), then we can approximate the mass of the \( i^{th} \) subregion of \( R \) by multiplying \( \Delta A_i \) by \( \delta(x_i, y_i) \), where \( (x_i, y_i) \) is a point in that subregion. That is, for a small enough subregion of \( R \), the density across that region is almost constant.

The total mass \( M \) of the lamina is approximately the sum of approximate masses of subregions:

\[
M \approx \sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} \delta(x_i, y_i) \Delta A_i.
\]

**Notes:**

- **Note:** Mass and weight are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.
Chapter 14  Multiple Integration

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating \( \delta(x, y) \) over \( R \) gives the mass of the lamina.

**Definition 105  Mass of a Lamina with Vairable Density**

Let \( \delta(x, y) \) be a continuous density function of a lamina corresponding to a plane region \( R \). The mass \( M \) of the lamina is

\[
M = \iint_R \delta(x, y) \, dA.
\]

**Example 1  Finding the mass of a lamina with constant density**

Find the mass of a square lamina, with side length 1, with a density of \( \delta = 3 \text{g/cm}^2 \).

**SOLUTION**  

We represent the lamina with a square region in the plane as shown in Figure 14.25. As the density is constant, it does not matter where we place the square.

Following Definition 105, the mass \( M \) of the lamina is

\[
M = \iint_R 3 \, dA = \int_0^1 \int_0^1 3 \, dx \, dy = 3 \int_0^1 \int_0^1 dx \, dy = 3\text{g}.
\]

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of 3\text{g/cm}^2.

**Example 2  Finding the mass of a lamina with variable density**

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 14.25), with variable density \( \delta(x, y) = (x + y + 2) \text{g/cm}^2 \).

Notes:
14.4 Center of Mass

The variable density $\delta$, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\delta(x, y)$ can be seen in Figure 14.26; notice how “same amount” of density is above $z = 3$ as below. We’ll comment on the significance of this momentarily.

The mass $M$ is found by integrating $\delta(x, y)$ over $R$. The order of integration is not important; we choose $dx\,dy$ arbitrarily. Thus:

$$M = \iint_R (x + y + 2)\,dA = \int_0^1 \int_0^1 (x + y + 2)\,dx\,dy$$

$$= \int_0^1 \left[ \frac{1}{2} x^2 + x(y + 2) \right]_0^1\,dy$$

$$= \int_0^1 \left[ \frac{5}{2} + y \right]\,dy$$

$$= \left[ \frac{5}{2} y + \frac{1}{2} y^2 \right]_0^1$$

$$= 3 \text{ g}.$$

It turns out that since the density of the lamina is so uniformly distributed “above and below” $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 1 and 2 are graphed in Figure 14.26, which illustrates this concept.

Example 3  Finding the weight of a lamina with variable density
Find the weight of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/ft}^2$. Compare this to the weight of the same lamina with density $\delta(x, y) = (2\sqrt{x^2 + y^2 + 1})\text{lb/ft}^2$.

Solution  A direct application of Definition 105 states that the weight of the lamina is $\iint_R \delta(x, y)\,dA$. Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function $\delta(x, y) = x^2 + y^2 + 1$ becomes $\delta(r, \theta) = (r\cos \theta)^2 + (r\sin \theta)^2 + 1 = r^2 + 1$. The circle is bounded by $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

Notes:
Thus the weight $W$ is:

$$W = \int_0^{2\pi} \int_0^2 (r^2 + 1) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left( \frac{1}{4}r^4 + \frac{1}{2}r^2 \right)^2 \, d\theta$$

$$= \int_0^{2\pi} (6) \, d\theta$$

$$= 12\pi \text{lb}.$$

Now compare this with the density function $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$. Converting this to polar coordinates gives $\delta(r, \theta) = 2\sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} + 1 = 2r + 1$. Thus the weight $W$ is:

$$W = \int_0^{2\pi} \int_0^2 (2r + 1) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left( \frac{2}{3}r^3 + \frac{1}{2}r^2 \right)^2 \, d\theta$$

$$= \int_0^{2\pi} \left( \frac{22}{3} \right) \, d\theta$$

$$= \frac{44}{3}\pi \text{lb}.$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 14.27.

Notice how $x^2 + y^2 + 1 \leq 2\sqrt{x^2 + y^2} + 1$ over the circle; this results in less weight.

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of “volume under a surface.” We interpreted $\iint_R f(x, y) \, dA$ as giving the volume under $f$ over $R$; we can understand $\iint_R \delta(x, y) \, dA$ in the same way. The “volume” under $\delta$ over $R$ is actually mass; by compressing the “volume” under $\delta$ onto the $x$-$y$ plane, we get “more mass” in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the center of mass, which we discuss next.
Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the center of mass, or center of gravity. It is as though all the mass is “centered” there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to “balance” the tires of your car: if they are “out of balance”, their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a weighted average. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade is not the average of these three grades: that is, it is not:

\[
\frac{0.9 + 0.73 + 0.85}{3} \approx 0.837 = 83.7\%.
\]

That is, you are probably not pulling a B in the course. Rather, your grades are weighted. Let’s say the homework is worth 10% of the grade, tests are 60% and
the exam is 30%. Then your final grade is:

\[(0.1)(0.9) + (0.6)(0.73) + (0.3)(0.85) = 0.783 = 78.3\% .\]

Each grade is multiplied by a **weight**.

In general, given values \(x_1, x_2, \ldots, x_n\) and weights \(w_1, w_2, \ldots, w_n\), the weighted average of the \(n\) values is

\[
\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.
\]

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don’t see the division by the sum of weights in that instance.

How this relates to center of mass is given in the following theorem.

**Theorem 124**  Center of Mass of Discrete Linear System

Let point masses \(m_1, m_2, \ldots, m_n\) be distributed along the \(x\)-axis at locations \(x_1, x_2, \ldots, x_n\), respectively. The center of mass \(\bar{x}\) of the system is located at

\[
\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.
\]

**Example 4**  Finding the center of mass of a discrete linear system

1. Point masses of 2g are located at \(x = -1, x = 2\) and \(x = 3\) are connected by a thin rod of negligible weight. Find the center of mass of the system.

2. Point masses of 10g, 2g and 1g are located at \(x = -1, x = 2\) and \(x = 3\), respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

**SOLUTION**

1. Following Theorem 124, we compute the center of mass as:

\[
\bar{x} = \frac{2(-1) + 2(2) + 2(3)}{2 + 2 + 2} = \frac{4}{3}.
\]

So the system would balance on a point placed at \(x = 4/3\), as illustrated in Figure 14.28(a).

---

Notes:
2. Again following Theorem 124, we find:

\[ x = \frac{10(-1) + 2(2) + 1(3)}{10 + 2 + 1} = \frac{-3}{13}. \]

Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 14.28(b).

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a moment of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 124, we are finding a weighted measure of distances from the y-axis, so we refer to this as the moment about the y-axis, represented by \( M_y \). Letting \( M \) be the total mass of the system, we have \( \bar{x} = M_y / M \).

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

**Definition 106  Moments about the x- and y- Axes.**

Let point masses \( m_1, m_2, \ldots, m_n \) be located at points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), respectively, in the x-y plane.

1. The moment about the y-axis, \( M_y \), is \( M_y = \sum_{i=1}^{n} m_i x_i \).

2. The moment about the x-axis, \( M_x \), is \( M_x = \sum_{i=1}^{n} m_i y_i \).

One can think that these definitions are “backwards” as \( M_y \) sums up “x” distances. But remember, “x” distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.
Theorem 125 Center of Mass of Discrete Planar System

Let point masses \( m_1, m_2, \ldots, m_n \) be located at points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), respectively, in the \( x-y \) plane, and let \( M = \sum_{i=1}^{n} m_i \).

The center of mass of the system is at \((\bar{x}, \bar{y})\), where

\[
\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}.
\]

Example 5 Finding the center of mass of a discrete planar system

Let point masses of 1kg, 2kg and 5kg be located at points \((2, 0), (1, 1)\) and \((3, 1)\), respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

**Solution** We follow Theorem 125 and Definition 106 to find \( M, M_x \) and \( M_y \): first, \( M = 1 + 2 + 5 = 8 \)kg. Next, we see that

\[
M_x = \sum_{i=1}^{n} m_i y_i = 1(0) + 2(1) + 5(1) = 7.
\]

\[
M_y = \sum_{i=1}^{n} m_i x_i = 1(2) + 2(1) + 5(3) = 19.
\]

Thus the center of mass is \((\bar{x}, \bar{y}) = \left( \frac{M_x}{M}, \frac{M_y}{M} \right) = \left( \frac{19}{8}, \frac{7}{8} \right) = (2.375, 0.875)\), illustrated in Figure 14.29.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region \( R \) in the \( x-y \) plane with density function \( \delta(x, y) \). Partition \( R \) into \( n \) subdivisions, each with area \( \Delta A_i \). As done before, we can approximate the mass of the \( i^{th} \) subregion with \( \delta(x_i, y_i) \Delta A_i \), where \((x_i, y_i)\) is a point inside the \( i^{th} \) subregion. We can approximate the moment of this subregion about the \( y \)-axis with \( x_i \delta(x_i, y_i) \Delta A_i \) – that is, by multiplying the approximate mass of the region by its approximate distance

Notes:
mass: \( M \approx n \sum_{i=1}^{n} \delta(x_i, y_i) \Delta A_i \) (as seen before)

moment about the x-axis: \( M_x \approx n \sum_{i=1}^{n} y_i \delta(x_i, y_i) \Delta A_i \)

moment about the y-axis: \( M_y \approx n \sum_{i=1}^{n} x_i \delta(x_i, y_i) \Delta A_i \)

By taking limits, where size of each subregion shrinks to 0 in both the x and y directions, we arrive at the double integrals given in the following theorem.

**Theorem 126 Center of Mass of a Planar Lamina, Moments**

Let a planar lamina be represented by a region \( R \) in the x-y plane with density function \( \delta(x, y) \).

1. mass: \( M = \iint_{R} \delta(x, y) \, dA \)

2. moment about the x-axis: \( M_x = \iint_{R} y \delta(x, y) \, dA \)

3. moment about the y-axis: \( M_y = \iint_{R} x \delta(x, y) \, dA \)

4. The center of mass of the lamina is \( (\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) \).

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute \( M, M_x \) and \( M_y \) and leave the details of the integration to the reader.

**Example 6 Finding the center of mass of a lamina**

Find the center mass of a square lamina, with side length 1, with a density of \( \delta = 3 \text{g/cm}^2 \). (Note: this is the lamina from Example 1.)
We represent the lamina with a square region in the plane as shown in Figure 14.30 as done previously.

Following Theorem 126, we find $M$, $M_x$ and $M_y$:

\[
M = \int_R (x + y + 2) \, dA = \int_0^1 \int_0^1 (x + y + 2) \, dx \, dy = 3g.
\]

\[
M_x = \int_R y(x + y + 2) \, dA = \int_0^1 \int_0^1 y(x + y + 2) \, dx \, dy = \frac{19}{12}.
\]

\[
M_y = \int_R x(x + y + 2) \, dA = \int_0^1 \int_0^1 x(x + y + 2) \, dx \, dy = \frac{19}{12}.
\]

Thus the center of mass is $(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{19}{36}, \frac{19}{36} \right)$. While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater $x$ and $y$ values pulls the center of mass from the center slightly towards the upper righthand corner.

**Example 8**  
Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the circle with radius 2 ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1) \text{lb/ft}^2$. (Note: this is one of the lamina used in Example 3.)
14.4 Center of Mass

**Solution** As done in Example 3, it is best to describe \( R \) using polar coordinates. Thus when we compute \( M_y \), we will integrate not \( x\delta(x, y) = x(x^2 + y^2 + 1) \), but rather \( (r \cos \theta)\delta(r \cos \theta, r \sin \theta) = (r \cos \theta)(r^2 + 1) \). We compute \( M, M_x \) and \( M_y \):

\[
M = \int_{0}^{2\pi} \int_{r_{in}}^{r_{out}} (r^2 + 1)r \, dr \, d\theta = 12\pi \text{lb}.
\]

\[
M_x = \int_{0}^{2\pi} \int_{r_{in}}^{r_{out}} (r \sin \theta)(r^2 + 1)r \, dr \, d\theta = 0.
\]

\[
M_y = \int_{0}^{2\pi} \int_{r_{in}}^{r_{out}} (r \cos \theta)(r^2 + 1)r \, dr \, d\theta = 0.
\]

Since \( R \) and the density of \( R \) are both symmetric about the \( x \) and \( y \) axes, it should come as no big surprise that the moments about each axis is 0. Thus the center of mass is \((x, y) = (0, 0)\).

**Example 9** Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the region \( R \) shown in Figure 14.31, half an annulus with outer radius 6 and inner radius 5, with constant density 2 lb/\( \text{ft}^2 \).

**Solution** Once again it will be useful to represent \( R \) in polar coordinates. Using the description of \( R \) and/or the illustration, we see that \( R \) is bounded by \( 5 \leq r \leq 6 \) and \( 0 \leq \theta \leq \pi \). As the lamina is symmetric about the \( y \)-axis, we should expect \( M_y = 0 \). We compute \( M, M_x \) and \( M_y \):

\[
M = \int_{0}^{\pi} \int_{5}^{6} (2) r \, dr \, d\theta = 11\pi \text{lb}.
\]

\[
M_x = \int_{0}^{\pi} \int_{5}^{6} (r \sin \theta)(2) r \, dr \, d\theta = \frac{364}{3}.
\]

\[
M_y = \int_{0}^{\pi} \int_{5}^{6} (r \cos \theta)(2) r \, dr \, d\theta = 0.
\]

Thus the center of mass is \((x, y) = (0, \frac{364}{33}) \approx (0, 3.51)\). The center of mass is indicated in Figure 14.31; note how it lies outside of \( R \).

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

---

Notes:
In Exercises 7–10, point masses are given along a line or in the plane. Find the center of mass \( \sum \) or \( (x,y) \), as appropriate. (All masses are in grams and distances are in cm.)

7. \( m_1 = 4 \) at \( x = 1 \); \( m_2 = 3 \) at \( x = 3 \); \( m_3 = 5 \) at \( x = 10 \)
8. \( m_1 = 2 \) at \( x = -3 \); \( m_2 = 2 \) at \( x = -1 \);
\( m_3 = 3 \) at \( x = 0 \); \( m_4 = 3 \) at \( x = 7 \)
9. \( m_1 = 2 \) at \( -2, -2 \); \( m_2 = 2 \) at \( 2, -2 \);
\( m_3 = 20 \) at \( 0, 4 \)
10. \( m_1 = 1 \) at \( -1, -1 \); \( m_2 = 2 \) at \( -1, 1 \);
\( m_3 = 2 \) at \( 1, 1 \); \( m_4 = 1 \) at \( 1, -1 \)

In Exercises 11–18, find the mass/weight of the lamina described by the region \( R \) in the plane and its density function \( \delta(x,y) \).

11. \( R \) is the rectangle with corners \( (1, -3), (1, 2), (7, 2) \) and \( (7, -3) \); \( \delta(x,y) = 5g/cm^2 \)
12. \( R \) is the rectangle with corners \( (1, -3), (1, 2), (7, 2) \) and \( (7, -3) \); \( \delta(x,y) = (x + y^2)g/cm^2 \)
13. \( R \) is the triangle with corners \( (-1, 0), (1, 0), (0, 1) \); \( \delta(x,y) = 2lb/in^2 \)
14. \( R \) is the triangle with corners \( (0, 0), (1, 0), (0, 1) \); \( \delta(x,y) = (x^2 + y^3 + 1)lb/in^2 \)
15. \( R \) is the circle centered at the origin with radius 2; \( \delta(x,y) = (x + y + 4)kg/m^2 \)
16. \( R \) is the circle sector bounded by \( x^2 + y^2 = 25 \) in the first quadrant; \( \delta(x,y) = (\sqrt{x^2 + y^2} + 1)kg/m^2 \)
17. \( R \) is the annulus in the first and second quadrants bounded by \( x^2 + y^2 = 9 \) and \( x^2 + y^2 = 36 \); \( \delta(x,y) = 2lb/ft^2 \)
18. \( R \) is the annulus in the first and second quadrants bounded by \( x^2 + y^2 = 9 \) and \( x^2 + y^2 = 36 \); \( \delta(x,y) = \sqrt{x^2 + y^2}lb/ft^2 \)

In Exercises 19–26, find the center of mass of the lamina described by the region \( R \) in the plane and its density function \( \delta(x,y) \).

Note: these are the same lamina as in Exercises 11–18.

19. \( R \) is the rectangle with corners \( (1, -3), (1, 2), (7, 2) \) and \( (7, -3) \); \( \delta(x,y) = 5g/cm^2 \)
20. \( R \) is the rectangle with corners \( (1, -3), (1, 2), (7, 2) \) and \( (7, -3) \); \( \delta(x,y) = (x + y^2)g/cm^2 \)
21. \( R \) is the triangle with corners \( (-1, 0), (1, 0), (0, 1) \); \( \delta(x,y) = 2lb/in^2 \)
22. \( R \) is the triangle with corners \( (0, 0), (1, 0), (0, 1) \); \( \delta(x,y) = (x^2 + y^3 + 1)lb/in^2 \)
23. \( R \) is the circle centered at the origin with radius 2; \( \delta(x,y) = (x + y + 4)kg/m^2 \)
24. \( R \) is the circle sector bounded by \( x^2 + y^2 = 25 \) in the first quadrant; \( \delta(x,y) = (\sqrt{x^2 + y^2} + 1)kg/m^2 \)
25. \( R \) is the annulus in the first and second quadrants bounded by \( x^2 + y^2 = 9 \) and \( x^2 + y^2 = 36 \); \( \delta(x,y) = 2lb/ft^2 \)
26. \( R \) is the annulus in the first and second quadrants bounded by \( x^2 + y^2 = 9 \) and \( x^2 + y^2 = 36 \); \( \delta(x,y) = \sqrt{x^2 + y^2}lb/ft^2 \)

The moment of inertia \( I \) is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis. \( I_x \) is the moment of inertia about the \( x \)-axis, \( I_y \) is the moment of inertia about the \( y \)-axis, and \( I_o \) is the moment of inertia about the origin. These are computed as follows:

- \( I_x = \int_R y^2 \, dm \)
- \( I_y = \int_R x^2 \, dm \)
- \( I_o = \int_R (x^2 + y^2) \, dm \)

In Exercises 27–30, a lamina corresponding to a planar region \( R \) is given with a mass of 16 units. For each, compute \( I_x, I_y, \) and \( I_o \).

27. \( R \) is the \( 4 \times 4 \) square with corners at \( (-2, -2) \) and \( (2, 2) \) with density \( \delta(x,y) = 1 \).
28. \( R \) is the \( 8 \times 2 \) rectangle with corners at \( (-4, -1) \) and \( (4, 1) \) with density \( \delta(x,y) = 1 \).
29. \( R \) is the \( 4 \times 2 \) rectangle with corners at \( (-2, -1) \) and \( (2, 1) \) with density \( \delta(x,y) = 2 \).
30. \( R \) is the circle with radius 2 centered at the origin with density \( \delta(x,y) = 4/\pi \).
14.5 Surface Area

In Section 10.1 we used definite integrals to compute the arc length of plane curves of the form \( y = f(x) \). We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of “arc length over an interval” to surfaces is “surface area over a region.”

Consider the surface \( z = f(x, y) \) over a region \( R \) in the \( x-y \) plane, shown in Figure 14.32(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region \( R \). We can find this area using the same basic technique we have used over and over: we’ll make an approximation, then using limits, we’ll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide \( R \) into \( n \) subregions. Here we subdivide \( R \) into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions \( \Delta x_i \) and \( \Delta y_i \), along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When \( \Delta x_i \) and \( \Delta y_i \) are small, the function is approximated well by the tangent plane at any point \( (x_i, y_i) \) in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area \( S_i \) of this region of the surface with the area \( T_i \) of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides \( \vec{u} \) and \( \vec{v} \), as shown. One of the applications of the cross product from Section 11.4 is that the area of this parallelogram is \( \| \vec{u} \times \vec{v} \| \). Once we can determine \( \vec{u} \) and \( \vec{v} \), we can determine the area.

\( \vec{u} \) is tangent to the surface in the direction of \( x \); therefore, from Section 13.7, \( \vec{u} \) is parallel to \( \langle 1, 0, f_x(x_i, y_i) \rangle \). The \( x \)-displacement of \( \vec{u} \) is \( \Delta x_i \), so we know that \( \vec{u} = \Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle \). Similar logic shows that \( \vec{v} = \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle \). Thus:

\[
\text{surface area } S_i \approx \text{area of } T_i = \| \vec{u} \times \vec{v} \| = \| \Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle \| = \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta x_i \Delta y_i.
\]

Note that \( \Delta x_i \Delta y_i = \Delta A_i \), the area of the \( i \)th subregion.

Summing up all \( n \) of the approximations to the surface area gives

\[
\text{surface area over } R = \sum_{i=1}^{n} \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta A_i.
\]

---

Notes:
Note: As before, we think of \( \iint_R dS \) as meaning “sum up lots of little surface areas over \( R \).”

The concept of surface area is defined here, for while we already have a notion of the area of a region in the plane, we did not yet have a solid grasp of what “the area of a surface in space” means.

**Definition 107 Surface Area**

Let \( z = f(x, y) \) where \( f_x \) and \( f_y \) are continuous over a closed, bounded region \( R \). The surface area \( S \) over \( R \) is

\[
S = \iint_R dS = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} \, dA.
\]

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

**Example 1 Finding the surface area of a plane over a triangle**

Let \( f(x, y) = 4 - x - 2y \), and let \( R \) be the region in the plane bounded by \( x = 0 \), \( y = 0 \) and \( y = 2 - x/2 \), as shown in Figure 14.33. Find the surface area of \( f \) over \( R \).

**Solution**

We follow Definition 107. We start by noting that \( f_x(x, y) = -1 \) and \( f_y(x, y) = -2 \). To define \( R \), we use bounds \( 0 \leq y \leq 2 - x/2 \) and \( 0 \leq x \leq 4 \). Therefore

\[
S = \iint_R dS = \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2} \, dy \, dx
\]

\[
= \int_0^4 \sqrt{6} \left(2 - \frac{x}{2}\right) \, dx
\]

\[
= 4\sqrt{6}.
\]

**Notes:**
Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the x-y plane, we find the length of the base to be $\sqrt{20}$. We can find the height using our knowledge of vectors: let $\vec{u}$ be the side in the x-z plane and let $\vec{v}$ be the side in the x-y plane. The height is then $\|\vec{u} - \text{proj}_\gamma \vec{u}\| = 4\sqrt{6}/5$. Geometry states that the area is thus

$$
\frac{1}{2} \cdot 4\sqrt{6}/5 \cdot \sqrt{20} = 4\sqrt{6}.
$$

We affirm the validity of our formula.

It is “common knowledge” that the surface area of a sphere of radius $r$ is $4\pi r^2$. We confirm this in the following example, which involves using our formula with polar coordinates.

**Example 2**  
**The surface area of a sphere.**

Find the surface area of the sphere with radius $a$ centered at the origin, whose top hemisphere has equation $f(x, y) = \sqrt{a^2 - x^2 - y^2}$.

**Solution**  
We start by computing partial derivatives and find

$$
f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}.
$$

As our function $f$ only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$
S = 2 \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} \, dA
= 2 \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}} \, dA.
$$

The region $R$ that we are integrating over is the disk, centered at the origin, with radius $a$: $x^2 + y^2 \leq a^2$. Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions $x = r \cos \theta$, $y = r \sin \theta$, $dA = r \, dr \, d\theta$ and bounds $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq a$, we have:

$$
S = 2 \int_0^a \int_0^{2\pi} \sqrt{1 + \frac{r^2 \cos^2 \theta + r^2 \sin^2 \theta}{a^2 - r^2 \cos^2 \theta - r^2 \sin^2 \theta}} \, r \, dr \, d\theta
= 2 \int_0^a \int_0^{2\pi} r \sqrt{1 + \frac{r^2}{a^2 - r^2}} \, dr \, d\theta
= 2 \int_0^a \int_0^{2\pi} r \sqrt{\frac{a^2}{a^2 - r^2}} \, dr \, d\theta.
$$

**Note:** The inner integral in Equation (14.1) is an improper integral, as the integrand of $\int_0^a r \sqrt{\frac{a^2}{a^2 - r^2}} \, dr$ is not defined at $r = a$. To properly evaluate this integral, one must use the techniques of Section 8.6.

The reason this need arises is that the function $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ fails the requirements of Definition 107, as $f_x$ and $f_y$ are not continuous on the boundary of the circle $x^2 + y^2 = a^2$.

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when $f_x$ and/or $f_y$ are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.
Apply substitution \( u = a^2 - r^2 \) and integrate the inner integral, giving
\[
= 2 \int_0^{2\pi} a^2 \, d\theta
= 4\pi a^2.
\]
Our work confirms our previous formula.

**Example 3** Finding the surface area of a cone

The general formula for a right cone with height \( h \) and base radius \( a \), as shown in Figure 14.34, is
\[
f(x, y) = h - \frac{h}{a} \sqrt{x^2 + y^2}.
\]

Find the surface area of this cone.

**SOLUTION** We begin by computing partial derivatives.
\[
f_x(x, y) = -\frac{xh}{a\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = -\frac{yh}{a\sqrt{x^2 + y^2}}.
\]

Since we are integrating over the disk \( x^2 + y^2 \leq a^2 \), we again use polar coordinates. Using the standard substitutions, our integrand becomes
\[
\sqrt{1 + \left( \frac{hr \cos \theta}{a\sqrt{r^2}} \right)^2 + \left( \frac{hr \sin \theta}{a\sqrt{r^2}} \right)^2}.
\]
This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just
\[
\sqrt{1 + \frac{h^2}{a^2}} = \frac{1}{a} \sqrt{a^2 + h^2}.
\]
Our polar bounds are \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq a \). Thus
\[
S = \int_0^{2\pi} \int_0^a \frac{1}{a} \sqrt{a^2 + h^2} \, dr \, d\theta
= \int_0^{2\pi} \left[ \frac{1}{2} \frac{1}{a} \sqrt{a^2 + h^2} \right]^a_0 \, d\theta
= \int_0^{2\pi} \frac{1}{2} a \sqrt{a^2 + h^2} \, d\theta
= \pi a \sqrt{a^2 + h^2}.
\]
This matches the formula found in the back of this text.

**Notes:**
Example 4  
Finding surface area over a region

Find the area of the surface \( f(x, y) = x^2 - 3y + 3 \) over the region \( R \) bounded by \(-x \leq y \leq x, 0 \leq x \leq 4\), as pictured in Figure 14.35.

**Solution**  
It is straightforward to compute \( f_x(x, y) = 2x \) and \( f_y(x, y) = -3 \). Thus the surface area is described by the double integral

\[
\iint_R \sqrt{1 + (2x)^2 + (-3)^2} \, dA = \iint_R \sqrt{10 + 4x^2} \, dA.
\]

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square–root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order \( dx \, dy \) requires us to evaluate \( \int \sqrt{10 + 4x^2} \, dx \). This can be done, though it involves Integration By Parts and \( \sinh^{-1} x \). Integrating with order \( dy \, dx \) has as its first integral \( \int \sqrt{10 + 4x^2} \, dy \), which is easy to evaluate: it is simply \( y\sqrt{10 + 4x^2} + C \). So we proceed with the order \( dy \, dx \); the bounds are already given in the statement of the problem.

\[
\iint_R \sqrt{10 + 4x^2} \, dA = \int_0^4 \int_{-x}^x \sqrt{10 + 4x^2} \, dy \, dx
\]

\[
= \int_0^4 \left( y\sqrt{10 + 4x^2} \right)_{-x}^x \, dx
\]

\[
= \int_0^4 (2x\sqrt{10 + 4x^2}) \, dx.
\]

Apply substitution with \( u = 10 + 4x^2 \):

\[
= \left( \frac{1}{6} (10 + 4x^2)^{3/2} \right)_{0}^{4}
\]

\[
= \frac{1}{3} (37\sqrt{74} - 5\sqrt{10}) \text{ units}^2.
\]

So while the region \( R \) over which we integrate has an area of 16 units\(^2\), the surface has a much greater area as its \( z \)-values change dramatically over \( R \).

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing how to set up the proper integrals brings one very close to being able to compute the needed

---

Notes:
value. Most of the work is actually done in just describing the region $R$ in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.


**Exercises 14.5**

**Terms and Concepts**

1. “Surface area” is analogous to what previously studied concept?
2. To approximate the area of a small portion of a surface, we computed the area of its _______ plane.
3. We interpret \( \iiint_R dS \) as “sum up lots of little _______ .”
4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?
5. Why do \( z = f(x, y) \) and \( z = g(x, y) = f(x, y) + h \), for some real number \( h \), have the same surface area over a region \( R \)?
6. Let \( z = f(x, y) \) and \( z = g(x, y) = 2f(x, y) \). Why is the surface area of \( g \) over a region \( R \) not twice the surface area of \( f \) over \( R \)?

**Problems**

In Exercises 7–10, set up the iterated integral that computes the surface area of the given surface over the region \( R \).

7. \( f(x, y) = \sin x \cos y; \ R \) is the rectangle with bounds \( 0 \leq x \leq 2\pi, \ 0 \leq y \leq 2\pi \).

8. \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2 + 1}}; \ R \) is the disk \( x^2 + y^2 \leq 9 \).

9. \( f(x, y) = x^2 - y^2; \ R \) is the rectangle with opposite corners \((-1, -1) \) and \((1, 1) \).

10. \( f(x, y) = \frac{1}{e^{x^2 + 1}}; \ R \) is the rectangle bounded by \(-5 \leq x \leq 5 \) and \( 0 \leq y \leq 1 \).

In Exercises 11–19, find the area of the given surface over the region \( R \).

11. \( f(x, y) = 3x - 7y + 2; \ R \) is the rectangle with opposite corners \((-1, 0) \) and \((1, 3) \).

12. \( f(x, y) = 2x + 2y + 2; \ R \) is the triangle with corners \((0, 0)\), \((1, 0)\) and \((0, 1)\).

13. \( f(x, y) = x^2 + y^2 + 10; \ R \) is the disk \( x^2 + y^2 \leq 16 \).

14. \( f(x, y) = -2x + 4y^2 + 7 \) over \( R \), the triangle bounded by \( y = -x, y = x, 0 \leq y \leq 1 \).

15. \( f(x, y) = x^2 + y^2 \) over \( R \), the triangle bounded by \( y = 2x, y = 0 \) and \( x = 2 \).

16. \( f(x, y) = \frac{1}{2}x^{3/2} + 2y^{3/2} \) over \( R \), the rectangle with opposite corners \((0, 0)\) and \((1, 1)\).

17. \( f(x, y) = 10 - 2\sqrt{x^2 + y^2} \) over \( R \), the disk \( x^2 + y^2 \leq 25 \). (This is the cone with height 10 and base radius 5; be sure to compare your result with the known formula.)

18. Find the surface area of the sphere with radius 5 by doubling the surface area of \( f(x, y) = \sqrt{25 - x^2 - y^2} \) over \( R \), the disk \( x^2 + y^2 \leq 25 \). (Be sure to compare your result with the known formula.)

19. Find the surface area of the ellipse formed by restricting the plane \( f(x, y) = cx + dy + h \) to the region \( R \), the disk \( x^2 + y^2 \leq 1 \), where \( c, d \) and \( h \) are some constants. Your answer should be given in terms of \( c \) and \( d \); why does the value of \( h \) not matter?
14.6 Volume Between Surfaces and Triple Integration

We learned in Section 14.2 how to compute the signed volume $V$ under a surface $z = f(x, y)$ over a region $R$: $V = \iint_R f(x, y) \, dA$. It follows naturally that if $f(x, y) \geq g(x, y)$ on $R$, then the volume between $f(x, y)$ and $g(x, y)$ on $R$ is

$$V = \iint_R f(x, y) \, dA - \iint_R g(x, y) \, dA = \iint_R (f(x, y) - g(x, y)) \, dA.$$ 

**Theorem 127 Volume Between Surfaces**

Let $f$ and $g$ be continuous functions on a closed, bounded region $R$, where $f(x, y) \geq g(x, y)$ for all $(x, y)$ in $R$. The volume $V$ between $f$ and $g$ over $R$ is

$$V = \iint_R (f(x, y) - g(x, y)) \, dA.$$ 

**Example 1 Finding volume between surfaces**

Find the volume of the space region bounded by the planes $z = 3x + y - 4$ and $z = 8 - 3x - 2y$ in the 1st octant. In Figure 14.36(a) the planes are drawn; in (b), only the defined region is given.

**Solution** We need to determine the region $R$ over which we will integrate. To do so, we need to determine where the planes intersect. They have common $z$-values when $3x + y - 4 = 8 - 3x - 2y$. Applying a little algebra, we have:

$$3x + y - 4 = 8 - 3x - 2y$$
$$6x + 3y = 12$$
$$2x + y = 4$$

The planes intersect along the line $2x + y = 4$. Therefore the region $R$ is bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$; we can convert these bounds to integration bounds of $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. Thus

$$V = \int_0^2 \int_0^{4-2x} (8 - 3x - 2y - (3x + y - 4)) \, dy \, dx$$
$$= \int_0^2 \int_0^{4-2x} (12 - 6x - 3y) \, dy \, dx$$
$$= 16 \text{units}^3.$$

The volume between the surfaces is 16 cubic units.

Notes:
In the preceding example, we found the volume by evaluating the integral

\[ \int_{0}^{4} \int_{0}^{4-2x} (8 - 3x - 2y - (3x + y - 4)) \, dy \, dx. \]

Note how we can rewrite the integrand as an integral, much as we did in Section 14.1:

\[ 8 - 3x - 2y - (3x + y - 4) = \int_{3x+y-4}^{8-3x-2y} dz. \]

Thus we can rewrite the double integral that finds volume as

\[ \int_{0}^{4} \int_{0}^{4-2x} (8 - 3x - 2y - (3x + y - 4)) \, dy \, dx = \int_{0}^{4} \int_{0}^{4-2x} \left( \int_{3x+y-4}^{8-3x-2y} dz \right) \, dy \, dx. \]

This no longer looks like a “double integral,” but more like a “triple integral.” Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region \( D \) in space, such as the one shown in Figure 14.37(a), we start with an approximation. Break \( D \) into \( n \) rectangular solids; the solids near the boundary of \( D \) may possibly not include portions of \( D \) and/or include extra space. In Figure 14.37(b), we zoom in on a portion of the boundary of \( D \) to show a rectangular solid that contains space not in \( D \); as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

The volume \( \Delta V_i \) of the \( i^{th} \) solid \( D_i \) is \( \Delta V_i = \Delta x_i \Delta y_i \Delta z_i \), where \( \Delta x_i \), \( \Delta y_i \), and \( \Delta z_i \) give the dimensions of the rectangular solid in the \( x \), \( y \) and \( z \) directions, respectively. By summing up the volumes of all \( n \) solids, we get an approximation of the volume \( V \) of \( D \):

\[ V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i. \]

Let \( \| \Delta D \| \) represent the length of the longest diagonal of rectangular solids in the subdivision of \( D \). As \( \| \Delta D \| \to 0 \), the volume of each solid goes to 0, as do each of \( \Delta x_i \), \( \Delta y_i \), and \( \Delta z_i \), for all \( i \). Our calculus experience tells us that taking a limit as \( \| \Delta D \| \to 0 \) turns our approximation of \( V \) into an exact calculation of \( V \). Before we state this result in a theorem, we use a definition to define some terms.

Notes:
Chapter 14  Multiple Integration

Definition 108  Triple Integrals, Iterated Integration (Part I)
Let $D$ be a closed, bounded region in space. Let $a$ and $b$ be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of $x$, and let $f_1(x, y)$ and $f_2(x, y)$ be continuous functions of $x$ and $y$.

1. The volume $V$ of $D$ is denoted by a **triple integral**, $V = \iiint_D dV$.

2. The iterated integral $\int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x, y)}^{f_2(x, y)} dz \right) dy \right) dx$ is evaluated as

$$\int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x, y)}^{f_2(x, y)} dz \right) dy \right) dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x, y)}^{f_2(x, y)} dz \right) dy \right) dx.$$

Evaluating the above iterated integral is **triple integration**.

Our informal understanding of the notation $\iiint_D dV$ is “sum up lots of little volumes over $D$,” analogous to our understanding of $\iint_R dA$ and $\int_R dm$. We now state the major theorem of this section.

Theorem 128  Triple Integration (Part I)
Let $D$ be a closed, bounded region in space and let $\Delta D$ be any subdivision of $D$ into $n$ rectangular solids, where the $i$th subregion $D_i$ has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume $\Delta V_i$.

1. The volume $V$ of $D$ is

$$V = \iiint_D dV = \lim_{\|\Delta D\| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{\|\Delta D\| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

2. If $D$ is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x, y) \leq f_2(x, y)$ on $D$, then

$$\iiint_D dV = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x, y)}^{f_2(x, y)} dz \right) dy \right) dx.$$

3. $V$ can be determined using iterated integration with other orders of integration (there are 6 total), as long as $D$ is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

Notes:
We evaluated the area of a plane region $R$ by iterated integration, where the bounds were “from curve to curve, then from point to point.” Theorem 128 allows us to find the volume of a space region with an iterated integral with bounds “from surface to surface, then from curve to curve, then from point to point.” In the iterated integral

$$
\int_{a}^{b} \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz \, dy \, dx,
$$

the bounds $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ define a region $R$ in the $x$-$y$ plane over which the region $D$ exists in space. However, these bounds are also defining surfaces in space; $x = a$ is a plane and $y = g_1(x)$ is a cylinder. The combination of these 6 surfaces enclose, and define, $D$.

Examples will help us understand triple integration, including integrating with various orders of integration.

**Example 2** Finding the volume of a space region with triple integration

Find the volume of the space region in the 1st octant bounded by the plane $z = 2 - y/3 - 2x/3$, shown in Figure 14.38(a), using the order of integration $dz \, dy \, dx$. Set up the triple integrals that give the volume in the other 5 orders of integration.

**Solution** Starting with the order of integration $dz \, dy \, dx$, we need to first find bounds on $z$. The region $D$ is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - y/3 - 2x/3$; $0 \leq z \leq 2 - y/3 - 2x/3$.

To find the bounds on $y$ and $x$, we “collapse” the region onto the $x$-$y$ plane, giving the triangle shown in Figure 14.38(b). (We know the equation of the line $y = 6 - 2x$ in two ways. First, by setting $z = 0$, we have $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the $x$-$y$ plane.)

Notes:
We define that region \( R \), in the integration order of \( dy \, dx \), with bounds \( 0 \leq y \leq 6 - 2x \) and \( 0 \leq x \leq 3 \). Thus the volume \( V \) of the region \( D \) is:

\[
V = \iiint_D dV
\]

\[
= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \, dy \, dz
\]

\[
= \int_0^3 \int_0^{6-2x} \left( \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy \, dz
\]

\[
= \int_0^3 \int_0^{6-2x} \left( 2 - \frac{1}{3}y - \frac{2}{3}x \right) dy \, dz.
\]

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

\[
= 6 \text{units}^3.
\]

The order \( dz \, dx \, dy \):

Now consider the volume using the order of integration \( dz \, dx \, dy \). The bounds on \( z \) are the same as before, \( 0 \leq z \leq 2 - y/3 - 2x/3 \). Collapsing the space region on the \( x-y \) plane as shown in Figure 14.38(b), we now describe this triangle with the order of integration \( dx \, dy \). This gives bounds \( 0 \leq x \leq 3 - y/2 \) and \( 0 \leq y \leq 6 \). Thus the volume is given by the triple integral

\[
V = \int_0^6 \int_0^{3-x/2} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \, dx \, dy.
\]

The order \( dx \, dy \, dz \):

Following our “surface to surface...” strategy, we need to determine the \( x \)-\( surfaces \) that bound our space region. To do so, approach the region “from behind,” in the direction of increasing \( x \). The first surface we hit as we enter the region is the \( y-z \) plane, defined by \( x = 0 \). We come out of the region at the plane \( z = 2 - y/3 - 2x/3 \); solving for \( x \), we have \( x = 3 - y/2 - 3z/2 \). Thus the bounds on \( x \) are: \( 0 \leq x \leq 3 - y/2 - 3z/2 \).

Now collapse the space region onto the \( y-z \) plane, as shown in Figure 14.39(a). (Again, we find the equation of the line \( z = 2 - y/3 \) by setting \( x = 0 \) in the equation \( x = 3 - y/2 - 3z/2 \).) We need to find bounds on this region with the order...
The curves that bound \( y \) are \( y = 0 \) and \( y = 6 - 3z \); the points that bound \( z \) are 0 and 2. Thus the triple integral giving volume is:

\[
0 \leq x \leq 3 - y/2 - 3z/2 \\
0 \leq y \leq 6 - 3z \\
0 \leq z \leq 2 
\]

The order \( dx \ dz \ dy \):

The \( x \)-bounds are the same as the order above. We now consider the triangle in Figure 14.39(a) and describe it with the order \( dz \ dy \): \( 0 \leq z \leq 2 - y/3 \) and \( 0 \leq y \leq 6 \). Thus the volume is given by:

\[
0 \leq x \leq 3 - y/2 - 3z/2 \\
0 \leq z \leq 2 - y/3 \\
0 \leq y \leq 6 
\]

The order \( dy \ dz \ dx \):

We now need to determine the \( y \)-surfaces that determine our region. Approaching the space region from “behind” and moving in the direction of increasing \( y \), we first enter the region at \( y = 0 \), and exit along the plane \( z = 2 - y/3 - 2x/3 \). Solving for \( y \), this plane has equation \( y = 6 - 2x - 3z \). Thus \( y \) has bounds \( 0 \leq y \leq 6 - 2x - 3z \).

Now collapse the region onto the \( x-z \) plane, as shown in Figure 14.39(b). The curves bounding this triangle are \( z = 0 \) and \( z = 2 - 2x/3 \); \( x \) is bounded by the points \( x = 0 \) to \( x = 3 \). Thus the triple integral giving volume is:

\[
0 \leq y \leq 6 - 2x - 3z \\
0 \leq z \leq 2 - 2x/3 \\
0 \leq x \leq 3 
\]

The order \( dy \ dx \ dz \):

The \( y \)-bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 14.39(b) using the order \( dy \ dx \ dz \). We see \( x \) is bounded by \( x = 0 \) and \( x = 3 - 3z/2 \); \( z \) is bounded between \( z = 0 \) and \( z = 2 \). This leads to the triple integral:

\[
0 \leq y \leq 6 - 2x - 3z \\
0 \leq x \leq 3 - 3z/2 \\
0 \leq z \leq 2 
\]

Notes:
This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region $D$. In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the $x$-$y$, $x$-$z$, and $y$-$z$ planes as we determined the “curve to curve, point to point” bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or projecting, was simple: the projection of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.

**Example 3** Finding the projection of a curve in space onto the coordinate planes

Consider the surfaces $z = 3 - x^2 - y^2$ and $z = 2y$, as shown in Figure 14.40(a).

The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

**Solution**

The two surfaces are $z = 3 - x^2 - y^2$ and $z = 2y$. To find where they intersect, it is natural to set them equal to each other: $3 - x^2 - y^2 = 2y$. This is an implicit function of $x$ and $y$ that gives all points $(x, y)$ in the $x$-$y$ plane where the $z$ values of the two surfaces are equal.

We can rewrite this implicit function by completing the square:

$$3 - x^2 - y^2 = 2y \quad \Rightarrow \quad y^2 + 2y + x^2 = 3 \quad \Rightarrow \quad (y + 1)^2 + x^2 = 4.$$

Thus in the $x$-$y$ plane the projection of the intersection is a circle with radius 2, centered at $(0, -1)$.

To project onto the $x$-$z$ plane, we do a similar procedure: find the $x$ and $z$ values where the $y$ values on the surface are the same. We start by solving the equation of each surface for $y$. In this particular case, it works well to actually solve for $y^2$:

$$z = 3 - x^2 - y^2 \quad \Rightarrow \quad y^2 = 3 - x^2 - z$$

$$z = 2y \quad \Rightarrow \quad y^2 = z^2/4.$$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \quad \Rightarrow \quad \frac{(z + 2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at $(0, -2)$ in the $x$-$z$ plane with a major axis of length 8 and a minor axis of length 4.
Finally, to project the curve of intersection into the \( y-z \) plane, we solve equation for \( x \). Since \( z = 2y \) is a cylinder that lacks the variable \( x \), it becomes our equation of the projection in the \( y-z \) plane.

All three projections are shown in Figure 14.40(b).

**Example 4** Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region \( D \) bounded by the surfaces \( x^2 + y^2 = 1 \), \( z = 0 \) and \( z = -y \), as shown in Figure 14.41(a), with the orders of integration \( dz \ dy \ dx \), \( dy \ dx \ dz \) and \( dx \ dz \ dy \).

**Solution** The order \( dz \ dy \ dx \):

The region \( D \) is bounded below by the plane \( z = 0 \) and above by the plane \( z = -y \). The cylinder \( x^2 + y^2 = 1 \) does not offer any bounds in the \( z \)-direction, as that surface is parallel to the \( z \)-axis. Thus \( 0 \leq z \leq -y \).

Collapsing the region into the \( x-y \) plane, we get part of the region bounded by the circle with equation \( x^2 + y^2 = 1 \) as shown in Figure 14.41(b). As a function of \( x \), this half circle has equation \( y = \sqrt{1 - x^2} \). Thus \( y \) is bounded below by \( -\sqrt{1 - x^2} \) and above by \( y = 0 \): \( -\sqrt{1 - x^2} \leq y \leq 0 \). The \( x \) bounds of the half circle are \(-1 \leq x \leq 1 \). All together, the bounds of integration and triple integral are as follows:

\[
0 \leq z \leq -y \quad -\sqrt{1 - x^2} \leq y \leq 0 \quad -1 \leq x \leq 1
\]

We evaluate this triple integral:

\[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} dz \ dy \ dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} (-y) \ dy \ dx
\]

\[
= \int_{-1}^{1} \left[ -\frac{1}{2}y^2 \right]_{-\sqrt{1-x^2}}^{0} \ dx = \int_{-1}^{1} \frac{1}{2} (1 - x^2) \ dx
\]

\[
= \left[ \frac{1}{3} (x - \frac{1}{3} x^3) \right]_{-1}^{1} = \frac{2}{3} \text{units}^3.
\]

With the order \( dy \ dx \ dz \):
The region is bounded “below” in the y-direction by the surface \( x^2 + y^2 = 1 \) \( \Rightarrow y = -\sqrt{1 - x^2} \) and “above” by the surface \( y = -z \). Thus the y bounds are \(-\sqrt{1 - x^2} \leq y \leq -z\).

Collapsing the region onto the x-z plane gives the region shown in Figure 14.42(a); this half disk is bounded by a circle with equation \( x^2 + z^2 = 1 \). (We find this curve by solving each surface for \( y^2 \), then setting them equal to each other. We have \( y^2 = 1 - x^2 \) and \( y = -z \Rightarrow y^2 = z^2 \). Thus \( x^2 + z^2 = 1 \).) It is bounded below by \( x = -\sqrt{1 - z^2} \) and above by \( x = \sqrt{1 - z^2} \), where \( z \) is bounded by \( 0 \leq z \leq 1 \).

All together, we have:

\[
-\sqrt{1 - x^2} \leq y \leq -z \\
-\sqrt{1 - z^2} \leq x \leq \sqrt{1 - z^2} \\
0 \leq z \leq 1
\]

\[
\Rightarrow \int_{0}^{1} \int_{-\sqrt{1 - z^2}}^{\sqrt{1 - z^2}} \int_{-\sqrt{1 - x^2}}^{-z} dy \, dx \, dz.
\]

With the order \( dx \, dz \, dy \):

\[
D \text{ is bounded below by the surface } x = -\sqrt{1 - y^2} \text{ and above by } \sqrt{1 - y^2}.
\]

We then collapse the region onto the y-z plane and get the triangle shown in Figure 14.42(b). (The hypotenuse is the line \( z = -y \), just as the plane.) Thus \( z \) is bounded by \( 0 \leq z \leq -y \) and \( y \) is bounded by \(-1 \leq y \leq 0\). This gives:

\[
-\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2} \\
0 \leq z \leq -y \\
-1 \leq y \leq 0
\]

\[
\Rightarrow \int_{-1}^{0} \int_{0}^{y} \int_{\sqrt{1 - y^2}}^{\sqrt{1 - y^2}} dx \, dz \, dy.
\]

The following theorem states two things that should make “common sense” to us. First, using the triple integral to find volume of a region \( D \) should always return a positive number; we are computing volume here, not signed volume. Secondly, to compute the volume of a “complicated” region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

**Theorem 129 Properties of Triple Integrals**

Let \( D \) be a closed, bounded region in space, and let \( D_1 \) and \( D_2 \) be non-overlapping regions such that \( D = D_1 \bigcup D_2 \).

1. \( \iiint_D dV \geq 0 \)

2. \( \iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV \).

**Notes:**
We use this latter property in the next example.

**Example 5** Finding the volume of a space region with triple integration

Find the volume of the space region \( D \) bounded by the coordinate planes, \( z = 1 - x/2 \) and \( z = 1 - y/4 \), as shown in Figure 14.43(a). Set up the triple integrals that find the volume of \( D \) in all 6 orders of integration.

**Solution** Following the bounds–determining strategy of “surface to surface, curve to curve, and point to point,” we can see that the most difficult orders of integration are the two in which we integrate with respect to \( z \) first, for there are two “upper” surfaces that bound \( D \) in the \( z \)-direction. So we start by noting that we have

\[
0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.
\]

We now collapse the region \( D \) onto the \( x-y \) axis, as shown in Figure 14.43(b). The boundary of \( D \), the line from \((0, 0, 1)\) to \((2, 4, 0)\), is shown in part (b) of the figure as a dashed line; it has equation \( y = 2x \). (We can recognize this in two ways: one, in collapsing the line from \((0, 0, 1)\) to \((2, 4, 0)\) onto the \( x-y \) plane, we simply ignore the \( z \)-values, meaning the line now goes from \((0, 0)\) to \((2, 4)\). Secondly, the two surfaces meet where \( z = 1 - x/2 = 1 - y/4 \); thus \( 1 - x/2 = 1 - y/4 \Rightarrow y = 2x \).)

We use the second property of Theorem 129 to state that

\[
\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,
\]

where \( D_1 \) and \( D_2 \) are the space regions above the plane regions \( R_1 \) and \( R_2 \), respectively. Thus we can say

\[
\iiint_D dV = \iint_{R_1} \left( \int_0^{1-x/2} dz \right) dA + \iint_{R_2} \left( \int_0^{1-y/4} dz \right) dA.
\]

All that is left is to determine bounds of \( R_1 \) and \( R_2 \), depending on whether we are integrating with order \( dx \; dy \; dz \) or \( dz \; dy \; dx \). We give the final integrals here, leaving it to the reader to confirm these results.

\[
dz \; dy \; dx: \quad \begin{array}{ll}
0 \leq z \leq 1 - x/2 & 0 \leq z \leq 1 - y/4 \\
0 \leq y \leq 2x & 2x \leq y \leq 4 \\
0 \leq x \leq 2 & 0 \leq x \leq 2
\end{array}
\]

\[
\iiint_D dV = \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz \; dy \; dx + \int_0^2 \int_0^4 \int_0^{1-y/4} dz \; dy \; dx
\]
Chapter 14  Multiple Integration

We give one more example of finding the volume of a space region.

**Example 6**  Finding the volume of a space region

Set up a triple integral that gives the volume of the space region $D$ bounded by $z = 2x^2 + 2$ and $z = 6 - 2x^2 - y^2$. These surfaces are plotted in Figure 14.45(a) and (b), respectively; the region $D$ is shown in part (c) of the figure.
The main point of this example is this: integrating with respect to \( z \) first is rather straightforward; integrating with respect to \( x \) first is not.

The order \( dz \, dy \, dx \):

The bounds on \( z \) are clearly \( 2x^2 + 2 \leq z \leq 6 - 2x^2 - y^2 \). Collapsing \( D \) onto the \( x-y \) plane gives the ellipse shown in Figure 14.45(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

\[
2x^2 + 2 = 6 - 2x^2 - y^2 \implies 4x^2 + y^2 = 4 \implies x^2 + \frac{y^2}{4} = 1.
\]

We can describe this ellipse with the bounds

\[
-\sqrt{4 - 4x^2} \leq y \leq \sqrt{4 - 4x^2} \quad \text{and} \quad -1 \leq x \leq 1.
\]

Thus we find volume as

\[
\int_{-1}^{1} \int_{\sqrt{4 - 4x^2}^-}^{\sqrt{4 - 4x^2}^+} \int_{2x^2 + 2}^{6 - 2x^2 - y^2} dz \, dy \, dx
\]

The order \( dy \, dz \, dx \):

Integrating with respect to \( y \) is not too difficult. Since the surface \( z = 2x^2 + 2 \) is a cylinder whose directrix is the \( y \)-axis, it does not create a border for \( y \). The paraboloid \( z = 6 - 2x^2 - y^2 \) does; solving for \( y \), we get the bounds

\[
-\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z}.
\]
Collapsing $D$ onto the $x$-$z$ axes gives the region shown in Figure 14.46(a); the lower curve is from the cylinder, with equation $z = 2x^2 + 2$. The upper curve is from the paraboloid; with $y = 0$, the curve is $z = 6 - 2x^2$. Thus bounds on $z$ are $2x^2 + 2 \leq z \leq 6 - 2x^2$; the bounds on $x$ are $-1 \leq x \leq 1$. Thus we have:

\[ -\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z} \]
\[ 2x^2 + 2 \leq z \leq 6 - 2x^2 \]
\[ -1 \leq x \leq 1 \]

\[ \Rightarrow \int_{-1}^{1} \int_{2x^2+2}^{\sqrt{6-2x^2}} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy \; dz \; dx. \]

The order $dx \; dz \; dy$:

This order takes more effort as $D$ must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of $x$; the cylinder creates bounds

\[ -\sqrt{z/2 - 1} \leq x \leq \sqrt{z/2 - 1} \]

for region $D_1$ and the paraboloid creates bounds

\[ -\sqrt{3 - y^2/2 - z^2/2} \leq x \leq \sqrt{3 - y^2/2 - z^2/2} \]

for region $D_2$.

Collapsing $D$ onto the $y$-$z$ axes gives the regions shown in Figure 14.46(b). We find the equation of the curve $z = 4 - y^2/2$ by noting that the equation of the ellipse seen in Figure 14.45(c) has equation

\[ x^2 + y^2/4 = 1 \Rightarrow x = \sqrt{1-y^2/4}. \]

Substitute this expression for $x$ in either surface equation, $z = 6 - 2x^2 - y^2$ or $z = 2x^2 + 2$. In both cases, we find

\[ z = 4 - \frac{1}{2}y^2. \]

Region $R_1$, corresponding to $D_1$, has bounds

\[ 2 \leq z \leq 4 - y^2/2, \quad -2 \leq y \leq 2 \]

and region $R_2$, corresponding to $D_2$, has bounds

\[ 4 - y^2/2 \leq z \leq 6 - y^2, \quad -2 \leq y \leq 2. \]

Thus the volume of $D$ is given by:

\[ \int_{-2}^{2} \int_{-2}^{2} \int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx \; dz \; dy + \int_{-2}^{2} \int_{-2}^{2} \int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx \; dz \; dy. \]

Notes:
If all one wanted to do in Example 6 was find the volume of the region $D$, one would have likely stopped at the first integration setup (with order $dz \, dy \, dx$) and computed the volume from there. However, we included the other two methods (1) to show that it could be done, “messy” or not, and (2) because sometimes we “have” to use a less desirable order of integration in order to actually integrate.

### Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond “area under the curve.” These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

**Definition 109**  
**Iterated Integration, (Part II)**  
Let $D$ be a closed, bounded region in space, over which $g_1(x)$, $g_2(x)$, $f_1(x, y)$, $f_2(x, y)$ and $h(x, y, z)$ are all continuous, and let $a$ and $b$ be real numbers.

The iterated integral
\[
\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x, y)}^{f_2(x, y)} h(x, y, z) \, dz \, dy \, dx
\]

is evaluated as
\[
\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x, y)}^{f_2(x, y)} h(x, y, z) \, dz \, dy \, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x, y)}^{f_2(x, y)} h(x, y, z) \, dz \right) \, dy \, dx.
\]

**Example 7**  
**Evaluating a triple integral of a function of three variables**  
Evaluate $\int_0^1 \int_{x^2}^{2x+3y} \int_{x^2-y}^{xy + 2xz} dz \, dy \, dx$.

**Solution**  
We evaluate this integral according to Definition 109.

---

Notes:
\[
\int_0^1 \int_{x^2}^x \int_{x^2-y}^{2x+3y} (xy + 2xz) \, dz \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^x \left( \int_{x^2-y}^{2x+3y} (xy + 2xz) \, dz \right) \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^x \left( xy + xz^2 \right) \left| _{x^2-y}^{2x+3y} \right. \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^x \left( xy(2x + 3y) + x(2x + 3y)^2 - \left( xy(x^2 - y) + x(x^2 - y)^2 \right) \right) \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^x \left( -x^5 + x^3y + 4x^3y + 14x^2y + 12xy^2 \right) \, dy \, dx.
\]

We continue as we have in the past, showing fewer steps.

\[
= \int_0^1 \left( -\frac{7}{2}x^5 - 8x^5 - \frac{7}{2}x^5 + 15x^4 \right) \, dx
\]
\[
= \frac{281}{336}
\]

We now know how to evaluate a triple integral of a function of three variables; we do not yet understand what it means. We build upon this understanding in a way very similar to how we have understood integration and double integration.

Let \( h(x, y, z) \) be a continuous function of three variables, defined over some space region \( D \). We can partition \( D \) into \( n \) rectangular–solid subregions, each with dimensions \( \Delta x_i \times \Delta y_i \times \Delta z_i \). Let \((x_i, y_i, z_i)\) be some point in the \( i \)th subregion, and consider the product \( h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i \). It is the product of a function value (that’s the \( h(x_i, y_i, z_i) \) part) and a small volume \( \Delta V_i \) (that’s the \( \Delta x_i \Delta y_i \Delta z_i \) part). One of the simplest understanding of this type of product is when \( h \) describes the density of an object, for then \( h \times \text{volume} = \text{mass} \).

We can sum up all \( n \) products over \( D \). Again letting \( ||\Delta D|| \) represent the length of the longest diagonal of the \( n \) rectangular solids in the partition, we can take the limit of the sums of products as \( ||\Delta D|| \to 0 \). That is, we can find

\[
S = \lim_{||\Delta D|| \to 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i.
\]

While this limit has lots of interpretations depending on the function \( h \), in the case where \( h \) describes density, \( S \) is the total mass of the object described by the region \( D \).

**Notes:**
We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

**Definition 110  Triple Integral**

Let \( w = h(x, y, z) \) be a continuous function over a closed, bounded space region \( D \), and let \( \Delta D \) be any partition of \( D \) into \( n \) rectangular solids with volume \( \Delta V_i \). The **triple integral of \( h \) over \( D \)** is

\[
\iiint_D h(x, y, z) \, dV = \lim_{\| \Delta D \| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i) \Delta V_i.
\]

The following theorem assures us that the above limit exists for continuous functions \( h \) and gives us a method of evaluating the limit.

**Theorem 130  Triple Integration (Part II)**

Let \( w = h(x, y, z) \) be a continuous function over a closed, bounded space region \( D \), and let \( \Delta D \) be any partition of \( D \) into \( n \) rectangular solids with volume \( V_i \).

1. The limit \( \lim_{\| \Delta D \| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i) \Delta V_i \) exists.
2. If \( D \) is defined as the region bounded by the planes \( x = a \) and \( x = b \), the cylinders \( y = g_1(x) \) and \( y = g_2(x) \), and the surfaces \( z = f_1(x, y) \) and \( z = f_2(x, y) \), where \( a < b, g_1(x) \leq g_2(x) \) and \( f_1(x, y) \leq f_2(x, y) \) on \( D \), then

\[
\iiint_D h(x, y, z) \, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z) \, dz \, dy \, dx.
\]

We now apply triple integration to find the centers of mass of solid objects.

**Mass and Center of Mass**

One may wish to review Section 14.4 for a reminder of the relevant terms and concepts.

**Note:** In the marginal note on page 862, we showed how the summation of rectangles over a region \( R \) in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum

\[
\sum_{i=1}^{n} h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i
\]

as a triple sum,

\[
\sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{m} h(x_i, y_j, z_k) \Delta x_i \Delta y_j \Delta z_k,
\]

which we evaluate as

\[
\sum_{k=1}^{p} \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{m} h(x_i, y_j, z_k) \right) \Delta y_j \right) \Delta z_k.
\]

Here we fix a \( k \) value, which establishes the \( z \)-height of the rectangular solids on one “level” of all the rectangular solids in the space region \( D \). The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.

This triple summation understanding leads to the \( \iiint_D \) notation of the triple integral, as well as the method of evaluation shown in Theorem 130.

---

Notes:
Definition 111  Mass, Center of Mass of Solids
Let a solid be represented by a region \( D \) in space with variable density function \( \delta(x, y, z) \).

1. The mass of the object is
   \[
   M = \iiint_D \, dm = \iiint_D \delta(x, y, z) \, dV.
   \]

2. The moment about the \( x\)-\( y \) plane is
   \[
   M_{xy} = \iiint_D z \delta(x, y, z) \, dV.
   \]

3. The moment about the \( x\)-\( z \) plane is
   \[
   M_{xz} = \iiint_D y \delta(x, y, z) \, dV.
   \]

4. The moment about the \( y\)-\( z \) plane is
   \[
   M_{yz} = \iiint_D x \delta(x, y, z) \, dV.
   \]

5. The center of mass of the object is
   \[
   (\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right). 
   \]

Example 8  Finding the center of mass of a solid
Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and \( z = 2 - y/3 - 2x/3 \), shown in Figure 14.47, with constant density \( \delta(x, y, z) = 3 \text{g/cm}^3 \). (Note: this space region was used in Example 2.)

**Solution**  We apply Definition 111. In Example 2, we found bounds for the order of integration \( dz \, dy \, dx \) to be \( 0 \leq z \leq 2 - y/3 - 2x/3 \), \( 0 \leq y \leq 6 - 2x \) and \( 0 \leq x \leq 3 \). We find the mass of the object:

\[
M = \iiint_D \delta(x, y, z) \, dV
= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3) \, dz \, dy \, dx
= 3 \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz \, dy \, dx
= 3(6) = 18 \text{g}.
\]

The evaluation of the triple integral is done in Example 2, so we skipped those steps above. Note how the mass of an object with constant density is simply “density \( \times \) volume.”
We now find the moments about the planes.

\[ M_{xy} = \iiint_D 3z \, dV \]
\[ = \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3z) \, dz \, dy \, dx \]
\[ = \int_0^3 \int_0^{6-2x} \frac{3}{2} (2 - y/3 - 2x/3)^2 \, dy \, dx \]
\[ = \int_0^3 -\frac{4}{9} (x - 3)^3 \, dx \]
\[ = 9. \]

We omit the steps of integrating to find the other moments.

\[ M_{yz} = \iiint_D 3x \, dV = \frac{27}{2}. \]
\[ M_{xz} = \iiint_D 3y \, dV = 27. \]

The center of mass is

\[ (x, y, z) = \left( \frac{27/2}{18}, \frac{27}{18}, \frac{9}{18} \right) = (0.75, 1.5, 0.5). \]

**Example 9  Finding the center of mass of a solid**

Find the center of mass of the solid represented by the region bounded by the planes \( z = 0 \) and \( z = -y \) and the cylinder \( x^2 + y^2 = 1 \), shown in Figure 14.48, with density function \( \delta(x, y, z) = 10 + x^2 + 5y - 5z \). (Note: this space region was used in Example 4.)

**Solution** As we start, consider the density function. It is symmetric about the \( y-z \) plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that \( x \) should be 0.

As one moves away from the origin in the \( y \) or \( z \) directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer.

We use the order of integration \( dz \, dy \, dx \), using the bounds found in Example 4.
As stated before, there are many uses for triple integration beyond finding volume. When \( h(x, y, z) \) describes a rate of change function over some space region \( D \), then \( \iiint_D h(x, y, z) \, dV \) gives the total change over \( D \). Our one specific example of this was computing mass; a density function is simply a “rate of mass change per volume” function. Integrating density gives total mass.

While knowing how to integrate is important, it is arguably much more important to know how to set up integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.
Exercises 14.6

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "_______ to ________, _________ to ________, and _________ to ________.”

2. Give an informal interpretation of what \( \iiint_D \, dV \) means.

3. Give two uses of triple integration.

4. If an object has a constant density \( \delta \) and a volume \( V \), what is its mass?

Problems

In Exercises 5–8, two surfaces \( f_1(x, y) \) and \( f_2(x, y) \) and a region \( R \) in the \( x, y \) plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over \( R \).

5. \( f_1(x, y) = 8 - x^2 - y^2, f_2(x, y) = 2x + y; \) 
   \( R \) is the square with corners \((-1, -1)\) and \((1, 1)\).

6. \( f_1(x, y) = x^2 + y^2, f_2(x, y) = -x^2 - y^2; \) 
   \( R \) is the square with corners \((0, 0)\) and \((2, 3)\).

7. \( f_1(x, y) = \sin x \cos y, f_2(x, y) = \cos x \sin y + 2; \) 
   \( R \) is the triangle with corners \((0, 0)\), \((\pi, 0)\) and \((\pi, \pi)\).

8. \( f_1(x, y) = 2x^2 + 2y^2 + 3, f_2(x, y) = 6 - x^2 - y^2; \) 
   \( R \) is the disk \( x^2 + y^2 \leq 1 \).

In Exercises 9–16, a domain \( D \) is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of \( D \) in all 6 orders of integration, and find the volume of \( D \) by evaluating the indicated triple integral.

9. \( D \) is bounded by the coordinate planes and \( z = 2 - 2x/3 - 2y \). 
   Evaluate the triple integral with order \( dz \, dy \, dx \).

10. \( D \) is bounded by the planes \( y = 0, y = 2, x = 1, z = 0 \) and \( z = (3 - x)/2 \). 
    Evaluate the triple integral with order \( dx \, dy \, dz \).

11. \( D \) is bounded by the planes \( x = 0, y = 2, z = -y \) and by \( z = y^2/2 \). 
    Evaluate the triple integral with the order \( dy \, dz \, dx \).

12. \( D \) is bounded by the planes \( z = 0, y = 9, x = 0 \) and by \( z = \sqrt{y^2 - 9x^2} \). 
    Do not evaluate any triple integral.
13. \( D \) is bounded by the planes \( x = 2, y = 1, z = 0 \) and \( z = 2x + 4y - 4 \).
   Evaluate the triple integral with the order \( dx \ dy \ dz \).

14. \( D \) is bounded by the plane \( z = 2y \) and by \( y = 4 - x^2 \).
   Evaluate the triple integral with the order \( dz \ dy \ dx \).

15. \( D \) is bounded by the coordinate planes and by \( y = 1 - x^2 \) and \( y = 1 - z^2 \).
   Do not evaluate any triple integral. Which order is easier to evaluate: \( dz \ dy \ dx \) or \( dy \ dz \ dx \)? Explain why.

16. \( D \) is bounded by the coordinate planes and by \( z = 1 - y/3 \) and \( z = 1 - x \).
   Evaluate the triple integral with order \( dx \ dy \ dz \).

In Exercises 17–20, evaluate the triple integral.

17. \( \int_{-\pi /2}^{\pi /2} \int_{0}^{x} \int_{0}^{x} (\cos x \sin y \sin z) \ dz \ dy \ dx \)

18. \( \int_{0}^{\pi} \int_{0}^{x} \int_{0}^{x+y} (x + y + z) \ dz \ dy \ dx \)

19. \( \int_{0}^{\pi} \int_{0}^{1} \int_{0}^{y} (\sin(yz)) \ dz \ dy \ dx \)

20. \( \int_{-\pi}^{\pi} \int_{-1}^{x} \int_{0}^{y^2} \left( z \frac{x^2 y + y^2 z}{e^{x^2+y^2}} \right) \ dz \ dy \ dx \)

In Exercises 21–24, find the center of mass of the solid represented by the indicated space region \( D \) with density function \( \delta(x, y, z) \).

21. \( D \) is bounded by the coordinate planes and \( z = 2 - 2x/3 - 2y; \ \delta(x, y, z) = 10g/cm^3 \).
   (Note: this is the same region as used in Exercise 9.)

22. \( D \) is bounded by the planes \( y = 0, y = 2, x = 1, z = 0 \) and \( z = (3 - x)/2; \ \delta(x, y, z) = 2g/cm^3 \).
   (Note: this is the same region as used in Exercise 10.)

23. \( D \) is bounded by the planes \( x = 2, y = 1, z = 0 \) and \( z = 2x + 4y - 4; \ \delta(x, y, z) = x^2lb/in^3 \).
   (Note: this is the same region as used in Exercise 13.)

24. \( D \) is bounded by the plane \( z = 2y \) and by \( y = 4 - x^2 \).
   \( \delta(x, y, z) = y^2lb/in^3 \).
   (Note: this is the same region as used in Exercise 14.)
14.7 Change of Variables in Multiple Integrals

Given the difficulty of evaluating multiple integrals, the reader may be wondering if it is possible to simplify those integrals using a suitable substitution for the variables. The answer is yes, though it is a bit more complicated than the substitution method which you learned in single-variable calculus.

Recall that if you are given, for example, the definite integral

\[ \int_{1}^{2} x^3 \sqrt{x^2 - 1} \, dx, \]

then you would make the substitution

\[ u = x^2 - 1 \Rightarrow x^2 = u + 1 \]
\[ du = 2x \, dx \]

which changes the limits of integration

\[ x = 1 \Rightarrow u = 0 \]
\[ x = 2 \Rightarrow u = 3 \]

so that we get

\[ \int_{1}^{2} x^3 \sqrt{x^2 - 1} \, dx = \int_{1}^{2} \frac{1}{2} x^2 \cdot 2x \sqrt{x^2 - 1} \, dx \]
\[ = \int_{0}^{3} \frac{1}{2} (u + 1) \sqrt{u} \, du \]
\[ = \frac{1}{2} \int_{0}^{3} (u^{3/2} + u^{1/2}) \, du \]
\[ = \frac{14 \sqrt{3}}{5}. \]

Let us take a different look at what happened when we did that substitution, which will give some motivation for how substitution works in multiple integrals. First, we let \( u = x^2 - 1 \). On the interval of integration \([1, 2] \), the function \( x \mapsto x^2 - 1 \) is strictly increasing (and maps \([1, 2] \) onto \([0, 3] \)) and hence has an inverse function (defined on the interval \([0, 3] \)). That is, on \([0, 3] \) we can define \( x \) as a function of \( u \), namely

\[ x = g(u) = \sqrt{u + 1}. \]

Then substituting that expression for \( x \) into the function \( f(x) = x^3 \sqrt{x^2 - 1} \) gives

\[ f(x) = f(g(u)) = (u + 1)^{3/2} \sqrt{u}. \]
and we see that
\[ \frac{dx}{du} = g'(u) \Rightarrow dx = g'(u) \, du \]
\[ dx = \frac{1}{2} (u + 1)^{-1/2} \, du, \]
so since
\[ g(0) = 1 \Rightarrow 0 = g^{-1}(1) \]
\[ g(3) = 2 \Rightarrow 3 = g^{-1}(2) \]

then performing the substitution as we did earlier gives
\[ \int_1^2 f(x) \, dx = \int_1^2 x^3 \sqrt{x^2 - 1} \, dx \]
\[ = \int_0^3 \frac{1}{2} (u + 1) \sqrt{u} \, du, \] which can be written as
\[ = \int_0^3 (u + 1)^{3/2} \sqrt{u} \cdot \frac{1}{2} (u + 1)^{-1/2} \, du, \] which means
\[ \int_1^2 f(x) \, dx = \int_{g^{-1}(1)}^{g^{-1}(2)} f(g(u)) \, g'(u) \, du. \]

In general, if \( x = g(u) \) is a one-to-one, differentiable function from an interval \([c, d]\) (which you can think of as being on the “u-axis”) onto an interval \([a, b]\) (on the x-axis), which means that \( g'(u) \neq 0 \) on the interval \((c, d)\), so that \( a = g(c) \) and \( b = g(d) \), then \( c = g^{-1}(a) \) and \( d = g^{-1}(b) \), and
\[ \int_a^b f(x) \, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u)) \, g'(u) \, du. \]

This is called the change of variable formula for integrals of single-variable functions, and it is what you were implicitly using when doing integration by substitution. This formula turns out to be a special case of a more general formula which can be used to evaluate multiple integrals. We will state the formulas for double and triple integrals involving real-valued functions of two and three variables, respectively. We will assume that all the functions involved are continuously differentiable and that the regions and solids involved all have “reasonable” boundaries. The proof of the following theorem is beyond the scope of the text.

Notes:
Theorem 131 Change of Variables Formula for Multiple Integrals

Let \( x = x(u, v) \) and \( y = y(u, v) \) define a one-to-one mapping of a region \( R' \) in the \( uv \)-plane onto a region \( R \) in the \( xy \)-plane such that the determinant

\[
J(u, v) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} \quad (14.2)
\]

is never 0 in \( R' \). Then

\[
\iint_R f(x, y) \, dA(x, y) = \iint_{R'} f(x(u, v), y(u, v)) \left| J(u, v) \right| \, dA(u, v). \quad (14.3)
\]

We use the notation \( dA(x, y) \) and \( dA(u, v) \) to denote the area element in the \( (x, y) \) and \( (u, v) \) coordinates, respectively.

Similarly, if \( x = x(u, v, w) \), \( y = y(u, v, w) \) and \( z = z(u, v, w) \) define a one-to-one mapping of a solid \( S' \) in \( uvw \)-space onto a solid \( S \) in \( xyz \)-space such that the determinant

\[
J(u, v, w) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix} \quad (14.4)
\]

is never 0 in \( S' \), then

\[
\iiint_S f(x, y, z) \, dV(x, y, z) = \iiint_{S'} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| J(u, v, w) \right| \, dV(u, v, w). \quad (14.5)
\]

The determinant \( J(u, v) \) in Equation (14.2) is called the Jacobian of \( x \) and \( y \) with respect to \( u \) and \( v \), and is sometimes written as

\[
J(u, v) = \frac{\partial (x, y)}{\partial (u, v)}.
\]

Similarly, the Jacobian \( J(u, v, w) \) of three variables is sometimes written as

\[
J(u, v, w) = \frac{\partial (x, y, z)}{\partial (u, v, w)}.
\]

Notes:
Notice that Equation (14.3) is saying that \( dA(x, y) = |J(u, v)| \, dA(u, v) \), which you can think of as a two-variable version of the relation \( dx = g'(u) \, du \) in the single-variable case.

The following example shows how the change of variables formula is used.

**Example 1 Change of Variables**

Evaluate \( \iint_R e^{x+y} \, dA \), where \( R = \{(x, y) : x \geq 0, y \geq 0, x+y \leq 1 \} \).

**Solution** First, note that evaluating this double integral without using substitution is probably impossible, at least in a closed form. By looking at the numerator and denominator of the exponent of \( e \), we will try the substitution \( u = x - y \) and \( v = x + y \). To use the change of variables Equation (14.3), we need to write both \( x \) and \( y \) in terms of \( u \) and \( v \). So solving for \( x \) and \( y \) gives \( x = \frac{1}{2}(u + v) \) and \( y = \frac{1}{2}(v - u) \). In Figure 14.49 below, we see how the mapping \( x = x(u, v) = \frac{1}{2}(u + v) \), \( y = y(u, v) = \frac{1}{2}(v - u) \) maps the region \( R' \) onto \( R \) in a one-to-one manner.

Now we see that

\[
J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \Rightarrow |J(u, v)| = \left| \frac{1}{2} \right| = \frac{1}{2},
\]

Notes:
14.7 Change of Variables in Multiple Integrals

so using horizontal slices in $R'$, we have

$$
\iint_R e^{x+y} \, dA = \iint_{R'} f(x(u,v), y(u,v)) \left| J(u,v) \right| \, dA
\leq \int_0^1 \int_{-\nu}^\nu e^{\frac{1}{2}} 1 \, du \, dv
\leq \int_0^1 \frac{\nu^2}{2} \left( e^{\frac{1}{2}} \right) \, dv
\leq \frac{\nu^2}{4} (e - e^{-1}) \bigg|_0^1 = \frac{1}{4} \left( e - \frac{1}{e} \right) = \frac{e^2 - 1}{4e}.
$$

The change of variables formula can be used to evaluate double integrals in polar coordinates. Letting

$$
x = x(r, \theta) = r \cos \theta \quad \text{and} \quad y = y(r, \theta) = r \sin \theta,
$$

we have

$$
J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r \Rightarrow \left| J(u, v) \right| = \left| r \right| = r,
$$

which verifies Key Idea 61.

In a similar fashion, it can be shown (see Exercises 5 and 6) that triple integrals in cylindrical and spherical coordinates take the following forms:

**Key Idea 62 Triple Integral in Cylindrical Coordinates**

$$
\iiint_S f(x, y, z) \, dx \, dy \, dz = \iiint_{S'} f(r \cos \theta, r \sin \theta, z) \, r \, dr \, d\theta \, dz. \quad (14.6)
$$

where the mapping $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ maps the solid $S'$ in $r \theta z$-space onto the solid $S$ in $xyz$-space in a one-to-one manner.

Notes:
Chapter 14  Multiple Integration

Key Idea 63  Triple Integral in Spherical Coordinates

\[ \iiint_{S'} f(x, y, z) \, dx \, dy \, dz = \iiint_{S} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta, \quad (14.7) \]

where the mapping \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \) maps the solid \( S' \) in \( \rho \phi \theta \)-space onto the solid \( S \) in \( xyz \)-space in a one-to-one manner.

Example 2  Finding the Volume of a Sphere

For \( a > 0 \), find the volume \( V \) inside the sphere \( S = x^2 + y^2 + z^2 = a^2 \).

**SOLUTION**  We see that \( S \) is the set \( \rho = a \) in spherical coordinates, so

\[
V = \iiint_{S} 1 \, dV = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{a} 1 \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
= \int_{0}^{2\pi} \int_{0}^{\pi} \left( \frac{a^3}{3} \right) \sin \phi \, d\phi \, d\theta \\
= \int_{0}^{2\pi} \left( -\frac{a^3}{3} \cos \phi \right) \Big|_{\phi=0}^{\phi=\pi} \, d\theta \\
= \int_{0}^{2\pi} 2\frac{a^3}{3} \, d\theta = \frac{4\pi a^3}{3}.
\]

This chapter investigated the natural follow–on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating “quadruple integrals” whose bounds describe a region in 4–dimensional space (which are very hard to visualize). We can also look back to “regular” integration where we found the area under a curve in the plane. A natural analogue to this is finding the “area under a curve,” where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of “integration.”

Notes:
Exercises 14.7

Problems

1. Find the volume $V$ inside the paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 4$.
2. Find the volume $V$ inside the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 3$.
3. Find the volume $V$ of the solid inside both $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.
4. Find the volume $V$ inside both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.
5. Prove Equation (14.6).
7. Evaluate $\iint_{R} \sin \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right) \, dA$, where $R$ is the triangle with vertices $(0, 0), (2, 0)$ and $(1, 1)$. (Hint: Use the change of variables $u = (x+y)/2, v = (x-y)/2$.)
8. Find the volume of the solid bounded by $z = x^2 + y^2$ and $z^2 = 4(x^2 + y^2)$.
9. Find the volume inside the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $0 \leq z \leq 2$.
10. Show that the volume inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4}{3}abc\pi$. (Hint: Use the change of variables $x = au, y = bv, z = cw$, then consider Example 2.)
11. Show that the Beta function, defined by
    
    $$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad \text{for } x > 0, y > 0,$$
    
    satisfies the relation $B(y, x) = B(x, y)$ for $x > 0, y > 0$.
12. Using the substitution $t = u/(u+1)$, show that the Beta function can be written as
    
    $$B(x, y) = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}} \, du, \quad \text{for } x > 0, y > 0.$$
15: Line and Surface Integrals

15.1 Line Integrals

In single-variable calculus you learned how to integrate a real-valued function \( f(x) \) over an interval \([a, b]\) in \( \mathbb{R}^1 \). This integral (usually called a Riemann integral) can be thought of as an integral over a path in \( \mathbb{R}^1 \), since an interval (or collection of intervals) is really the only kind of “path” in \( \mathbb{R}^1 \). You may also recall that if \( f(x) \) represented the force applied along the \( x \)-axis to an object at position \( x \) in \([a, b]\), then the work \( W \) done in moving that object from position \( x = a \) to \( x = b \) was defined as the integral:

\[
W = \int_a^b f(x) \, dx
\]

In this section, we will see how to define the integral of a function (either real-valued or vector-valued) of two variables over a general path (i.e. a curve) in \( \mathbb{R}^2 \). This definition will be motivated by the physical notion of work. We will begin with real-valued functions of two variables.

In physics, the intuitive idea of work is that

\[
\text{Work} = \text{Force} \times \text{Distance}.
\]

Suppose that we want to find the total amount \( W \) of work done in moving an object along a curve \( C \) in \( \mathbb{R}^2 \) with a smooth parametrization \( x = x(t), y = y(t), a \leq t \leq b \), with a force \( f(x, y) \) which varies with the position \( (x, y) \) of the object and is applied in the direction of motion along \( C \) (see Figure 15.1 below).

![Figure 15.1: Curve C: x = x(t), y = y(t) for t in [a, b]](image)

We will assume for now that the function \( f(x, y) \) is continuous and real-valued, so we only consider the magnitude of the force. Partition the interval \([a, b]\) as follows:

\[
a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b, \text{ for some integer } n \geq 2
\]

As we can see from Figure 15.1, over a typical subinterval \([t_{i-1}, t_i]\) the distance \( \Delta s_i \) traveled along the curve is approximately \( \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \), by the Pythagorean theorem.
Theorem. Thus, if the subinterval is small enough then the work done in moving
the object along that piece of the curve is approximately

$$\text{Force} \times \text{Distance} \approx f(x_i^*, y_i^*) \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

where \((x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))\) for some \(t_i^*\) in \([t_{i-1}, t_i]\), and so

$$W \approx \sum_{i=1}^{n} f(x_i^*, y_i^*) \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

is approximately the total amount of work done over the entire curve. But since

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \Delta t_i,$$

where \(\Delta t_i = t_i - t_{i-1}\), then

$$W \approx \sum_{i=1}^{n} f(x_i^*, y_i^*) \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \Delta t_i.$$

Taking the limit of that sum as the length of the largest subinterval goes to 0, the
sum over all subintervals becomes the integral from \(t = a\) to \(t = b\), \(\Delta x_i \Delta t_i\)
and \(\Delta y_i \Delta t_i\) become \(x'(t)\) and \(y'(t)\), respectively, and \(f(x_i^*, y_i^*)\) becomes \(f(x(t), y(t))\), so that

$$W = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt.$$  

The integral on the right side of the above equation gives us our idea of how
to define, for \textit{any} real-valued function \(f(x, y)\), the integral of \(f(x, y)\) along the
curve \(C\), called a \textit{line integral}:

**Definition 112 Line Integral of a Real Valued Function**

For a real-valued function \(f(x, y)\) and a curve \(C\) in \(\mathbb{R}^2\), parametrized by
\(x = x(t), \ y = y(t), \ a \leq t \leq b\), the \textit{line integral of \(f(x, y)\) along \(C\) with
respect to arc length} \(s\) is

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt.$$  

The symbol \(ds\) is the differential of the arc length function

$$s = s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2} \, du,$$  

Notes:
which you may recognize from Section 10.3 as the length of the curve $C$ over the interval $[a, t]$, for all $t$ in $[a, b]$. That is,

$$ds = s'(t) \, dt = \sqrt{x'(t)^2 + y'(t)^2} \, dt,$$

by the Fundamental Theorem of Calculus.

For a general real-valued function $f(x, y)$, what does the line integral $\int_C f(x, y) \, ds$ represent? The preceding discussion of $ds$ gives us a clue. You can think of differentials as infinitesimal lengths. So if you think of $f(x, y)$ as the height of a picket fence along $C$, then $f(x, y) \, ds$ can be thought of as approximately the area of a section of that fence over some infinitesimally small section of the curve, and thus the line integral $\int_C f(x, y) \, ds$ is the total area of that picket fence (see Figure 15.2).

![Figure 15.2: Area of shaded rectangle = height \times width \approx f(x, y) \, ds](image)

Watch the video:

**Example 1**  
**Using the Line Integral**

Use a line integral to show that the lateral surface area $A$ of a right circular cylinder of radius $r$ and height $h$ is $2\pi rh$.

**Solution**  
We will use the right circular cylinder with base circle $C$ given by $x^2 + y^2 = r^2$ and with height $h$ in the positive $z$ direction (see Figure 15.3). Parametrize $C$ as follows:

$$x = x(t) = r \cos t, \quad y = y(t) = r \sin t, \quad 0 \leq t \leq 2\pi$$

![Figure 15.3: Figure for Example 1](image)

Notes:
Let \( f(x, y) = h \) for all \((x, y)\). Then

\[
A = \int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt
\]

\[
= \int_0^{2\pi} h \sqrt{(-r \sin t)^2 + (r \cos t)^2} \, dt
\]

\[
= h \int_0^{2\pi} r \sqrt{\sin^2 t + \cos^2 t} \, dt
\]

\[
= rh \int_0^{2\pi} 1 \, dt = 2\pi rh.
\]

Note in Example 1 that if we had traversed the circle \( C \) twice, i.e. let \( t \) vary from 0 to \( 4\pi \), then we would have gotten an area of \( 4\pi rh \), i.e. twice the desired area, even though the curve itself is still the same (namely, a circle of radius \( r \)). Also, notice that we traversed the circle in the counter-clockwise direction. If we had gone in the clockwise direction, using the parametrization

\[
x = x(t) = r \cos(2\pi - t), \quad y = y(t) = r \sin(2\pi - t), \quad 0 \leq t \leq 2\pi, \quad (15.1)
\]

then it is easy to verify (see Exercise 12) that the value of the line integral is unchanged.

In general, it can be shown (see Exercise 15) that reversing the direction in which a curve \( C \) is traversed leaves \( \int_C f(x, y) \, ds \) unchanged, for any \( f(x, y) \). If a curve \( C \) has a parametrization \( x = x(t), \ y = y(t), \ a \leq t \leq b \), then denote by \(-C\) the same curve as \( C \) but traversed in the opposite direction. Then \(-C\) is parametrized by

\[
x = x(a + b - t), \quad y = y(a + b - t), \quad a \leq t \leq b, \quad (15.2)
\]

and we have

\[
\int_C f(x, y) \, ds = \int_{-C} f(x, y) \, ds.
\]

Notice that our definition of the line integral was with respect to the arc length parameter \( s \). We can also define

\[
\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt
\]

as the line integral of \( f(x, y) \) along \( C \) with respect to \( x \), and

\[
\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt
\]

Notes:
as the line integral of $f(x, y)$ along $C$ with respect to $y$.

In the derivation of the formula for a line integral, we used the idea of work as force multiplied by distance. However, we know that force is actually a vector. So it would be helpful to develop a vector form for a line integral. For this, suppose that we have a function $\vec{f}(x, y)$ defined on $\mathbb{R}^2$ by

$$\vec{f}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

for some continuous real-valued functions $P(x, y)$ and $Q(x, y)$ on $\mathbb{R}^2$. Such a function $\vec{f}$ is called a vector field on $\mathbb{R}^2$. It is defined at points in $\mathbb{R}^2$, and its values are vectors in $\mathbb{R}^2$. For a curve $C$ with a smooth parametrization $x = x(t), y = y(t), a \leq t \leq b$, let

$$\vec{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}$$

be the position vector for a point $(x(t), y(t))$ on $C$. Then $\vec{r}'(t) = x'(t) \mathbf{i} + y'(t) \mathbf{j}$ and so

$$\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy = \int_a^b P(x(t), y(t)) \, x'(t) \, dt + \int_a^b Q(x(t), y(t)) \, y'(t) \, dt$$

$$= \int_a^b (P(x(t), y(t)) \, x'(t) + Q(x(t), y(t)) \, y'(t)) \, dt$$

$$= \int_a^b \vec{f}(x(t), y(t)) \cdot \vec{r}'(t) \, dt$$

by definition of $\vec{f}(x, y)$. Notice that the function $\vec{f}(x(t), y(t)) \cdot \vec{r}'(t)$ is a real-valued function on $[a, b]$, so the last integral on the right looks somewhat similar to our earlier definition of a line integral. This leads us to the following definition:

**Definition 113** Line Integral of a Vector Valued Function

For a vector field $\vec{f}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ and a curve $C$ with a smooth parametrization $x = x(t), y = y(t), a \leq t \leq b$, the line integral of $f$ along $C$ is

$$\int_C \vec{f} \cdot d\vec{r} = \int_C P(x, y) \, dx + \int_C Q(x, y) \, dy$$

$$= \int_a^b \vec{f}(x(t), y(t)) \cdot \vec{r}'(t) \, dt,$$

(15.3)

(15.4)

where $\vec{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}$ is the position vector for points on $C$. 

Notes:
We use the notation \( d\vec{r} = \vec{r}'(t) \, dt = dx \hat{i} + dy \hat{j} \) to denote the **differential** of the vector-valued function \( \vec{r} \). The line integral in Definition 113 is often called a *line integral of a vector field* to distinguish it from the line integral in Definition 112 which is called a *line integral of a scalar field*. For convenience we will often write

\[
\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy = \int_C P(x, y) \, dx + Q(x, y) \, dy,
\]

where it is understood that the line integral along \( C \) is being applied to both \( P \) and \( Q \). The quantity \( P(x, y) \, dx + Q(x, y) \, dy \) is known as a **differential form**.

**Note:** We defined total differential in Definition 89 in Section 13.4 as \( dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy \).

Recall that if the points on a curve \( C \) have position vector \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \), then \( \vec{r}'(t) \) is a tangent vector to \( C \) at the point \((x(t), y(t))\) in the direction of increasing \( t \) (which we call the *direction of \( C \)). Since \( C \) is a smooth curve, then \( \vec{r}'(t) \neq \vec{0} \) on \([a, b]\) and hence

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}
\]

is the unit tangent vector to \( C \) at \((x(t), y(t))\). Putting Definitions 112 and 113 together we get the following theorem:

**Theorem 132  Line Integrals and Tangent Vectors**

For a vector field \( \vec{f}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j} \) and a curve \( C \) with a smooth parametrization \( x = x(t), y = y(t), a \leq t \leq b \) and position vector \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \),

\[
\int_C \vec{f} \cdot d\vec{r} = \int_C \vec{f} \cdot \vec{T} \, ds,
\]

where \( \vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \) is the unit tangent vector to \( C \) at \((x(t), y(t))\).

If the vector field \( \vec{f}(x, y) \) represents the force moving an object along a curve \( C \), then the work \( W \) done by this force is

\[
W = \int_C \vec{f} \cdot \vec{T} \, ds = \int_C \vec{f} \cdot d\vec{r}.
\]

**Notes:**
Example 2  Evaluating Line Integrals

Evaluate \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \), where:

1. \( C : x = t, \quad y = 2t, \quad 0 \leq t \leq 1 \)
2. \( C : x = t, \quad y = 2t^2, \quad 0 \leq t \leq 1 \)

Solution  Figure 15.4 shows both curves.

1. Since \( x'(t) = 1 \) and \( y'(t) = 2 \), then

\[
\int_C (x^2 + y^2) \, dx + 2xy \, dy = \int_0^1 ((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)y'(t)) \, dt \\
= \int_0^1 ((t^2 + 4t^2)(1) + 2t(2t)(2)) \, dt \\
= \int_0^1 13t^2 \, dt \\
= \frac{13}{3} t^3 \bigg|_0^1 = \frac{13}{3}
\]

2. Since \( x'(t) = 1 \) and \( y'(t) = 4t \), then

\[
\int_C (x^2 + y^2) \, dx + 2xy \, dy = \int_0^1 ((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)y'(t)) \, dt \\
= \int_0^1 ((t^2 + 4t^4)(1) + 2t(2t^2)(4t)) \, dt \\
= \int_0^1 (t^2 + 20t^4) \, dt \\
= \frac{t^3}{3} + 4t^5 \bigg|_0^1 = \frac{1}{3} + 4 = \frac{13}{3}
\]

So in both cases, if the vector field \( \vec{f}(x, y) = (x^2 + y^2) \hat{i} + 2xy \hat{j} \) represents the force moving an object from \((0, 0)\) to \((1, 2)\) along the given curve \( C \), then the work done is \( \frac{13}{3} \). This may lead you to think that work (and more generally, the line integral of a vector field) is independent of the path taken. However, as we will see in the next section, this is not always the case.

Notes:
Although we defined line integrals over a single smooth curve, if $C$ is a piecewise smooth curve, that is

$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$

is the union of smooth curves $C_1, \ldots, C_n$, then we can define

$$\int_C \vec{f} \cdot d\vec{r} = \int_{C_1} \vec{f} \cdot d\vec{r}_1 + \int_{C_2} \vec{f} \cdot d\vec{r}_2 + \cdots + \int_{C_n} \vec{f} \cdot d\vec{r}_n$$

where each $\vec{r}_i$ is the position vector of the curve $C_i$.

**Example 3** A Piecewise Smooth Line Integral

Evaluate $\int_C (x^2 + y^2) \, dx + 2xy \, dy$, where $C$ is the polygonal path from $(0, 0)$ to $(0, 2)$ to $(1, 2)$.

**SOLUTION** Write $C = C_1 \cup C_2$, where $C_1$ is the curve given by $x = 0$, $y = t$, $0 \leq t \leq 2$ and $C_2$ is the curve given by $x = t$, $y = 2$, $0 \leq t \leq 1$ (see Figure 15.5). Then

$$\int_C (x^2 + y^2) \, dx + 2xy \, dy$$

$$= \int_{C_1} (x^2 + y^2) \, dx + 2xy \, dy$$

$$+ \int_{C_2} (x^2 + y^2) \, dx + 2xy \, dy$$

$$= \int_0^2 ((0^2 + t^2)(0) + 2(0)\tau(1)) \, dt + \int_0^1 ((t^2 + 4)(1) + 2t(2)(0)) \, dt$$

$$= \int_0^2 0 \, dt + \int_0^1 (t^2 + 4) \, dt$$

$$= \frac{t^3}{3} + 4t \bigg|_0^1 = \frac{1}{3} + 4 = \frac{13}{3}$$

Line integral notation varies quite a bit. For example, in physics it is common to see the notation $\int_a^b \vec{f} \cdot d\vec{l}$, where it is understood that the limits of integration $a$ and $b$ are for the underlying parameter $t$ of the curve, and the letter $l$ signifies length. Also, the formulation $\int_{C} \vec{f} \cdot \vec{T} \, ds$ from Theorem 132 is often preferred in physics since it emphasizes the idea of integrating the tangential component $\vec{f} \cdot \vec{T}$ of $\vec{f}$ in the direction of $\vec{T}$ (i.e., in the direction of $C$), which is a useful physical interpretation of line integrals.

Notes:
Exercises 15.1

Problems

In Exercises 1–4, calculate \( \int_C f(x, y) \, ds \) for the given function \( f(x, y) \) and curve \( C \).

1. \( f(x, y) = xy; \quad C: x = \cos t, y = \sin t, 0 \leq t \leq \pi/2 \)
2. \( f(x, y) = \frac{x}{x^2 + 1}; \quad C: x = t, y = 0, 0 \leq t \leq 1 \)
3. \( f(x, y) = 2x + y; \quad C: \) polygonal path from \((0, 0)\) to \((3, 0)\) to \((3, 2)\)
4. \( f(x, y) = x + y^2; \quad C: \) path from \((0, 0)\) clockwise along the circle \(x^2 + y^2 = 4\) to the point \((-2, 0)\) and then back to \((2, 0)\) along the x-axis

Use a line integral to find the lateral surface area of the part of the cylinder \(x^2 + y^2 = 4\) below the plane \(x + 2y + z = 6\) and above the xy-plane.

In Exercises 6–11, calculate \( \int_C \vec{F} \cdot d\vec{r} \) for the given vector field \( \vec{F}(x, y) \) and curve \( C \).

6. \( \vec{F}(x, y) = \vec{i} - \vec{j}; \quad C: x = 3t, y = 2t, 0 \leq t \leq 1 \)
7. \( \vec{F}(x, y) = y \vec{i} - x \vec{j}; \quad C: x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \)
8. \( \vec{F}(x, y) = x \vec{i} + y \vec{j}; \quad C: x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \)
9. \( \vec{F}(x, y) = (x^2 - y) \vec{i} + (x - y^2) \vec{j}; \quad C: x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \)

10. \( \vec{F}(x, y) = xy^2 \vec{i} + xy^3 \vec{j}; \quad C: \) the polygonal path from \((0, 0)\) to \((1, 0)\) to \((0, 1)\) to \((0, 0)\)
11. \( \vec{F}(x, y) = (x^2 + y^2) \vec{i}; \quad C: x = 2 + \cos t, y = \sin t, 0 \leq t \leq 2\pi \)
12. Verify that the value of the line integral in Example 1 is unchanged when using the parametrization of the circle \( C \) given in Equation (15.1).
13. Show that if \( \vec{F} \perp \vec{r}'(t) \) at each point \( \vec{r}(t) \) along a smooth curve \( C \), then \( \int_C \vec{F} \cdot d\vec{r} = 0 \).
14. Show that if \( \vec{F} \) points in the same direction as \( \vec{r}'(t) \) at each point \( \vec{r}(t) \) along a smooth curve \( C \), then \( \int_C \vec{F} \cdot d\vec{r} = \int_C \left| \vec{F} \right| \, ds \).
15. Prove that \( \int_C f(x, y) \, ds = \int_{-C} f(x, y) \, ds \). (Hint: Use Equation (15.2).)
16. Let \( C \) be a smooth curve with arc length \( L \), and suppose that \( \vec{F}(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j} \) is a vector field such that \( \left| \vec{F}(x, y) \right| \leq M \) for all \((x, y)\) on \( C \). Show that \( \int_C \vec{F} \cdot d\vec{r} \leq ML \). (Hint: Recall that \( \int_a^b g(x) \, dx \leq \int_a^b |g(x)| \, dx \) for Riemann integrals.)
17. Prove that the Riemann integral \( \int_a^b f(x) \, dx \) is a special case of a line integral.
15.2 Properties of Line Integrals

We know from the previous section that for line integrals of real-valued functions (scalar fields), reversing the direction in which the integral is taken along a curve does not change the value of the line integral:

\[
\int_C f(x, y) \, ds = \int_{-C} f(x, y) \, ds
\]

For line integrals of vector fields, however, the value does change. To see this, let \( \vec{f}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j} \) be a vector field, with \( P \) and \( Q \) continuously differentiable functions. Let \( C \) be a smooth curve parametrized by \( x = x(t), y = y(t), a \leq t \leq b \), with position vector \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \) (we will usually abbreviate this by saying that \( C : \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \) is a smooth curve). We know that the curve \(-C \) traversed in the opposite direction is parametrized by \( x = x(a + b - t), y = y(a + b - t), a \leq t \leq b \). Then

\[
\int_{-C} P(x, y) \, dx = \int_a^b P(x(a + b - t), y(a + b - t)) \frac{d}{dt}(x(a + b - t)) \, dt \\
= \int_a^b P(x(a + b - t), y(a + b - t)) (-x'(a + b - t)) \, dt \quad \text{(by the Chain Rule)} \\
= \int_b^a P(x(u), y(u)) (-x'(u)) (-du) \quad \text{(by letting } u = a + b - t) \\
= \int_b^a P(x(u), y(u)) x'(u) \, du \\
= -\int_a^b P(x(u), y(u)) x'(u) \, du, \quad \text{since } \int_b^a = -\int_a^b, \text{ so} \\
\int_{-C} P(x, y) \, dx = -\int_C P(x, y) \, dx
\]

since we are just using a different letter \( u \) for the line integral along \( C \). A similar argument shows that

\[
\int_{-C} Q(x, y) \, dy = -\int_C Q(x, y) \, dy,
\]

Notes:
and hence
\[ \int_C \vec{f} \cdot d\vec{r} = \int_C P(x, y) \, dx + \int_C Q(x, y) \, dy \]
\[ = - \int_C P(x, y) \, dx + - \int_C Q(x, y) \, dy \]
\[ = - \left( \int_C P(x, y) \, dx + \int_C Q(x, y) \, dy \right) \]
\[ = \int_C \vec{f} \cdot d\vec{r}. \]

The above formula can be interpreted in terms of the work done by a force \( \vec{f}(x, y) \) (treated as a vector) moving an object along a curve \( C \): the total work performed moving the object along \( C \) from its initial point to its terminal point, and then back to the initial point moving backwards along the same path, is zero. This is because when force is considered as a vector, direction is accounted for.

The preceding discussion shows the importance of always taking the direction of the curve into account when using line integrals of vector fields. For this reason, the curves in line integrals are sometimes referred to as directed curves or oriented curves.

Recall that our definition of a line integral required that we have a parametrization \( x = x(t), y = y(t), a \leq t \leq b \) for the curve \( C \). But as we know, any curve has infinitely many parameterizations. So could we get a different value for a line integral using some other parametrization of \( C \), say, \( x = \tilde{x}(u), y = \tilde{y}(u), c \leq u \leq d \)? If so, this would mean that our definition is not well-defined. Luckily, it turns out that the value of a line integral of a vector field is unchanged as long as the direction of the curve \( C \) is preserved by whatever parametrization is chosen:

**Theorem 133  Line Integral is Independent of Parameterization**

Let \( \vec{f}(x, y) = P(x, y) \, i + Q(x, y) \, j \) be a vector field, and let \( C \) be a smooth curve parametrized by \( x = x(t), y = y(t), a \leq t \leq b \). Suppose that \( t = \alpha(u) \) for \( c \leq u \leq d \), such that \( a = \alpha(c), b = \alpha(d) \), and \( \alpha'(u) > 0 \) on the open interval \( (c, d) \) (i.e., \( \alpha(u) \) is strictly increasing on \( [c, d] \)). Then \( \int_C \vec{f} \cdot d\vec{r} \) has the same value for the parameterizations \( x = x(t), y = y(t), a \leq t \leq b \) and \( x = \tilde{x}(u) = x(\alpha(u)), y = \tilde{y}(u) = y(\alpha(u)), c \leq u \leq d \).

**Proof**

Since \( \alpha(u) \) is strictly increasing and maps \( [c, d] \) onto \( [a, b] \), then we know that

Notes:
Chapter 15  Line and Surface Integrals

\[ t = \alpha(u) \] has an inverse function \( u = \alpha^{-1}(t) \) defined on \([a, b] \) such that \( c = \alpha^{-1}(a), \) \( d = \alpha^{-1}(b), \) and \( \frac{du}{dt} = \frac{1}{\alpha'(u)}. \) Also, \( dt = \alpha'(u) \, du, \) and by the Chain Rule

\[ \frac{dx}{du} \left( \alpha(u) \right) = \frac{dx}{dt} \frac{dt}{du} = \alpha'(u) \Rightarrow x'(t) = \frac{x'(u)}{\alpha'(u)} \]

so making the substitution \( t = \alpha(u) \) gives

\[
\int_a^b P(x(t), y(t)) \dot{x}(t) \, dt = \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} P(x(\alpha(u)), y(\alpha(u))) \frac{x'(u)}{\alpha'(u)} (\alpha'(u) \, du)
\]

\[
= \int_c^d P(x(u), y(u)) \dot{x}'(u) \, du,
\]

which shows that \( \int_C P(x, y) \, dx \) has the same value for both parameterizations. A similar argument shows that \( \int_C Q(x, y) \, dy \) has the same value for both parameterizations, and hence \( \int_C \vec{F} \cdot d\vec{r} \) has the same value. \( \square \)

Notice that the condition \( \alpha'(u) > 0 \) in Theorem 133 means that the two parameterizations move along \( C \) in the same direction. That was not the case with the “reverse” parametrization for \( -C \): for \( u = a + b - t \) we have \( t = \alpha(u) = a + b - u \Rightarrow \alpha'(u) = -1 < 0. \)

Example 1  Re-evaluating a Line Integral
Evaluate the line integral

\[
\int_C (x^2 + y^2) \, dx + 2xy \, dy
\]

from Example 15.1.2 in Section 15.1, along the curve \( C : x = t, \ y = 2t^2, \ 0 \leq t \leq 1, \) where \( t = \sin u \) for \( 0 \leq u \leq \pi/2. \)

SOLUTION  First, we notice that \( 0 = \sin 0, \ 1 = \sin(\pi/2), \) and \( \frac{dt}{du} = \cos u > 0 \) on \((0, \pi/2).\) So by Theorem 133 we know that if \( C \) is parametrized by

\[
x = \sin u, \quad y = 2\sin^2 u, \quad 0 \leq u \leq \pi/2
\]

then \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \) should have the same value as we found in Exam-
Example 15.1.2, namely $\frac{13}{3}$. And we can indeed verify this:

$$\int_C (x^2 + y^2) \, dx + 2xy \, dy$$

$$= \int_0^{\pi/2} \left( (\sin^2 u + (2 \sin^2 u)^2) \cos u + 2(\sin u)(2 \sin^2 u)4 \sin u \cos u \right) \, du$$

$$= \int_0^{\pi/2} \left( \sin^3 u + 20 \sin^4 u \right) \cos u \, du$$

$$= \frac{\sin^3 u}{3} + 4 \sin^5 u \bigg|_0^{\pi/2}$$

$$= \frac{1}{3} + 4 = \frac{13}{3}$$

In other words, the line integral is unchanged whether $t$ or $u$ is the parameter for $C$.

By a closed curve, we mean a curve $C$ whose initial point and terminal point are the same, i.e., for $C: x = x(t), y = y(t), a \leq t \leq b$, we have $(x(a),y(a)) = (x(b),y(b))$.

A simple closed curve is a closed curve which does not intersect itself. Note that any closed curve can be regarded as a union of simple closed curves (think of the loops in a figure eight). We use the special notation

$$\oint_C f(x, y) \, ds \quad \text{and} \quad \oint_C \mathbf{F} \cdot d\mathbf{r}$$

to denote line integrals of scalar and vector fields, respectively, along closed curves. In some older texts you may see the notation $\oint_C$ or $\oint_C$ to indicate a line integral traversing a closed curve in a counterclockwise or clockwise direction, respectively.

So far, the examples we have seen of line integrals (e.g., Example 15.1.2) have had the same value for different curves joining the initial point to the terminal
point. That is, the line integral has been independent of the path joining the two points. As we mentioned before, this is not always the case. The following theorem gives a necessary and sufficient condition for this path independence:

**Theorem 134 Path Independence of Line Integrals**

In a region $R$, the line integral \( \int_C \vec{f} \cdot d\vec{r} \) is independent of the path between any two points in $R$ if and only if \( \oint_C \vec{f} \cdot d\vec{r} = 0 \) for every closed curve $C$ which is contained in $R$.

**Proof**

Suppose that \( \oint_C \vec{f} \cdot d\vec{r} = 0 \) for every closed curve $C$ which is contained in $R$. Let $P_1$ and $P_2$ be two distinct points in $R$. Let $C_1$ be a curve in $R$ going from $P_1$ to $P_2$, and let $C_2$ be another curve in $R$ going from $P_1$ to $P_2$, as in Figure 15.7.

Then $C = C_1 \cup -C_2$ is a closed curve in $R$ (from $P_1$ to $P_1$), and so $\oint_C \vec{f} \cdot d\vec{r} = 0$. Thus,

\[
0 = \oint_C \vec{f} \cdot d\vec{r} = \int_{C_1} \vec{f} \cdot d\vec{r} + \int_{-C_2} \vec{f} \cdot d\vec{r} = \int_{C_1} \vec{f} \cdot d\vec{r} - \int_{C_2} \vec{f} \cdot d\vec{r},
\]

and so \( \int_{C_1} \vec{f} \cdot d\vec{r} = \int_{C_2} \vec{f} \cdot d\vec{r} \). This proves path independence.

Conversely, suppose that the line integral \( \int_C \vec{f} \cdot d\vec{r} \) is independent of the path between any two points in $R$. Let $C$ be a closed curve contained in $R$. Let $P_1$ and $P_2$ be two distinct points on $C$. Let $C_1$ be a part of the curve $C$ that goes from $P_1$ to $P_2$, and let $C_2$ be the remaining part of $C$ that goes from $P_1$ to $P_2$, again as in

Notes:
Figure 15.7. Then by path independence we have

\[ \int_{C_1} \vec{f} \cdot d\vec{r} = \int_{C_2} \vec{f} \cdot d\vec{r} \]

\[ \int_{C_1} \vec{f} \cdot d\vec{r} - \int_{C_2} \vec{f} \cdot d\vec{r} = 0 \]

\[ \int_{C_1} \vec{f} \cdot d\vec{r} + \int_{-C_2} \vec{f} \cdot d\vec{r} = 0, \text{ so} \]

\[ \int_{C} \vec{f} \cdot d\vec{r} = 0 \]

since \( C = C_1 \cup -C_2 \). □

Clearly, the above theorem does not give a practical way to determine path independence, since it is impossible to check the line integrals around all possible closed curves in a region. What it mostly does is give an idea of the way in which line integrals behave, and how seemingly unrelated line integrals can be related (in this case, a specific line integral between two points and all line integrals around closed curves). We will now prove the following sufficient condition for path independence of line integrals:

**Theorem 135  Fundamental Theorem of Line Integrals**

Let \( \vec{f}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j} \) be a vector field in some region \( R \) without holes, with \( P \) and \( Q \) continuously differentiable functions on \( R \). Let \( C \) be a smooth curve in \( R \) parametrized by \( x = x(t), y = y(t), a \leq t \leq b. \)

Suppose that there is a real-valued function \( F(x, y) \) such that \( \nabla F = \vec{f} \) on \( R \). Then

\[ \int_{C} \vec{f} \cdot d\vec{r} = F(B) - F(A), \]

where \( A = (x(a), y(a)) \) and \( B = (x(b), y(b)) \) are the endpoints of \( C \). Thus, the line integral is independent of the path between its endpoints, since it depends only on the values of \( F \) at those endpoints.

**Proof**

**Notes:**
By definition of $\int_C \vec{f} \cdot d\vec{r}$, we have

$$\int_C \vec{f} \cdot d\vec{r} = \int_a^b \left( P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) \right) dt$$

$$= \int_a^b \left( \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} \right) dt \text{ (since } \nabla F = \vec{f} \Rightarrow \frac{\partial F}{\partial x} = P \text{ and } \frac{\partial F}{\partial y} = Q \}$$

$$= \int_a^b F'(x(t), y(t)) dt \text{ (by Theorem 110)}$$

$$= F(x(t), y(t)) \bigg|_a^b = F(B) - F(A)$$

by the Fundamental Theorem of Calculus.

Theorem 135 can be thought of as the line integral version of the Fundamental Theorem of Calculus. A real-valued function $F(x, y)$ such that $\nabla F(x, y) = \vec{f}(x, y)$ is called a potential for $\vec{f}$. A conservative vector field is one which has a potential.

---

**Example 2**

Using the Fundamental Theorem of Line Integrals

Recall from Examples 15.1.2 and 15.1.3 in Section 15.1 that the line integral $\int_C (x^2 + y^2) \, dx + 2xy \, dy$ was found to have the value $\frac{13}{2}$ for three different curves $C$ going from the point $(0, 0)$ to the point $(1, 2)$. Use Theorem 135 to show that this line integral is indeed path independent.

**Solution**

We need to find a real-valued function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = x^2 + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = 2xy.$$

Suppose that $\frac{\partial F}{\partial x} = x^2 + y^2$, then we must have $F(x, y) = \frac{1}{3}x^3 + xy^2 + g(y)$ for some function $g(y)$. So $\frac{\partial F}{\partial y} = 2xy + g'(y)$ satisfies the condition $\frac{\partial F}{\partial y} = 2xy$ if $g'(y) = 0$, i.e., $g(y) = K$, where $K$ is a constant. Since any choice for $K$ will do (why?), we pick $K = 0$. Thus, a potential $F(x, y)$ for $\vec{F}(x, y) = (x^2 + y^2) \vec{i} + 2xy \vec{j}$ exists, namely

$$F(x, y) = \frac{1}{3}x^3 + xy^2.$$
Hence the line integral \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \) is path independent.

Note that we can also verify that the value of the line integral of \( \vec{f} \) along any curve \( C \) going from \((0, 0)\) to \((1, 2)\) will always be \( \frac{13}{3} \), since by Theorem 135

\[
\int_C \vec{f} \cdot d\vec{r} = F(1, 2) - F(0, 0) = \frac{1}{3}(1)^3 + (1)(2)^2 - (0 + 0) = \frac{1}{3} + 4 = \frac{13}{3}.
\]

A consequence of Theorem 135 in the special case where \( C \) is a closed curve, so that the endpoints \( A \) and \( B \) are the same point, is the following:

**Theorem 136  Closed Line Integrals of Conservative Fields**

If a vector field \( \vec{f} \) has a potential in a region \( R \) without holes, then \( \int_C \vec{f} \cdot d\vec{r} = 0 \) for any closed curve \( C \) in \( R \) (i.e., \( \int_C \nabla F \cdot d\vec{r} = 0 \) for any real-valued function \( F(x, y) \)).

**Example 3  Calculating a Closed Line Integral of a Conservative Field**

Evaluate \( \int_C x \, dx + y \, dy \) for \( C : x = 2 \cos t, \ y = 3 \sin t, \ 0 \leq t \leq 2\pi \).

**SOLUTION**

The vector field \( \vec{f}(x, y) = x \hat{i} + y \hat{j} \) has a potential \( F(x, y) \):

\[
\frac{\partial F}{\partial x} = x \Rightarrow F(x, y) = \frac{1}{2}x^2 + g(y), \text{ so}
\]

\[
\frac{\partial F}{\partial y} = y \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{1}{2}y^2 + K
\]

for any constant \( K \), so \( F(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 \) is a potential for \( \vec{f}(x, y) \). Thus,

\[
\int_C x \, dx + y \, dy = \int_C \vec{f} \cdot d\vec{r} = 0
\]

by Theorem 136, since the curve \( C \) is closed (it is the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)).
Problems

1. Evaluate \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \) for \( C : x = \cos t, y = \sin t, 0 \leq t \leq \frac{\pi}{2} \).

2. Evaluate \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \) for \( C : x = \cos t, y = \sin t, 0 \leq t \leq \pi \).

3. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = y \vec{i} - x \vec{j} \)? If so, find one.

4. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = x \vec{i} - y \vec{j} \)? If so, find one.

5. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = xy \vec{i} + x^2 \vec{j} \)? If so, find one.

6. Let \( \vec{f}(x, y) \) and \( \vec{g}(x, y) \) be vector fields, let \( a \) and \( b \) be constants, and let \( C \) be a curve in \( \mathbb{R}^2 \). Show that
   \[
   \int_C (a \vec{f} + b \vec{g}) \cdot d\vec{r} = a \int_C \vec{f} \cdot d\vec{r} + b \int_C \vec{g} \cdot d\vec{r}.
   \]

7. Let \( C \) be a curve whose arc length is \( L \). Show that \( \int_C 1 \, ds = L \).

8. Let \( f(x, y) \) and \( g(x, y) \) be continuously differentiable real-valued functions in a region \( R \). Show that
   \[
   \int_C (f \nabla g) \cdot d\vec{r} = -\int_C (g \nabla f) \cdot d\vec{r}
   \]
   for any closed curve \( C \) in \( R \).

9. Let \( \vec{f}(x, y) = \frac{y}{x^2+y^2} \vec{i} + \frac{x}{x^2+y^2} \vec{j} \) for all \( (x, y) \neq (0,0) \), and \( C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \).
   (a) Show that \( \vec{f} = \nabla F \), for \( F(x, y) = \tan^{-1}(y/x) \).
   (b) Show that \( \int_C \vec{f} \cdot d\vec{r} = 2\pi \). Does this contradict Theorem 136? Explain.

10. Let \( g(x) \) and \( h(y) \) be differentiable functions, and let \( \vec{f}(x, y) = h(y) \vec{i} + g(x) \vec{j} \). Is it possible for \( \vec{f} \) to have a potential \( F(x, y) \)? If so, find an example. You may assume that \( F \) would be smooth. (Hint: Consider the mixed partial derivatives of \( F \).)
15.3 Green's Theorem

We will now see a way of evaluating the line integral of a smooth vector field around a simple closed curve. A vector field \( \mathbf{f}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \) is smooth if its component functions \( P(x, y) \) and \( Q(x, y) \) are smooth. We will use Green's Theorem (sometimes called Green's Theorem in the plane) to relate the line integral around a closed curve with a double integral over the region inside the curve:

**Theorem 137 Green's Theorem**

Let \( R \) be a region in \( \mathbb{R}^2 \) whose boundary is a simple closed curve \( C \) which is piecewise smooth. Let \( \mathbf{f}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \) be a smooth vector field defined on both \( R \) and \( C \). Then

\[
\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA, \tag{15.5}
\]

where \( C \) is traversed so that \( R \) is always on the left side of \( C \).

**Proof**

We will prove the theorem in the case for a simple region \( R \), that is, where the boundary curve \( C \) can be written as \( C = C_1 \cup C_2 \) in two distinct ways:

\[
\begin{align*}
C_1 &= \text{the curve } y = y_1(x) \text{ from the point } X_1 \text{ to the point } X_2 \quad (15.6) \\
C_2 &= \text{the curve } y = y_2(x) \text{ from the point } X_2 \text{ to the point } X_1, \quad (15.7)
\end{align*}
\]

where \( X_1 \) and \( X_2 \) are the points on \( C \) farthest to the left and right, respectively; and

\[
\begin{align*}
C_1 &= \text{the curve } x = x_1(y) \text{ from the point } Y_1 \text{ to the point } Y_2 \quad (15.8) \\
C_2 &= \text{the curve } x = x_2(y) \text{ from the point } Y_2 \text{ to the point } Y_1, \quad (15.9)
\end{align*}
\]

where \( Y_1 \) and \( Y_2 \) are the lowest and highest points, respectively, on \( C \). See Figure 15.8.

Notes:
Integrate $P(x, y)$ around $C$ using the representation $C = C_1 \cup C_2$ given by Equations (15.6) and (15.7). Since $y = y_1(x)$ along $C_1$ (as $x$ goes from $a$ to $b$) and $y = y_2(x)$ along $C_2$ (as $x$ goes from $b$ to $a$), as we see from Figure 15.8, then we have

\[
\oint_C P(x, y) \, dx = \int_{C_1} P(x, y) \, dx + \int_{C_2} P(x, y) \, dx
\]

\[
= \int_a^b P(x, y_1(x)) \, dx + \int_a^b P(x, y_2(x)) \, dx
\]

\[
= \int_a^b P(x, y_1(x)) \, dx - \int_a^b P(x, y_2(x)) \, dx
\]

\[
= -\int_a^b (P(x, y_2(x)) - P(x, y_1(x))) \, dx
\]

\[
= -\left( \int_a^b P(x, y) \bigg|_{y=y_2(x)}^{y=y_1(x)} \right) \, dx
\]

\[
= -\int_a^b \int_{y_2(x)}^{y_1(x)} \frac{\partial P(x, y)}{\partial y} \, dy \, dx \text{ (by the Fundamental Theorem of Calculus)}
\]

\[
= -\int_R \frac{\partial P}{\partial y} \, dA.
\]

Likewise, integrate $Q(x, y)$ around $C$ using the representation $C = C_1 \cup C_2$ given by Equations (15.8) and (15.9). Since $x = x_1(y)$ along $C_1$ (as $y$ goes from $d$ to $c$) and $x = x_2(y)$ along $C_2$ (as $y$ goes from $c$ to $d$), as we see from Figure 15.8, then

Notes:
we have
\[
\int_C Q(x, y) \, dy = \int_{C_1} Q(x, y) \, dy + \int_{C_2} Q(x, y) \, dy
\]
\[
= \int_d^c Q(x_1(y), y) \, dy + \int_c^d Q(x_2(y), y) \, dy
\]
\[
= -\int_c^d Q(x_1(y), y) \, dy + \int_c^d Q(x_2(y), y) \, dy
\]
\[
= \int_c^d (Q(x_2(y), y) - Q(x_1(y), y)) \, dy
\]
\[
= \int_c^d \left( Q(x, y) \right)_{x=x_1(y)}^{x=x_2(y)} \, dy
\]
\[
= \int_c^d \int_{c(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x} \, dx \, dy \quad \text{(by the Fundamental Theorem of Calculus)}
\]
\[
= \iiint_R \frac{\partial Q}{\partial x} \, dA.
\]

Putting this together, we have
\[
\oint_C \vec{f} \cdot d\vec{r} = \int_C P(x, y) \, dx + \int_C Q(x, y) \, dy
\]
\[
= -\iint_R \frac{\partial P}{\partial y} \, dA + \iint_R \frac{\partial Q}{\partial x} \, dA
\]
\[
= \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA. \quad \square
\]

Though we proved Green’s Theorem only for a simple region \( R \), the theorem can also be proved for more general regions (say, a union of simple regions).

Watch the video:
Green’s Theorem at
https://youtu.be/a_zdFvYXX_c

Notes:
Example 1  **Using Green’s Theorem**

Evaluate \( \oint_C (x^2 + y^2) \, dx + 2xy \, dy \), where \( C \) is the boundary (traversed counterclockwise) of the region \( R = \{(x, y): 0 \leq x \leq 1, 2x^2 \leq y \leq 2x\} \).

**SOLUTION**  
\( R \) is the shaded region in Figure 15.9. By Green’s Theorem, for \( P(x, y) = x^2 + y^2 \) and \( Q(x, y) = 2xy \), we have

\[
\oint_C (x^2 + y^2) \, dx + 2xy \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[
= \iint_R (2y - 2y) \, dA = \iint_R 0 \, dA = 0.
\]

We actually already knew that the answer was zero. Recall from Exercise 15.2.2 in Section 15.2 that the vector field \( \vec{f}(x, y) = (x^2 + y^2) \vec{i} + 2xy \vec{j} \) has a potential function \( F(x, y) = \frac{1}{2}x^3 + xy^2 \), and so \( \vec{f} \cdot \vec{r'} = 0 \) by Theorem 136.

Example 2  **Green’s Theorem with a Hole**

Let \( \vec{f}(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j} \), where

\[
P(x, y) = -\frac{y}{x^2 + y^2} \quad \text{and} \quad Q(x, y) = \frac{x}{x^2 + y^2},
\]

and let \( R = \{(x, y): 0 < x^2 + y^2 \leq 1\} \). For the boundary curve \( C: x^2 + y^2 = 1 \), traversed counterclockwise, it was shown in Exercise 9(b) in Section 15.2 that \( \oint_C \vec{f} \cdot \vec{r'} = 2\pi \). But

\[
\frac{\partial Q}{\partial x} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial P}{\partial y} \Rightarrow \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_R 0 \, dA = 0.
\]

This would seem to contradict Green’s Theorem. However, note that \( R \) is not the entire region enclosed by \( C \), since the point \((0, 0)\) is not contained in \( R \). That is, \( R \) has a “hole” at the origin, so Green’s Theorem does not apply.

If we modify the region \( R \) to be the **annulus** \( R = \{(x, y): 1/4 \leq x^2 + y^2 \leq 1\} \) (see Figure 15.10), and take the “boundary” \( C \) of \( R \) to be \( C = C_1 \cup C_2 \), where \( C_1 \) is the unit circle \( x^2 + y^2 = 1 \) traversed counterclockwise and \( C_2 \) is the circle \( x^2 + y^2 = 1/4 \) traversed **clockwise**, then it can be shown (see Exercise 8) that

\[
\oint_C \vec{f} \cdot \vec{r'} = 0.
\]

We would still have \( \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = 0 \), so for this \( R \) we would have

\[
\oint_C \vec{f} \cdot \vec{r'} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA,
\]

**Notes:**
which shows that Green’s Theorem holds for the annular region $R$.

It turns out that Green’s Theorem can be extended to multiply connected regions, that is, regions like the annulus in Example 2, which have one or more regions cut out from the interior, as opposed to discrete points being cut out. For such regions, the “outer” boundary and the “inner” boundaries are traversed so that $R$ is always on the left side.

![Figure 15.11: Multiply connected regions](image)

The intuitive idea for why Green’s Theorem holds for multiply connected regions is shown in Figure 15.11 above. The idea is to cut “slits” between the boundaries of a multiply connected region $R$ so that $R$ is divided into subregions which do not have any “holes”. For example, in Figure 15.11(a) the region $R$ is the union of the regions $R_1$ and $R_2$, which are divided by the slits indicated by the dashed lines. Those slits are part of the boundary of both $R_1$ and $R_2$, and we traverse them in the manner indicated by the arrows. Notice that along each slit the boundary of $R_1$ is traversed in the opposite direction as that of $R_2$, which means that the line integrals of $\vec{f}$ along those slits add to 0. Since $R_1$ and $R_2$ do not have holes in them, then Green’s Theorem holds in each subregion, so that

$$\oint_{\text{bdy of } R_1} \vec{f} \cdot d\vec{r} = \iint_{R_1} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$ and $$\oint_{\text{bdy of } R_2} \vec{f} \cdot d\vec{r} = \iint_{R_2} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

But since the line integrals along the slits are opposite each other, we have

$$\oint_{C_1 \cup C_2} \vec{f} \cdot d\vec{r} = \oint_{\text{bdy of } R_1} \vec{f} \cdot d\vec{r} + \oint_{\text{bdy of } R_2} \vec{f} \cdot d\vec{r},$$

and so

$$\oint_{C_1 \cup C_2} \vec{f} \cdot d\vec{r} = \iint_{R_1} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{R_2} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Notes:
which shows that Green’s Theorem holds in the region $R$. A similar argument shows that the theorem holds in the region with two holes shown in Figure 15.11(b).

We know from Theorem 136 that when a smooth vector field $\vec{f}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j}$ on a region $R$ (whose boundary is a piecewise smooth, simple closed curve $C$) has a potential in $R$, then $\int_C \vec{f} \cdot d\vec{r} = 0$. And if the potential $F(x, y)$ is smooth in $R$, then $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$, and so we know that

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ in } R.$$  

Conversely, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $R$ then

$$\int_C \vec{f} \cdot d\vec{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_R 0 \, dA = 0.$$

For a simply connected region $R$ (i.e. a region with no holes), the following can be shown:

**Theorem 138  Equivalence of Path Independence**

The following statements are equivalent for a simply connected region $R$ in $\mathbb{R}^2$:

1. $\vec{f}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j}$ has a smooth potential $F(x, y)$ in $R$

2. $\int_C \vec{f} \cdot d\vec{r}$ is independent of the path for any curve $C$ in $R$

3. $\int_C \vec{f} \cdot d\vec{r} = 0$ for every simple closed curve $C$ in $R$

4. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $R$ (in this case, the differential form $P \, dx + Q \, dy$ is exact)

---

Notes:
Exercises 15.3

Problems

In Exercises 1–4, use Green’s Theorem to evaluate the given line integral around the curve C, traversed counterclockwise.

1. \( \int_C (x^2 - y^2) \, dx + 2xy \, dy \); C is the boundary of \( R = \{(x, y) : 0 \leq x \leq 1, 2x^2 \leq y \leq 2x \} \)

2. \( \int_C x^2 y \, dx + 2xy \, dy \); C is the boundary of \( R = \{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq x \} \)

3. \( \int_C 2y \, dx - 3x \, dy \); C is the circle \( x^2 + y^2 = 1 \)

4. \( \int_C (e^{x^2} + y^2) \, dx + (e^{x^2} + x^2) \, dy \); C is the boundary of the triangle with vertices \((0, 0), (4, 0)\) and \((0, 4)\)

5. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = (y^2 + 3x^2) \hat{i} + 2xy \hat{j} \)? If so, find one.

6. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = (x^3 \cos(xy) + 2x \sin(xy)) \hat{i} + x^2 y \cos(xy) \hat{j} \)? If so, find one.

7. Is there a potential \( F(x, y) \) for \( \vec{f}(x, y) = (8xy + 3) \hat{i} + 4(x^2 + y) \hat{j} \)? If so, find one.

8. Show that for any constants \( a, b \) and any closed simple curve \( C, \int_C a \, dx + b \, dy = 0. \)

9. For the vector field \( \vec{f} \) as in Example 2, show directly that \( \oint_C \vec{f} \cdot d\vec{r} = 0 \), where C is the boundary of the annulus \( R = \{(x, y) : 1/4 \leq x^2 + y^2 \leq 1 \} \) traversed so that \( R \) is always on the left.

10. Evaluate \( \int_C e^x \sin y \, dx + (y^3 + e^x \cos y) \, dy \), where C is the boundary of the rectangle with vertices \((1, -1), (1, 1), (-1, 1)\) and \((-1, -1)\), traversed counterclockwise.

11. For a region \( R \) bounded by a simple closed curve C, show that the area \( A \) of \( R \) is

\[
A = - \int_C y \, dx = \int_C x \, dy = \frac{1}{2} \int_C x \, dy - y \, dx,
\]

where C is traversed so that \( R \) is always on the left. (Hint: Use Green’s Theorem and the fact that \( A = \iint_R 1 \, dA \).)
15.4 Surface Integrals and the Divergence Theorem

In Section 15.1 we learned how to integrate along a curve. We will now learn how to perform integration over a surface in \( \mathbb{R}^3 \), such as a sphere or a paraboloid. Recall from Section 12.1 how we identified points \((x, y, z)\) on a curve \( C \) in \( \mathbb{R}^3 \), parametrized by \( x = x(t), y = y(t), z = z(t), a \leq t \leq b \), with the terminal points of the position vector

\[
\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad \text{for } t \in [a, b].
\]

The idea behind a parametrization of a curve is that it “transforms” a subset of \( \mathbb{R}^1 \) (normally an interval \([a, b]\)) into a curve in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) (see Figure 15.12).

Similar to how we used a parametrization of a curve to define the line integral along the curve, we will use a parametrization of a surface to define a surface integral. We will use two variables, \( u \) and \( v \), to parametrize a surface \( \Sigma \) in \( \mathbb{R}^3 \): \( x = x(u, v), y = y(u, v), z = z(u, v) \), for \((u, v)\) in some region \( R \) in \( \mathbb{R}^2 \) (see Figure 15.13).

Notes:
In this case, the position vector of a point on the surface \( \Sigma \) is given by the vector-valued function

\[
\vec{r}(u, v) = x(u, v)i + y(u, v)j + z(u, v)k \quad \text{for} \ (u, v) \in \mathbb{R}.
\]

Since \( \vec{r}(u, v) \) is a function of two variables, define the partial derivatives \( \frac{\partial \vec{r}}{\partial u} \) and \( \frac{\partial \vec{r}}{\partial v} \) for \((u, v)\) in \( \mathbb{R} \) by

\[
\begin{align*}
\frac{\partial \vec{r}}{\partial u}(u, v) &= \frac{\partial x}{\partial u}(u, v)i + \frac{\partial y}{\partial u}(u, v)j + \frac{\partial z}{\partial u}(u, v)k, \\
\frac{\partial \vec{r}}{\partial v}(u, v) &= \frac{\partial x}{\partial v}(u, v)i + \frac{\partial y}{\partial v}(u, v)j + \frac{\partial z}{\partial v}(u, v)k.
\end{align*}
\]

The parametrization of \( \Sigma \) can be thought of as “transforming” a region in \( \mathbb{R}^2 \) (in the uv-plane) into a 2-dimensional surface in \( \mathbb{R}^3 \). This parametrization of the surface is sometimes called a “patch”, based on the idea of “patching” the region \( R \) onto \( \Sigma \) in the grid-like manner shown in Figure 15.13.

In fact, those gridlines in \( \mathbb{R} \) lead us to how we will define a surface integral over \( \Sigma \). Along the vertical gridlines in \( \mathbb{R} \), the variable \( u \) is constant. So those lines get mapped to curves on \( \Sigma \), and the variable \( u \) is constant along the position vector \( \vec{r}(u, v) \). Thus, the tangent vector to those curves at a point \((u, v)\) is \( \frac{\partial \vec{r}}{\partial u} \).

Similarly, the horizontal gridlines in \( \mathbb{R} \) get mapped to curves on \( \Sigma \) whose tangent vectors are \( \frac{\partial \vec{r}}{\partial v} \).

Now take a point \((u, v)\) in \( \mathbb{R} \) as, say, the lower left corner of one of the rectangular grid sections in \( \mathbb{R} \), as shown in Figure 15.13. Suppose that this rectangle has a small width and height of \( \Delta u \) and \( \Delta v \), respectively. The corner points of that rectangle are \((u, v), (u + \Delta u, v), (u + \Delta u, v + \Delta v)\) and \((u, v + \Delta v)\). So the area of that rectangle is \( A = \Delta u \Delta v \). Then that rectangle gets mapped by the parametrization onto some section of the surface \( \Sigma \) which, for \( \Delta u \) and \( \Delta v \) small enough, will have a surface area (call it \( \sigma \)) that is very close to the area of the parallelogram which has adjacent sides \( \vec{r}(u + \Delta u, v) - \vec{r}(u, v) \) (corresponding to the line segment from \((u, v)\) to \((u + \Delta u, v)\) in \( \mathbb{R} \)) and \( \vec{r}(u, v + \Delta v) - \vec{r}(u, v) \) (corresponding to the line segment from \((u, v)\) to \((u, v + \Delta v)\) in \( \mathbb{R} \)). By combining our usual notion of a partial derivative (see Definition 86 in Section 13.3) with that of the derivative of a vector-valued function (see Definition 73 in Section 12.2) applied to a function of two variables, we have

\[
\begin{align*}
\frac{\partial \vec{r}}{\partial u} &\approx \frac{\vec{r}(u + \Delta u, v) - \vec{r}(u, v)}{\Delta u}, \quad \text{and} \\
\frac{\partial \vec{r}}{\partial v} &\approx \frac{\vec{r}(u, v + \Delta v) - \vec{r}(u, v)}{\Delta v}.
\end{align*}
\]
and so the surface area element $d\sigma$ is approximately

$$\|(\vec{r}(u + \Delta u, v) - \vec{r}(u, v)) \times (\vec{r}(u, v + \Delta v) - \vec{r}(u, v))\|$$

$$\approx \left\| (\Delta u \frac{\partial \vec{r}}{\partial u}) \times (\Delta v \frac{\partial \vec{r}}{\partial v}) \right\| \Delta u \Delta v$$

by Equation (11.5). Thus, the total surface area $S$ of $\Sigma$ is approximately the sum of all the quantities $\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \Delta u \Delta v$, summed over the rectangles in $R$. Taking the limit of that sum as the diagonal of the largest rectangle goes to 0 gives

$$S = \iint_R \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \, du \, dv.$$

We will write the double integral on the right using the special notation

$$\iint_{\Sigma} d\sigma = \iint_R \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \, du \, dv.$$

This is a special case of a surface integral over the surface $\Sigma$, where the surface area element $d\sigma$ can be thought of as $1 \, d\sigma$. Replacing 1 by a general real-valued function $f(x, y, z)$ defined in $\mathbb{R}^3$, we have the following:

**Definition 114 Scalar Surface Integral**

Let $\Sigma$ be a surface in $\mathbb{R}^3$ parametrized by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, for $(u, v)$ in some region $R$ in $\mathbb{R}^2$. Let $\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ be the position vector for any point on $\Sigma$, and let $f(x, y, z)$ be a real-valued function defined on some subset of $\mathbb{R}^3$ that contains $\Sigma$. The surface integral of $f(x, y, z)$ over $\Sigma$ is

$$\iint_{\Sigma} f(x, y, z) \, d\sigma = \iiint_R f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \, du \, dv.$$

In particular, the surface area $S$ of $\Sigma$ is

$$S = \iint_{\Sigma} 1 \, d\sigma.$$

Sometimes, the notation $\iint_{\Sigma} f(x, y, z) \, d\sigma$ is used instead of $\iint_{\Sigma} f(x, y, z) \, d\sigma$ when $\Sigma$ is a closed surface. Especially in physics texts, it is common to see simply $\iint_{\Sigma}$ instead of $\iint_{\Sigma}$.  

Notes:
Example 1: Computing a Surface Integral

A torus $T$ is a surface obtained by revolving a circle of radius $a$ in the $yz$-plane around the $z$-axis, where the circle’s center is at a distance $b$ from the $z$-axis ($0 < a < b$), as in Figure 15.14. Find the surface area of $T$.

**SOLUTION**

For any point on the circle, the line segment from the center of the circle to that point makes an angle $u$ with the $y$-axis in the positive $y$ direction (see Figure 15.14(a)). And as the circle revolves around the $z$-axis, the line segment from the origin to the center of that circle sweeps out an angle $v$ with the positive $x$-axis (see Figure 15.14(b)). Thus, the torus can be parametrized as:

$$
x = (b + a \cos u) \cos v, \quad y = (b + a \cos u) \sin v, \quad z = a \sin u,
$$

where $0 \leq u \leq 2\pi$, and $0 \leq v \leq 2\pi$. So for the position vector

$$
\vec{r}(u, v) = x(u, v) \vec{i} + y(u, v) \vec{j} + z(u, v) \vec{k}
$$

$$
= (b + a \cos u) \cos v \vec{i} + (b + a \cos u) \sin v \vec{j} + a \sin u \vec{k}
$$

we see that

$$
\frac{\partial \vec{r}}{\partial u} = -a \sin u \cos v \vec{i} - a \sin u \sin v \vec{j} + a \cos u \vec{k}
$$

$$
\frac{\partial \vec{r}}{\partial v} = -(b + a \cos u) \sin v \vec{i} + (b + a \cos u) \cos v \vec{j} + 0 \vec{k},
$$

---

**Notes:**
and so computing the cross product gives

\[
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -a(b + a \cos u) \cos v \cos u \hat{i} - a(b + a \cos u) \sin v \cos u \hat{j} - a(b + a \cos u) \sin u \hat{k},
\]

which has magnitude

\[
\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = a(b + a \cos u).
\]

Thus, the surface area of \( T \) is

\[
S = \int_{\Sigma} 1 \, d\sigma
= \int_{0}^{2\pi} \int_{0}^{\pi} \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \, du \, dv
= \int_{0}^{2\pi} \int_{0}^{\pi} a(b + a \cos u) \, du \, dv
= \int_{0}^{2\pi} \left( abu + a^2 \sin u \right|_{u=0}^{u=2\pi} \right) \, dv
= \int_{0}^{2\pi} 2\pi ab \, dv
= 4\pi^2 ab
\]

Since \( \frac{\partial \vec{r}}{\partial u} \) and \( \frac{\partial \vec{r}}{\partial v} \) are tangent to the surface \( \Sigma \) (i.e. lie in the tangent plane to \( \Sigma \) at each point on \( \Sigma \)), then their cross product \( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \) is perpendicular to the tangent plane to the surface at each point of \( \Sigma \). Thus,

\[
\iint_{\Sigma} f(x, y, z) \, d\sigma = \iint_{\Sigma} f(x(u, v), y(u, v), z(u, v)) \left\| \vec{n} \right\| \, d\sigma,
\]

where \( \vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \). We say that \( \vec{n} \) is a \textbf{normal vector} to \( \Sigma \).

Recall that normal vectors to a plane can point in two opposite directions. By an \textbf{outward unit normal vector} to a surface \( \Sigma \), we will mean the unit vector that is normal to \( \Sigma \) and points away from the “top” (or “outer” part) of the surface. This is a hazy definition, but the picture in Figure 15.15 gives a better idea of what outward normal vectors look like, in the case of a sphere. With this idea in mind, we make the following definition of a surface integral of a 3-dimensional vector field over a surface:
Definition 115  Vector Surface Integral
Let \( \Sigma \) be a surface in \( \mathbb{R}^3 \) and let \( \mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k} \) be a vector field defined on some subset of \( \mathbb{R}^3 \) that contains \( \Sigma \). The surface integral of \( \mathbf{f} \) over \( \Sigma \) is
\[
\iint_{\Sigma} \mathbf{f} \cdot d\mathbf{a} = \iint_{\Sigma} \mathbf{f} \cdot \mathbf{n} \, d\sigma,
\]
where, at any point on \( \Sigma \), \( \mathbf{n} \) is the outward unit normal vector to \( \Sigma \).

Note in the above definition that the dot product inside the integral on the right is a real-valued function, and hence we can use Definition 114 to evaluate the integral.

Example 2  Evaluating a Surface Integral
Evaluate the surface integral \( \iint_{\Sigma} \mathbf{f} \cdot d\mathbf{a} \), where \( \mathbf{f}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \) and \( \Sigma \) is the part of the plane \( x + y + z = 1 \) with \( x \geq 0 \), \( y \geq 0 \), and \( z \geq 0 \), with the outward unit normal \( \mathbf{n} \) pointing in the positive \( z \) direction (see Figure 15.16).

Solution  Since the vector \( \mathbf{v} = (1, 1, 1) \) is normal to the plane \( x + y + z = 1 \) (why?), then dividing \( \mathbf{v} \) by its length yields the outward unit normal vector \( \mathbf{n} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \). We now need to parametrize \( \Sigma \). As we can see from Figure 15.16, projecting \( \Sigma \) onto the \( xy \)-plane yields a triangular region \( R = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x \} \). Thus, using \((u, v)\) instead of \((x, y)\), we see that
\[
x = u, \ y = v, \ z = 1 - (u + v), \text{ for } 0 \leq u \leq 1, 0 \leq v \leq 1 - u
\]
is a parametrization of \( \Sigma \) over \( R \) (since \( z = 1 -(x+y) \) on \( \Sigma \)). So on \( \Sigma \),
\[
\mathbf{f} \cdot \mathbf{n} = (yz, xz, xy) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}}(yz + xz + xy)
\]
\[
= \frac{1}{\sqrt{3}}((x+y)z + xy) = \frac{1}{\sqrt{3}}((u+v)(1-(u+v)) + uv)
\]
\[
= \frac{1}{\sqrt{3}}((u+v) - (u+v)^2 + uv)
\]
for \((u, v)\) in \( R \), and for \( \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} = u\mathbf{i} + v\mathbf{j} + (1 - (u + v))\mathbf{k} \) we have
\[
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1) \quad \Rightarrow \quad \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{3}.
\]

Sometimes, the notation \( \iint_{\Sigma} \mathbf{f} \cdot d\mathbf{a} \) is used instead of \( \iint_{\Sigma} \mathbf{f} \cdot d\mathbf{a} \) when \( \Sigma \) is a closed surface. Especially in physics texts, it is common to see simply \( \iint_{\Sigma} \) instead of \( \iint_{\Sigma} \).

Notes:
Thus, integrating over $R$ using vertical slices (e.g. as indicated by the dashed line in Figure 15.16) gives

$$
\int_{\Sigma} f \cdot d\vec{\sigma} = \int_{\Sigma} \vec{f} \cdot \vec{n} d\sigma \\
= \int_{r} (\vec{f}(x(u, v), y(u, v), z(u, v)) \cdot \vec{n}) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dv du \\
= \int_{0}^{1} \int_{0}^{1-u} \frac{1}{\sqrt{3}} ((u + v) - (u + v)^2 + uv) \sqrt{3} dv du \\
= \int_{0}^{1} \left( \frac{(u + v)^2}{2} - \frac{(u + v)^3}{3} + \frac{uv^2}{2} \right)_{v=0}^{v=1-u} du \\
= \int_{0}^{1} \left( \frac{1}{6} \frac{u}{2} - \frac{3u^2}{2} + \frac{5u^3}{6} \right) du \\
= \frac{u}{6} + \frac{u^2}{4} - \frac{u^3}{2} + \frac{5u^4}{24} \bigg|_{0}^{1} = \frac{1}{8}.
$$

Notice that we divided $\vec{n}$ by its norm, and later multiplied by the same factor with $\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|$. This is generally the case:

$$
\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}, \\
\sigma = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dv du \\
\vec{n} d\sigma = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} dv du,
$$

which simplifies the calculation.

Computing surface integrals can often be tedious, especially when the formula for the outward unit normal vector at each point of $\Sigma$ changes. The following theorem provides an easier way in the case when $\Sigma$ is a closed surface, that is, when $\Sigma$ encloses a bounded solid in $\mathbb{R}^3$. For example, spheres, cubes, and ellipsoids are closed surfaces, but planes and paraboloids are not.

Notes:
Theorem 139  Divergence Theorem

Let \( \Sigma \) be a closed surface in \( \mathbb{R}^3 \) which bounds a solid \( S \), and let \( \vec{f}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k} \) be a vector field defined on some subset of \( \mathbb{R}^3 \) that contains \( \Sigma \). Then

\[
\iint_{\Sigma} \vec{f} \cdot d\vec{\sigma} = \iiint_{S} \text{div}\vec{f}dV, \tag{15.11}
\]

where

\[
\text{div}\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \tag{15.12}
\]

is called the divergence of \( \vec{f} \).

The proof of the Divergence Theorem is very similar to the proof of Green’s Theorem, i.e. it is first proved for the simple case when the solid \( S \) is bounded above by one surface, bounded below by another surface, and bounded laterally by one or more surfaces. The proof can then be extended to more general solids.

In Definition 95 in Section 13.6, we defined the operator \( \nabla \) as a vector in \( \mathbb{R}^3 \), namely

\[
\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.
\]

This definition allows us to write \( \text{div}\vec{f} = \nabla \cdot \vec{f} \).

Example 3  Using the Divergence Theorem

Evaluate \( \iint_{\Sigma} \vec{f} \cdot d\vec{\sigma} \), where \( \vec{f}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k} \) and \( \Sigma \) is the unit sphere \( x^2 + y^2 + z^2 = 1 \).

**Solution**  We see that \( \text{div}\vec{f} = 1 + 1 + 1 = 3 \), so

\[
\iint_{\Sigma} \vec{f} \cdot d\vec{\sigma} = \iiint_{S} \text{div}\vec{f}dV = \iiint_{S} 3dV = 3 \iiint_{S} 1dV = 3 \text{vol}(S) = 3 \cdot \frac{4\pi(1)^3}{3} = 4\pi.
\]

In physical applications, the surface integral \( \iint_{\Sigma} \vec{f} \cdot d\vec{\sigma} \) is often referred to as the flux of \( \vec{f} \) through the surface \( \Sigma \). For example, if \( \vec{f} \) represents the velocity field of a fluid, then the flux is the net quantity of fluid to flow through the surface \( \Sigma \) per unit time. A positive flux means there is a net flow out of the surface (i.e. in the direction of the outward unit normal vector \( \vec{n} \)), while a negative flux indicates a net flow inward (in the direction of \( -\vec{n} \)).

Notes:
The term divergence comes from interpreting \( \nabla \cdot \mathbf{f} \) as a measure of how much a vector field “diverges” from a point. This is best seen by using another definition of \( \nabla \cdot \mathbf{f} \) which is equivalent to the definition given by Equation (15.12). Namely, for a point \((x, y, z)\) in \(\mathbb{R}^3\),

\[
\nabla \cdot \mathbf{f}(x, y, z) = \lim_{V \to 0} \frac{1}{V} \int_{\Sigma} \mathbf{f} \cdot d\mathbf{\sigma}, \tag{15.13}
\]

where \(V\) is the volume enclosed by a closed surface \(\Sigma\) around the point \((x, y, z)\). In the limit, \(V \to 0\) means that we take smaller and smaller closed surfaces around \((x, y, z)\), which means that the volumes they enclose are going to zero. It can be shown that this limit is independent of the shapes of those surfaces. Notice that the limit being taken is of the ratio of the flux through a surface to the volume enclosed by that surface, which gives a rough measure of the flow “leaving” a point, as we mentioned. Vector fields which have zero divergence are often called solenoidal fields.

The following theorem is a simple consequence of Equation (15.13).

**Theorem 140 Zero Flux**

If the flux of a vector field \(\mathbf{f}\) is zero through every closed surface containing a given point, then \(\nabla \cdot \mathbf{f} = 0\) at that point.

**Proof**

By Equation (15.13), at the given point \((x, y, z)\) we have

\[
\nabla \cdot \mathbf{f}(x, y, z) = \lim_{V \to 0} \frac{1}{V} \int_{\Sigma} \mathbf{f} \cdot d\mathbf{\sigma} \text{ for closed surfaces } \Sigma \text{ containing } (x, y, z), \text{ so}
\]

\[
= \lim_{V \to 0} \frac{1}{V} (0) \text{ by our assumption that the flux through each } \Sigma \text{ is zero, so}
\]

\[
= \lim_{V \to 0} 0 = 0. \quad \square
\]

This section and the previous introduced four new types of integrals, which we gather in Key Idea 64.

---

**Notes:**
15.4 Surface Integrals and the Divergence Theorem

Key Idea 64 Integrating Parameterized Curves and Surfaces

<table>
<thead>
<tr>
<th></th>
<th>Integrating Parameterized Curves and Surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{c}: \mathbb{R} \to \mathbb{R}^3$</td>
<td>parameterizes a curve</td>
</tr>
<tr>
<td>$\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$</td>
<td>parameterizes a surface</td>
</tr>
<tr>
<td>$f: \mathbb{R}^3 \to \mathbb{R}$</td>
<td>scalar line integral: $\int_{\vec{C}(D)} f,ds$</td>
</tr>
<tr>
<td></td>
<td>$= \int_D f(\vec{c}(t)) |\vec{c}'(t)| ,dt$</td>
</tr>
<tr>
<td></td>
<td>scalar surface integral: $\iint_{\vec{R}(D)} f,d\sigma$</td>
</tr>
<tr>
<td></td>
<td>$= \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| ,du ,dv$</td>
</tr>
<tr>
<td>$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$</td>
<td>vector line integral: $\int_{\vec{C}(D)} \vec{F} \cdot d\vec{r}$</td>
</tr>
<tr>
<td></td>
<td>$= \int_D \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) ,dt$</td>
</tr>
<tr>
<td></td>
<td>vector surface integral: $\iint_{\vec{R}(D)} \vec{F} \cdot d\vec{\sigma}$</td>
</tr>
<tr>
<td></td>
<td>$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) ,du ,dv$</td>
</tr>
</tbody>
</table>

If we are in the bottom left entry, where $\vec{c}: \mathbb{R} \to \mathbb{R}^3$ and $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, we may be able to use the Fundamental Theorem for Gradient Vector Fields: $\int_{\vec{C}} \nabla \phi \cdot d\vec{s} = \phi(Q) - \phi(P)$. 

Notes:
### Exercises 15.4

**Problems**

In Exercises 1–4, use the Divergence Theorem to evaluate the surface integral \( \iint_S \vec{f} \cdot d\sigma \) of the given vector field \( \vec{f}(x, y, z) \) over the surface \( \Sigma \).

1. \( \vec{f}(x, y, z) = x\hat{i} + 2y\hat{j} + 3z\hat{k}, \Sigma: x^2 + y^2 + z^2 = 9 \)
2. \( \vec{f}(x, y, z) = xi + yj + zk, \Sigma: \text{boundary of the solid cube} \quad S = \{(x, y, z): 0 \leq x, y, z \leq 1 \} \)
3. \( \vec{f}(x, y, z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}, \Sigma: x^2 + y^2 + z^2 = 1 \)
4. \( \vec{f}(x, y, z) = 2\hat{i} + 3y\hat{j} + 5k, \Sigma: x^2 + y^2 + z^2 = 1 \)

5. Show that the flux of any constant vector field through any closed surface is zero.
6. Evaluate the surface integral from Exercise 2 without using the Divergence Theorem, i.e. using only Definition 114, as in Example 2. Note that there will be a different outward unit normal vector to each of the six faces of the cube.
7. Evaluate the surface integral \( \iint_S \vec{f} \cdot d\sigma \), where \( \vec{f}(x, y, z) = x^2\hat{i} + xy\hat{j} + zk \) and \( \Sigma \) is the part of the plane \( 6x + 3y + 2z = 6 \) with \( x \geq 0, y \geq 0, \) and \( z \geq 0, \) with the outward unit normal \( \vec{n} \) pointing in the positive z direction.
8. Use a surface integral to show that the surface area of a sphere of radius \( r \) is \( 4\pi r^2 \). (Hint: Use spherical coordinates to parametrize the sphere.)
9. Use a surface integral to show that the surface area of a right circular cone of radius \( R \) and height \( h \) is \( \pi R \sqrt{R^2 + R^2} \).
   (Hint: Use the parametrization \( x = r \cos \theta, y = r \sin \theta, z = \frac{h}{2}, \) for \( 0 \leq r \leq R \) and \( 0 \leq \theta \leq 2\pi \).)
10. The ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) can be parametrized using **ellipsoidal coordinates**

\[
\begin{align*}
    x &= a \sin \phi \cos \theta, \\
    y &= b \sin \phi \sin \theta, \\
    z &= c \cos \phi,
\end{align*}
\]

for \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \). Show that the surface area \( S \) of the ellipsoid is

\[
S = \int_0^\pi \int_0^{2\pi} \sin \phi \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \sin^2 \phi \, d\theta \, d\phi.
\]

(Note: The above double integral can not be evaluated by elementary means. For specific values of \( a, b \) and \( c \) it can be evaluated using numerical methods. An alternative is to express the surface area in terms of elliptic integrals.)
11. Use Definition 114 to prove that the surface area \( S \) over a region \( R \) in \( \mathbb{R}^2 \) of a surface \( z = f(x, y) \) is given by the formula

\[
S = \iint_R \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA.
\]

(Hint: Think of the parametrization of the surface.)
15.5 Stokes’ Theorem

So far the only types of line integrals which we have discussed are those along curves in $\mathbb{R}^2$. But the definitions and properties which were covered in Sections 15.1 and 15.2 can easily be extended to include functions of three variables, so that we can now discuss line integrals along curves in $\mathbb{R}^3$.

**Definition 116 Scalar Line Integral**

For a real-valued function $f(x, y, z)$ and a curve $C$ in $\mathbb{R}^3$, parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \leq t \leq b$, the line integral of $f(x, y, z)$ along $C$ with respect to arc length $s$ is

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

The line integral of $f(x, y, z)$ along $C$ with respect to $x$ is

$$\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t)) x'(t) \, dt.$$

The line integral of $f(x, y, z)$ along $C$ with respect to $y$ is

$$\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t)) y'(t) \, dt.$$

The line integral of $f(x, y, z)$ along $C$ with respect to $z$ is

$$\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt.$$

Similar to the two-variable case, if $f(x, y, z) \geq 0$ then the line integral $\int_C f(x, y, z) \, ds$ can be thought of as the total area of the “picket fence” of height $f(x, y, z)$ at each point along the curve $C$ in $\mathbb{R}^3$.

Vector fields in $\mathbb{R}^3$ are defined in a similar fashion to those in $\mathbb{R}^2$, which allows us to define the line integral of a vector field along a curve in $\mathbb{R}^3$.

Notes:
Vector Line Integral

For a vector field \( \mathbf{f}(x, y, z) = P(x, y, z) \hat{i} + Q(x, y, z) \hat{j} + R(x, y, z) \hat{k} \) and a curve \( C \) in \( \mathbb{R}^3 \) with a smooth parametrization \( x = x(t), y = y(t), z = z(t), a \leq t \leq b \), the line integral of \( \mathbf{f} \) along \( C \) is

\[
\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C P(x, y, z) \, dx + \int_C Q(x, y, z) \, dy + \int_C R(x, y, z) \, dz
\]

where \( \mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \) is the position vector for points on \( C \).

Similar to the two-variable case, if \( \mathbf{f}(x, y, z) \) represents the force applied to an object at a point \( (x, y, z) \) then the line integral \( \int_C \mathbf{f} \cdot d\mathbf{r} \) represents the work done by that force in moving the object along the curve \( C \) in \( \mathbb{R}^3 \).

Some of the most important results we will need for line integrals in \( \mathbb{R}^3 \) are stated below without proof (the proofs are similar to their two-variable equivalents).

Vector Line Integral Along a Curve

For a vector field \( \mathbf{f}(x, y, z) = P(x, y, z) \hat{i} + Q(x, y, z) \hat{j} + R(x, y, z) \hat{k} \) and a curve \( C \) with a smooth parametrization \( x = x(t), y = y(t), z = z(t), a \leq t \leq b \) and position vector \( \mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \),

\[
\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \mathbf{f} \cdot \mathbf{T} \, ds,
\]

where \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \) is the unit tangent vector to \( C \) at \( (x(t), y(t), z(t)) \).

Notes:
Theorem 142  Fundamental Theorem of Line Integrals in Three Dimensions
Let \( \vec{f}(x, y, z) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k} \) be a vector field in some solid \( S \), with \( P \), \( Q \) and \( R \) continuously differentiable functions on \( S \). Let \( C \) be a smooth curve in \( S \) parametrized by \( x = x(t), y = y(t), z = z(t), a \leq t \leq b \). Suppose that there is a real-valued function \( F(x, y, z) \) such that \( \nabla F = \vec{f} \) on \( S \). Then
\[
\int_C \vec{f} \cdot d\vec{r} = F(B) - F(A),
\]  
where \( A = (x(a), y(a), z(a)) \) and \( B = (x(b), y(b), z(b)) \) are the endpoints of \( C \).

Theorem 143  Zero Line Integral
If a vector field \( \vec{f} \) has a potential in a solid \( S \) without holes, then \( \int_C \vec{f} \cdot d\vec{r} = 0 \) for any closed curve \( C \) in \( S \) (i.e., \( \int_C \nabla F \cdot d\vec{r} = 0 \) for any real-valued function \( F(x, y, z) \)).

Example 1  Evaluating a Line Integral
Let \( f(x, y, z) = z \) and let \( C \) be the curve in \( \mathbb{R}^3 \) parametrized by
\[
\begin{align*}
  x &= t \sin t, \\
  y &= t \cos t, \\
  z &= t,
\end{align*}
\]
\( 0 \leq t \leq 8\pi \).
Evaluate \( \int_C f(x, y, z) \, ds \). (Note: \( C \) is called a conical helix. See Figure 15.17).

Figure 15.17: Conical helix \( C \) for Example 1.
Example 1  
Let \( f(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \) be a vector field in \( \mathbb{R}^3 \). Using the same curve \( C \) from Example 1, evaluate \( \int_C f \cdot d\mathbf{r} \).

**Solution**  
It is easy to see that \( F(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + z^2 \) is a potential for \( \vec{f}(x, y, z) \) (i.e., \( \nabla F = \vec{f} \)). So by Theorem 142 we know that

\[
\int_C \vec{f} \cdot d\mathbf{r} = F(B) - F(A), \quad \text{where } A = (x(0), y(0), z(0)) \text{ and } B = (x(8\pi), y(8\pi), z(8\pi)),
\]

so

\[
\begin{align*}
\int_C \vec{f} \cdot d\mathbf{r} &= F(B) - F(A) \\
&= F(8\pi \sin 8\pi, 8\pi \cos 8\pi, 8\pi) - F(0 \sin 0, 0 \cos 0, 0) \\
&= F(0, 8\pi, 8\pi) - F(0, 0, 0) \\
&= 0 + \frac{(8\pi)^2}{2} + (8\pi)^2 - (0 + 0 + 0) = 96\pi^2.
\end{align*}
\]

We will now discuss a generalization of Green’s Theorem in \( \mathbb{R}^2 \) to orientable surfaces in \( \mathbb{R}^3 \), called Stokes’ Theorem. A surface \( \Sigma \) in \( \mathbb{R}^3 \) is **orientable** if there is a continuous vector field \( \vec{n} \) in \( \mathbb{R}^3 \) such that \( \vec{n} \) is nonzero and normal to \( \Sigma \) (i.e. perpendicular to the tangent plane) at each point of \( \Sigma \). We say that such an \( \vec{n} \) is a **normal vector field**.

For example, the unit sphere \( x^2 + y^2 + z^2 = 1 \) is orientable, since the continuous vector field \( \vec{n}(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \) is nonzero and normal to the sphere at any point.
each point. In fact, \(-\vec{n}(x, y, z)\) is another normal vector field (see Figure 15.18). We see in this case that \(\vec{n}(x, y, z)\) is what we have called an outward normal vector, and \(-\vec{n}(x, y, z)\) is an inward normal vector. These “outward” and “inward” normal vector fields on the sphere correspond to an “outer” and “inner” side, respectively, of the sphere. That is, we say that the sphere is a \textit{two-sided} surface. Roughly, “two-sided” means “orientable”. Other examples of two-sided, and hence orientable, surfaces are cylinders, paraboloids, ellipsoids, and planes.

You may be wondering what kind of surface would \textit{not} have two sides. An example is the \textit{Möbius strip}, which is constructed by taking a thin rectangle and connecting its ends at the opposite corners, resulting in a “twisted” strip (see Figure 15.19).

![Figure 15.19: Möbius strip](image)

If you imagine walking along a line down the center of the Möbius strip, as in Figure 15.19(b), then you arrive back at the same place from which you started but upside down! That is, your \textit{orientation} changed even though your motion was continuous along that center line. Informally, thinking of your vertical direction as a normal vector field along the strip, there is a discontinuity at your starting point (and, in fact, at every point) since your vertical direction takes two different values there. The Möbius strip has only \textit{one side}, and hence is nonorientable.

For an orientable surface \(\Sigma\) which has a boundary curve \(C\), pick a unit normal vector \(\vec{n}\) such that if you walked along \(C\) with your head pointing in the direction of \(\vec{n}\), then the surface would be on your left. We say in this situation that \(\vec{n}\) is a \textit{positive unit normal vector} and that \(C\) is traversed \(\vec{n}\)-\textit{positively}. We can now state Stokes’ Theorem:

Notes:
Theorem 144    Stokes' Theorem

Let $\Sigma$ be an orientable surface in $\mathbb{R}^3$ whose boundary is a simple closed curve $C$, and let $\vec{f}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ be a smooth vector field defined on some subset of $\mathbb{R}^3$ that contains $\Sigma$. Then

$$\oint_C \vec{f} \cdot d\vec{r} = \iint_\Sigma (\text{curl} \, \vec{f}) \cdot \vec{n} \, d\sigma, \quad (15.15)$$

where

$$\text{curl} \, \vec{f} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}, \quad (15.16)$$

$\vec{n}$ is a positive unit normal vector over $\Sigma$, and $C$ is traversed $\vec{n}$-positively.

The formula for $\text{curl} \, \vec{f}$ is unfortunately complicated. If we recall that we have defined the operator $\nabla$ as a vector in $\mathbb{R}^3$ by

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k},$$

then we can write $\text{curl} \, \vec{f} = \nabla \times \vec{f}$.

Proof

As the general case is beyond the scope of this text, we will prove the theorem only for the special case where $\Sigma$ is the graph of $z = z(x, y)$ for some smooth real-valued function $z(x, y)$, with $(x, y)$ varying over a region $D$ in $\mathbb{R}^2$. Projecting $\Sigma$ onto the $xy$-plane, we see that the closed curve $C$ (the boundary curve of $\Sigma$) projects onto a closed curve $C_D$ which is the boundary curve of $D$ (see Figure 15.20). Assuming that $C$ has a smooth parametrization, its projection $C_D$ in the $xy$-plane also has a smooth parametrization, say

$$C_D : x = x(t), \ y = y(t), \ a \leq t \leq b,$$

and so $C$ can be parametrized (in $\mathbb{R}^3$) as

$$C : x = x(t), \ y = y(t), \ z = z(x(t), y(t)) , \ a \leq t \leq b,$$

since the curve $C$ is part of the surface $z = z(x, y)$. Now, by the Chain Rule (Theorem 110 in Section 13.5), for $z = z(x(t), y(t))$ as a function of $t$, we know that

$$z'(t) = \frac{\partial z}{\partial x} x'(t) + \frac{\partial z}{\partial y} y'(t).$$

Notes:
and so
\[ \oint_{C} \vec{f} \cdot d\vec{r} = \int_{C} P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz \]
\[ = \int_{a}^{b} \left( P(x'(t)) + Q(y'(t)) + R \left( \frac{\partial z}{\partial x} x'(t) + \frac{\partial z}{\partial y} y'(t) \right) \right) \, dt \]
\[ = \int_{a}^{b} \left( \left( P + R \frac{\partial z}{\partial x} \right) x'(t) + \left( Q + R \frac{\partial z}{\partial y} \right) y'(t) \right) \, dt \]
\[ = \int_{C_0} \tilde{P}(x, y) \, dx + \tilde{Q}(x, y) \, dy, \]
where
\[ \tilde{P}(x, y) = P(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial x}(x, y), \]
and
\[ \tilde{Q}(x, y) = Q(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y) \]
for \((x, y)\) in \(D\). Thus, by Green's Theorem applied to the region \(D\), we have
\[ \oint_{C} \vec{f} \cdot d\vec{r} = \iint_{D} \left( \frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y} \right) \, dA. \]  \hspace{1cm} (15.17)

Thus,
\[ \frac{\partial \tilde{Q}}{\partial x} = \frac{\partial}{\partial x} \left( Q(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y) \right), \]
so by the Product Rule we get
\[ = \frac{\partial}{\partial x} (Q(x, y, z(x, y))) + \left( \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} \right) \frac{\partial z}{\partial y}(x, y) + R(x, y, z(x, y)) \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y}(x, y) \right). \]

Now, by Theorem 111, we have
\[ \frac{\partial}{\partial x} (Q(x, y, z(x, y))) = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} \cdot 0 + \frac{\partial Q}{\partial z} \cdot \frac{\partial z}{\partial x} \]
\[ = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x}. \]

Similarly,
\[ \frac{\partial}{\partial x} (R(x, y, z(x, y))) = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x}. \]

Notes:
Thus,
\[
\begin{align*}
\frac{\partial Q}{\partial x} &= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \left( \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial y} + R(x, y, z(x, y)) \frac{\partial^2 z}{\partial x \partial y} \\
&= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x \partial y}.
\end{align*}
\]

In a similar fashion, we can calculate
\[
\frac{\partial P}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial y \partial y}.
\]

So subtracting gives
\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left( \frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y} \right) \frac{\partial z}{\partial x} + \left( \frac{\partial R}{\partial z} - \frac{\partial P}{\partial y} \right) \frac{\partial z}{\partial y} + \left( \frac{\partial Q}{\partial \partial y} - \frac{\partial P}{\partial \partial y} \right)
\]
(15.18)
since \( \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \) by the smoothness of \( z = z(x, y) \). Hence, by Equation (15.17),
\[
\int_c \mathbf{f} \cdot d\mathbf{r} = \int_D \left( - \left( \frac{\partial R}{\partial z} - \frac{\partial Q}{\partial \partial z} \right) \frac{\partial z}{\partial x} - \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial \partial z} \right) \frac{\partial z}{\partial y} + \left( \frac{\partial Q}{\partial \partial x} - \frac{\partial P}{\partial \partial y} \right) \right) dA
\]
(15.19)
after factoring out a \(-1\) from the terms in the first two products in Equation (15.18).

Now, recall from Section 13.7 that the vector \( \mathbf{n} = -\frac{\partial y}{\partial x} \mathbf{i} - \frac{\partial z}{\partial y} \mathbf{j} + \mathbf{k} \) is normal to the tangent plane to the surface \( z = z(x, y) \) at each point of \( \Sigma \). Thus,
\[
\mathbf{n} = \frac{\mathbf{r} - \frac{\partial x}{\partial x} \mathbf{i} - \frac{\partial y}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left( \frac{\partial x}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial y} \right)^2}} = \mathbf{n}
\]
is in fact a positive unit normal vector to \( \Sigma \) (see Figure 15.20). Hence, using the parametrization \( \mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + z(x, y) \mathbf{k} \), for \( (x, y) \) in \( D \), of the surface \( \Sigma \), we have \( \frac{\partial x}{\partial x} = \mathbf{i} + \frac{\partial x}{\partial x} \mathbf{k} \) and \( \frac{\partial y}{\partial y} = \mathbf{j} + \frac{\partial y}{\partial y} \mathbf{k} \), and so
\[
\left\| \frac{\partial x}{\partial x} \times \frac{\partial y}{\partial y} \right\| = \sqrt{1 + \left( \frac{\partial x}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial y} \right)^2}.
\]
So we see that using Equation (15.16) for \( \text{curl} \mathbf{f} \), we have
\[
\left\int_\Sigma (\text{curl} \mathbf{f}) \cdot \mathbf{n} d\sigma \right.
\]
\[
= \int_D (\text{curl} \mathbf{f}) \cdot \mathbf{n} \left\| \frac{\partial x}{\partial x} \times \frac{\partial y}{\partial y} \right\| dA
\]
\[
= \int_D \left( \left( \frac{\partial R}{\partial z} - \frac{\partial Q}{\partial \partial z} \right) \frac{\partial z}{\partial x} - \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial \partial z} \right) \frac{\partial z}{\partial y} + \left( \frac{\partial Q}{\partial \partial x} - \frac{\partial P}{\partial \partial y} \right) \right) dA,
\]
which, upon comparing to Equation (15.19), proves the Theorem. \qed

Note: The condition in Stokes’ Theorem that the surface \( \Sigma \) have a (continuously varying) positive unit normal vector \( \vec{n} \) and a boundary curve \( C \) traversed \( \vec{n} \)-positively can be expressed more precisely as follows: if \( \vec{r}(t) \) is the position vector for \( C \) and \( \vec{T}(t) = \vec{r}'(t) / \| \vec{r}'(t) \| \) is the unit tangent vector to \( C \), then the vectors \( \vec{T}, \vec{n}, \vec{T} \times \vec{n} \) form a right-handed system.

Also, it should be noted that Stokes’ Theorem holds even when the boundary curve \( C \) is piecewise smooth.

**Example 3**  
**Verifying Stokes’ Theorem**

Verify Stokes’ Theorem for \( \vec{f}(x,y,z) = z \vec{i} + x \vec{j} + y \vec{k} \) when \( \Sigma \) is the paraboloid \( z = x^2 + y^2 \) such that \( z \leq 1 \) (see Figure 15.21).

**Solution**  
The positive unit normal vector to the surface \( z = z(x,y) = x^2 + y^2 \) is given by

\[
\vec{n} = \frac{-\frac{\partial z}{\partial x} \vec{i} - \frac{\partial z}{\partial y} \vec{j} + \vec{k}}{\sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2}} = \frac{-2x \vec{i} - 2y \vec{j} + \vec{k}}{\sqrt{1 + 4x^2 + 4y^2}}
\]

and \( \text{curl} \vec{f} = (1-0) \vec{i} + (1-0) \vec{j} + (1-0) \vec{k} = \vec{i} + \vec{j} + \vec{k} \), so

\[
(\text{curl} \vec{f}) \cdot \vec{n} = (2x - 2y + 1) / \sqrt{1 + 4x^2 + 4y^2}.
\]

Since \( \Sigma \) can be parametrized as \( \vec{r}(x,y) = x \vec{i} + y \vec{j} + (x^2 + y^2) \vec{k} \) for \( (x,y) \) in the region \( D = \{ (x,y) : x^2 + y^2 \leq 1 \} \), then

\[
\iint_{\Sigma} (\text{curl} \vec{f}) \cdot \vec{n} \, d\sigma = \iint_{D} (\text{curl} \vec{f}) \cdot \vec{n} \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| \, dA
\]

\[
= \iint_{D} \frac{-2x - 2y + 1}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2} \, dA
\]

\[
= \iint_{D} (-2x - 2y + 1) \, dA, \quad \text{so switching to polar coordinates gives}
\]

\[
= \int_{0}^{2\pi} \int_{0}^{1} (-2r \cos \theta - 2r \sin \theta + 1) \, r \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{1} (-2r^2 \cos \theta - 2r^2 \sin \theta + r) \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \left( \frac{-2r^3}{3} \cos \theta - \frac{-2r^3}{3} \sin \theta + \frac{r^2}{2} \right)_{r=0}^{1} \, d\theta
\]

\[
= \int_{0}^{2\pi} \left( -\frac{2}{3} \cos \theta - \frac{2}{3} \sin \theta + \frac{1}{2} \right) \, d\theta
\]

\[
= -\frac{2}{3} \sin \theta + \frac{1}{2} \cos \theta + \frac{1}{2} \theta \bigg|_{0}^{2\pi} = \pi.
\]

---

Notes:
Chapter 15 Line and Surface Integrals

The boundary curve $C$ is the unit circle $x^2 + y^2 = 1$ laying in the plane $z = 1$ (see Figure 15.21), which can be parametrized as $x = \cos t, y = \sin t, z = 1$ for $0 \leq t \leq 2\pi$. So

$$\oint_C \vec{f} \cdot d\vec{r} = \int_0^{2\pi} ((1)(-\sin t) + (\cos t)(\cos t) + (\sin t)(0)) \, dt$$

$$= \int_0^{2\pi} (-\sin t + \frac{1 + \cos 2t}{2}) \, dt \quad \text{(here we used } \cos^2 t = \frac{1 + \cos 2t}{2} \text{)}$$

$$= \cos t + \frac{t}{2} + \frac{\sin 2t}{4} \bigg|_0^{2\pi} = \pi.$$  

So we see that $\oint_C \vec{f} \cdot d\vec{r} = \iint_{\Sigma} (\text{curl } \vec{f}) \cdot \vec{n} \, d\sigma$, as predicted by Stokes' Theorem.

The line integral in the preceding example was far simpler to calculate than the surface integral, but this will not always be the case.

**Example 4 Using Stokes' Theorem**

Let $\Sigma$ be the elliptic paraboloid $z = \frac{x^2}{4} + \frac{y^2}{9}$ for $z \leq 1$, and let $C$ be its boundary curve. Calculate $\oint_C \vec{f} \cdot d\vec{r}$ for $\vec{f}(x, y, z) = (9xz + 2y)\hat{i} + (2x + y^2)\hat{j} + (-2y^2 + 2z)\hat{k}$, where $C$ is traversed counterclockwise.

**Solution** The surface is similar to the one in Example 3, except now the boundary curve $C$ is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ laying in the plane $z = 1$. In this case, using Stokes' Theorem is easier than computing the line integral directly. As in Example 3, at each point $(x, y, z(x, y))$ on the surface $z = z(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$ the vector

$$\vec{n} = \frac{-\frac{\partial z}{\partial x} \hat{i} - \frac{\partial z}{\partial y} \hat{j} + \hat{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} = \frac{-\frac{x}{2} \hat{i} - \frac{2y}{9} \hat{j} + \hat{k}}{\sqrt{1 + \frac{x^2}{4} + \frac{4y^2}{9}}},$$

is a positive unit normal vector to $\Sigma$. And calculating the curl of $\vec{f}$ gives

$$\text{curl } \vec{f} = (-4y - 0)\hat{i} + (9x - 0)\hat{j} + (2 - 2)\hat{k} = -4y \hat{i} + 9x \hat{j} + 0 \hat{k},$$

so

$$\text{(curl } \vec{f}) \cdot \vec{n} = \frac{(-4y)(-\frac{x}{2}) + (9x)(-\frac{2y}{9}) + (0)(1)}{\sqrt{1 + \frac{x^2}{4} + \frac{4y^2}{9}}} = \frac{2xy - 2xy + 0}{\sqrt{1 + \frac{x^2}{4} + \frac{4y^2}{9}}} = 0,$$

and so by Stokes' Theorem

$$\oint_C \vec{f} \cdot d\vec{r} = \iint_{\Sigma} (\text{curl } \vec{f}) \cdot \vec{n} \, d\sigma = \iint_{\Sigma} 0 \, d\sigma = 0.$$  

Notes:
In physical applications, for a simple closed curve \( C \) the line integral \( \oint_C \vec{f} \cdot d\vec{r} \) is often called the circulation of \( \vec{f} \) around \( C \). For example, if \( \vec{E} \) represents the electrostatic field due to a point charge, then it turns out that \( \text{curl} \vec{E} = \vec{0} \), which means that the circulation \( \oint_C \vec{E} \cdot d\vec{r} = 0 \) by Stokes’ Theorem. Vector fields which have zero curl are often called irrotational fields.

In fact, the term curl was created by the 19th century Scottish physicist James Clerk Maxwell in his study of electromagnetism, where it is used extensively. In physics, the curl is interpreted as a measure of circulation density. This is best seen by using another definition of curl \( \vec{f} \) which is equivalent to the definition given by Equation (15.16). Namely, for a point \((x, y, z)\) in \( \mathbb{R}^3 \),

\[
\hat{n} \cdot (\text{curl} \vec{f})(x, y, z) = \lim_{S \to 0} \frac{1}{S} \oint_C \vec{f} \cdot d\vec{r},
\]

where \( S \) is the surface area of a surface \( \Sigma \) containing the point \((x, y, z)\) and with a simple closed boundary curve \( C \) and positive unit normal vector \( \hat{n} \) at \((x, y, z)\). In the limit, think of the curve \( C \) shrinking to the point \((x, y, z)\), which causes \( \Sigma \), the surface it bounds, to have smaller and smaller surface area. That ratio of circulation to surface area in the limit is what makes the curl a rough measure of circulation density (i.e., circulation per unit area).

An idea of how the curl of a vector field is related to rotation is shown in Figure 15.22. Suppose we have a vector field \( \vec{f}(x, y, z) \) which is always parallel to the xy-plane at each point \((x, y, z)\) and that the vectors grow larger the further the point \((x, y, z)\) is from the y-axis. For example, \( \vec{f}(x, y, z) = (1 + x^2) \hat{j} \). Think of the vector field as representing the flow of water, and imagine dropping two wheels with paddles into that water flow, as in Figure 15.22. Since the

Notes:
flow is stronger (i.e., the magnitude of \( \vec{f} \) is larger) as you move away from the 
\( y \)-axis, then such a wheel would rotate counterclockwise if it were dropped to 
the right of the \( y \)-axis, and it would rotate clockwise if it were dropped to the 
left of the \( y \)-axis. In both cases the curl would be nonzero (curl \( \vec{f}(x, y, z) = 2x \hat{k} \) 
in our example) and would obey the right-hand rule, that is, curl \( \vec{f}(x, y, z) \) points 
in the direction of your thumb as you cup your right hand in the direction of the 
rotation of the wheel. So the curl points outward (in the positive \( z \)-direction) if 
\( x > 0 \) and points inward (in the negative \( z \)-direction) if \( x < 0 \). Notice that if all 
the vectors had the same direction and the same magnitude, then the wheels 
would not rotate and hence there would be no curl (which is why such fields are 
called irrotational, meaning no rotation).

Finally, by Stokes’ Theorem, we know that if \( C \) is a simple closed curve in 
some solid region \( S \) in \( \mathbb{R}^3 \) and if \( \vec{f}(x, y, z) \) is a smooth vector field such that curl \( \vec{f} = \vec{0} \) in \( S \), then

\[
\oint_C \vec{f} \cdot d\vec{r} = \iint_S (\text{curl} \vec{f}) \cdot \hat{n} \, d\sigma = \iint_S 0 \cdot \hat{n} \, d\sigma = \iint_S 0 \, d\sigma = 0,
\]

where \( \Sigma \) is any orientable surface inside \( S \) whose boundary is \( C \) (such a surface 
is sometimes called a capping surface for \( C \)). So similar to the two-variable case, 
we have a three-dimensional version of a result from Section 15.3, for solid 
regions in \( \mathbb{R}^3 \) which are simply connected (i.e. regions having no holes):

**Theorem 145 Irrotational Equivalences**
The following statements are equivalent for a simply connected solid 
region \( S \) in \( \mathbb{R}^3 \):

1. \( \vec{f}(x, y, z) = P(x, y, z) \hat{i} + Q(x, y, z) \hat{j} + R(x, y, z) \hat{k} \) has a smooth 
potential \( F(x, y, z) \) in \( S \)

2. \( \int_C \vec{f} \cdot d\vec{r} \) is independent of the path for any curve \( C \) in \( S \)

3. \( \int_C \vec{f} \cdot d\vec{r} = 0 \) for every simple closed curve \( C \) in \( S \)

4. \( \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \) in \( S \); i.e., curl \( \vec{f} = \vec{0} \) in \( S \), or 
the differential form \( P \, dx + Q \, dy + R \, dz \) is exact

**Example 5 Determining Irrotation**
Determine if the vector field \( \vec{f}(x, y, z) = xyz \hat{i} + xz \hat{j} + xy \hat{k} \) has a potential in \( \mathbb{R}^3 \).

Notes:
SOLUTION: Since \( \mathbb{R}^3 \) is simply connected, we just need to check whether \( \text{curl} \vec{f} = 0 \) throughout \( \mathbb{R}^3 \), that is,

\[
\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
\]

throughout \( \mathbb{R}^3 \), where \( P(x, y, z) = xyz \), \( Q(x, y, z) = xz \), and \( R(x, y, z) = xy \). But we see that

\[
\frac{\partial P}{\partial z} = xy, \quad \frac{\partial R}{\partial x} = y \quad \Rightarrow \quad \frac{\partial P}{\partial z} \neq \frac{\partial R}{\partial x}
\]

for some \((x, y, z)\) in \( \mathbb{R}^3 \).

Thus, \( \vec{f}(x, y, z) \) does not have a potential in \( \mathbb{R}^3 \).

The theorems in this chapter all relate an integral over a domain to an integral over the boundary of the domain. This means that we need to pay special attention to the orientation of the domain. We’ll occasionally use the symbol \( \partial \) to indicate the boundary of something (so the boundary of \( D \) would be \( \partial D \)). This is the same symbol as a partial derivative, so you’ll have to look at the context to figure out which definition is being used.

We also have four new theorems about these types of integrals. These are similar to the Fundamental Theorem of Calculus, so we’ll put all five together into Key Idea 65.

**Key Idea 65  Fundamental Theorems Relating Integrals and Domains**

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Equation</th>
<th>Orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental Theorem of Calculus</td>
<td>( f(b) - f(a) = \int_a^b f'(x) , dx )</td>
<td></td>
</tr>
<tr>
<td>Fundamental Theorem of Gradient Fields</td>
<td>( \phi(Q) - \phi(P) = \int_C \nabla \phi \cdot ds )</td>
<td>( C ) goes from ( P ) to ( Q )</td>
</tr>
<tr>
<td>Green’s Theorem</td>
<td>( \oint_D P , dx + Q , dy = \iint_D Q_x - P_y , dA )</td>
<td>( \partial D ) oriented counterclockwise</td>
</tr>
<tr>
<td>Stokes’ Theorem</td>
<td>( \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{A} )</td>
<td>( \partial S ) oriented with ( S ) to the left</td>
</tr>
<tr>
<td>Divergence Theorem</td>
<td>( \iiint_{\partial W} \vec{F} \cdot d\vec{A} = \iiint_W \text{div} \vec{F} , dV )</td>
<td>( \partial W ) oriented outwards</td>
</tr>
</tbody>
</table>

Notes:
**Exercises 15.5**

**Problems**

In Exercises 1–3, calculate \( \int_C f(x, y, z) \, ds \) for the given function \( f(x, y, z) \) and curve \( C \).

1. \( f(x, y, z) = z; \quad C : x = \cos t, y = \sin t, z = t, \ 0 \leq t \leq 2\pi \)
2. \( f(x, y, z) = \frac{x}{y} + y + 2xz; \quad C : x = t^2, y = t, z = 1, \ 1 \leq t \leq 2 \)
3. \( f(x, y, z) = z^2; \quad C : x = t \sin t, y = t \cos t, z = \frac{2t^2}{\pi} \cos^{1/2} t, \ 0 \leq t \leq 1 \)

In Exercises 4–9, calculate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for the given vector field \( \mathbf{F}(x, y, z) \) and curve \( C \).

4. \( \mathbf{F}(x, y, z) = i - j + k; \quad C : x = 3t, y = 2t, z = t, \ 0 \leq t \leq 1 \)
5. \( \mathbf{F}(x, y, z) = y i - x j + z k; \quad C : x = \cos t, y = \sin t, z = t, \ 0 \leq t \leq 2\pi \)
6. \( \mathbf{F}(x, y, z) = x i + y j + z k; \quad C : x = \cos t, y = \sin t, z = 2, \ 0 \leq t \leq 2\pi \)
7. \( \mathbf{F}(x, y, z) = (y - 2z) i + xy j + (2xz + y) k; \quad C : x = t, y = 2t, z = t^2 - 1, \ 0 \leq t \leq 1 \)
8. \( \mathbf{F}(x, y, z) = yz i +xz j + xy k; \quad C : \) the polygonal path from \((0, 0, 0)\) to \((1, 2, 0)\)
9. \( \mathbf{F}(x, y, z) = xy i + (z - x) j + 2yz k; \quad C : \) the polygonal path from \((0, 0, 0)\) to \((1, 0, 0)\) to \((1, 2, 0)\) to \((1, 2, -2)\)

In Exercises 10–13, state whether or not the vector field \( \mathbf{F}(x, y, z) \) has a potential in \( \mathbb{R}^3 \) (you do not need to find the potential itself).

10. \( \mathbf{F}(x, y, z) = y i - x j + z k \)
11. \( \mathbf{F}(x, y, z) = a i + b j + c k; \quad (a, b, c \) constant\)
12. \( \mathbf{F}(x, y, z) = (x + y) i + x j + z^2 k \)
13. \( \mathbf{F}(x, y, z) = xy i - (x - yz^2) j + y^2 z k \)

In Exercises 14–15, verify Stokes’ Theorem for the given vector field \( \mathbf{F}(x, y, z) \) and surface \( \Sigma \).

14. \( \mathbf{F}(x, y, z) = 2y i - x j + z k; \quad \Sigma : x^2 + y^2 + z^2 = 1, z \geq 0 \)
15. \( \mathbf{F}(x, y, z) = xy i + xz j + yz k; \quad \Sigma : z = x^2 + y^2, z \leq 1 \)

16. Construct a Möbius strip from a piece of paper, then draw a line down its center (like the dotted line in Figure 15.19(b)).
   Cut the Möbius strip along that center line completely around the strip. How many surfaces does this result in? How would you describe them? Are they orientable?

17. Use a computer algebra system to plot the Möbius strip parametrized as:
   
   \[ r(u, v) = \left( \cos u \left( 1 + v \cos \frac{u}{2} \right), \sin u \left( 1 + v \cos \frac{u}{2} \right), v \sin \frac{u}{2} \right), \]
   
   where \( 0 \leq u \leq 2\pi, -\frac{1}{2} \leq v \leq \frac{1}{2} \)

18. Let \( \Sigma \) be a closed surface and \( \mathbf{F}(x, y, z) \) a smooth vector field. Show that \( \iint_{\Sigma} (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 0 \). (Hint: Split \( \Sigma \) in half.)

19. Show that Green’s Theorem is a special case of Stokes’ Theorem.
15.6 Gradient, Divergence, Curl and Laplacian

In this final section we will establish some relationships between the gradient, divergence and curl, and we will also introduce a new quantity called the Laplacian. We will then show how to write these quantities in cylindrical and spherical coordinates.

For a real-valued function \( f(x, y, z) \) on \( \mathbb{R}^3 \), the gradient \( \nabla f(x, y, z) \) is a vector-valued function on \( \mathbb{R}^3 \), that is, its value at a point \((x, y, z)\) is the vector
\[
\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle
\]
in \( \mathbb{R}^3 \), where each of the partial derivatives is evaluated at the point \((x, y, z)\). So in this way, you can think of the symbol \( \nabla \) as being “applied” to a real-valued function \( f \) to produce a vector \( \nabla f \).

In Section 15.4 and Section 15.5, we noted that the divergence and curl can also be expressed in terms of the symbol \( \nabla \) by thinking of \( \nabla \) as a vector in \( \mathbb{R}^3 \), namely
\[
\n = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}.
\]
(15.20)

Here, the symbols \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial z} \) are to be thought of as “partial derivative operators” that will get “applied” to a real-valued function, say \( f(x, y, z) \), to produce the partial derivatives \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) and \( \frac{\partial f}{\partial z} \). For instance, \( \frac{\partial f}{\partial x} \) “applied” to \( f(x, y, z) \) produces \( \frac{\partial f}{\partial x} \).

Is \( \nabla \) really a vector? Strictly speaking, no, since \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial z} \) are not actual numbers. But it helps to think of \( \nabla \) as a vector, especially with the divergence and curl, as we will soon see. The process of “applying” \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \) to a real-valued function \( f(x, y, z) \) is normally thought of as multiplying the quantities:
\[
\left( \frac{\partial}{\partial x} \right) (f) = \frac{\partial f}{\partial x}, \quad \left( \frac{\partial}{\partial y} \right) (f) = \frac{\partial f}{\partial y}, \quad \left( \frac{\partial}{\partial z} \right) (f) = \frac{\partial f}{\partial z}
\]

For this reason, \( \nabla \) is often referred to as the “del operator”, since it “operates” on functions.

For example, it is often convenient to write the divergence \( \text{div} \vec{f} \) as \( \nabla \cdot \vec{f} \), since for a vector field \( \vec{f}(x, y, z) = f_1(x, y, z) \hat{i} + f_2(x, y, z) \hat{j} + f_3(x, y, z) \hat{k} \), the dot product

---

Notes:
of \vec{f} with \nabla (thought of as a vector) makes sense:

\[ \nabla \cdot \vec{f} = \left( \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot \left( f_1(x, y, z) \vec{i} + f_2(x, y, z) \vec{j} + f_3(x, y, z) \vec{k} \right) \]
\[ = \left( \frac{\partial}{\partial x} \right) f_1 + \left( \frac{\partial}{\partial y} \right) f_2 + \left( \frac{\partial}{\partial z} \right) f_3 \]
\[ = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \]
\[ = \text{div} \vec{f} \quad (15.21) \]

We can also write curl\vec{f} in terms of \nabla, namely as \nabla \times \vec{f}, since for a vector field \vec{f}(x, y, z) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}, we have:

\[ \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix} \]
\[ = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \vec{j} + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \]
\[ = \text{curl} \vec{f} \quad (15.22) \]

For a real-valued function \( f(x, y, z) \), the gradient \( \nabla f(x, y, z) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \) is a vector field, so we can take its divergence:

\[ \text{div} \nabla f = \nabla \cdot \nabla f \]
\[ = \left( \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot \left( \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right) \]
\[ = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} \right) \]
\[ = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

Note that this is a real-valued function, to which we will give a special name:
Definition 118   Laplacian
For a real-valued function \( f(x, y, z) \), the Laplacian of \( f \), denoted by \( \Delta f \), is given by
\[
\Delta f(x, y, z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\]

Often the notation \( \nabla^2 f \) is used for the Laplacian instead of \( \Delta f \), using the convention \( \nabla^2 = \nabla \cdot \nabla \).

Example 1    Computing a Laplacian
Let \( \mathbf{r}(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \) be the position vector field on \( \mathbb{R}^3 \). Then \( ||\mathbf{r}(x, y, z)||^2 = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2 \) is a real-valued function. Find

1. the gradient of \( ||\mathbf{r}||^2 \)
2. the divergence of \( \mathbf{r} \)
3. the curl of \( \mathbf{r} \)
4. the Laplacian of \( ||\mathbf{r}||^2 \)

Solution
1. \( \nabla ||\mathbf{r}||^2 = 2x \hat{i} + 2y \hat{j} + 2z \hat{k} = 2\mathbf{r} \)
2. \( \nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3 \)
3. \[
\nabla \times \mathbf{r} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & y & z 
\end{vmatrix}
= (0-0) \hat{i} - (0-0) \hat{j} + (0-0) \hat{k} = \mathbf{0}
\]

Notes:
4. \[ \Delta ||\vec{r}||^2 = \frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2) + \frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2) + \frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2) = 2 + 2 + 2 = 6. \]

Note that we could have calculated \( \Delta ||\vec{r}||^2 \) another way, using the \( \nabla \) notation along with the first two parts:

\[ \Delta ||\vec{r}||^2 = \nabla \cdot \nabla ||\vec{r}||^2 = \nabla \cdot 2\vec{r} = 2\nabla \cdot \vec{r} = 2(3) = 6 \]

Notice that in Example 1 if we take the curl of the gradient of \( ||\vec{r}||^2 \) we get

\[ \nabla \times (\nabla ||\vec{r}||^2) = \nabla \times 2\vec{r} = 2\nabla \times \vec{r} = 2\vec{0} = \vec{0}. \]

The following theorem shows that this will be the case in general:

---

**Theorem 146  Gradient is Irrotational**

For any smooth real-valued function \( f(x, y, z) \), \( \nabla \times (\nabla f) = \vec{0} \).

**Proof**

We see by the smoothness of \( f \) that

\[
\nabla \times (\nabla f) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}
\end{vmatrix} \vec{k} = \vec{0},
\]

since the mixed partial derivatives in each component are equal. \( \square \)

Another way of stating Theorem 146 is that potentials are irrotational.

---

**Theorem 147  Potentials are Irrotational**

If a vector field \( \vec{f}(x, y, z) \) has a potential, then \( \text{curl} \vec{f} = \vec{0} \).

Notes:
Also, notice that in Example 1 if we take the divergence of the curl of \( \vec{r} \) we trivially get

\[
\nabla \cdot (\nabla \times \vec{r}) = \nabla \cdot \vec{0} = 0.
\]

The following theorem shows that this will be the case in general:

**Theorem 148** Curl is Divergence Free

For any smooth vector field \( \vec{f}(x, y, z) \),

\[
\nabla \cdot (\nabla \times \vec{f}) = 0.
\]

The proof is straightforward and left as Exercise 24.

**Theorem 149** Flux of a Curl Through a Closed Surface

The flux of the curl of a smooth vector field \( \vec{f}(x, y, z) \) through any closed surface is zero.

**Proof**

Let \( \Sigma \) be a closed surface which bounds a solid \( S \). The flux of \( \nabla \times \vec{f} \) through \( \Sigma \) is

\[
\iint_{\Sigma} (\nabla \times \vec{f}) \cdot d\sigma = \iiint_{S} \nabla \cdot (\nabla \times \vec{f}) dV \quad \text{(by the Divergence Theorem)}
\]

\[
= \iiint_{S} 0 dV \quad \text{(by Theorem 148)}
\]

\[
= 0. \quad \square
\]

There is another method for proving Theorem 146 which can be useful, and is often used in physics. Namely, if the surface integral \( \iint_{\Sigma} f(x, y, z) d\sigma = 0 \) for all surfaces \( \Sigma \) in some solid region (usually all of \( \mathbb{R}^3 \)), then we must have \( f(x, y, z) = 0 \) throughout that region. The proof is not trivial, and physicists do not usually bother to prove it. But the result is true, and can also be applied to double and triple integrals.

For instance, to prove Theorem 146, assume that \( f(x, y, z) \) is a smooth real-valued function on \( \mathbb{R}^3 \). Let \( C \) be a simple closed curve in \( \mathbb{R}^3 \) and let \( \Sigma \) be any capping surface for \( C \) (i.e., \( \Sigma \) is orientable and its boundary is \( C \)). Since \( \nabla f \) is a vector field, then

\[
\iiint_{\Sigma} (\nabla \times (\nabla f)) \cdot \vec{n} d\sigma = \oint_{C} \nabla f \cdot d\vec{r} \quad \text{by Stokes' Theorem, so}
\]

\[
= 0 \quad \text{by Theorem 143.}
\]

Notes:
Since the choice of $\Sigma$ was arbitrary, then we must have $(\nabla \times (\nabla f)) \cdot \vec{n} = 0$ throughout $\mathbb{R}^3$, where $\vec{n}$ is any unit vector. Using $\vec{i}$, $\vec{j}$ and $\vec{k}$ in place of $\vec{n}$, we see that we must have $\nabla \times (\nabla f) = \vec{0}$ in $\mathbb{R}^3$, which completes the proof.

**Example 2  Maxwell's Equation**

A system of electric charges has a charge density $\rho(x, y, z)$ and produces an electrostatic field $\vec{E}(x, y, z)$ at points $(x, y, z)$ in space. Gauss' Law states that

$$\int_{\Sigma} \vec{E} \cdot d\sigma = 4\pi \int_{S} \rho dV$$

for any closed surface $\Sigma$ which encloses the charges, with $S$ being the solid region enclosed by $\Sigma$. Show that $\nabla \cdot \vec{E} = 4\pi \rho$. This is one of Maxwell's Equations.

**SOLUTION**

By the Divergence Theorem, we have

$$\iiint_{S} \nabla \cdot \vec{E} dV = \iiint_{\Sigma} \vec{E} \cdot d\sigma$$

$$= 4\pi \iiint_{S} \rho dV \quad \text{by Gauss' Law}$$

Combining the integrals gives

$$\iiint_{S} (\nabla \cdot \vec{E} - 4\pi \rho) dV = 0,$$

so

$$\nabla \cdot \vec{E} - 4\pi \rho = 0 \quad \text{since } \Sigma \text{ and hence } S \text{ was arbitrary, so } \nabla \cdot \vec{E} = 4\pi \rho.$$

Often (especially in physics) it is convenient to use other coordinate systems when dealing with quantities such as the gradient, divergence, curl and Laplacian. We will present the formulas for these in cylindrical and spherical coordinates.

Recall from Section 11.7 that a point $(x, y, z)$ can be represented in cylindrical coordinates $(r, \theta, z)$, where $x = r \cos \theta$, $y = r \sin \theta$, $z = z$. At each point $(r, \theta, z)$, let $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ be unit vectors in the direction of increasing $r$, $\theta$, $z$, respectively (see Figure 15.23(a)). Then $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ form an orthonormal set of vectors. Note, by the right-hand rule, that $\vec{e}_z \times \vec{e}_r = \vec{e}_\theta$. 

Notes:
15.6 Gradient, Divergence, Curl and Laplacian

Similarly, a point \((x, y, z)\) can be represented in spherical coordinates \((\rho, \theta, \phi)\), where \(x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi\). At each point \((\rho, \theta, \phi)\), let \(\vec{e}_\rho, \vec{e}_\theta, \vec{e}_\phi\) be unit vectors in the direction of increasing \(\rho, \theta, \phi\), respectively (see Figure 15.23(b)). Then the vectors \(\vec{e}_\rho, \vec{e}_\theta, \vec{e}_\phi\) are orthonormal. By the right-hand rule, we see that \(\vec{e}_\theta \times \vec{e}_\rho = \vec{e}_\phi\).

We can now summarize the expressions for the gradient, divergence, curl and Laplacian in Cartesian, cylindrical and spherical coordinates in the following tables:

---

### Key Idea 66 Vector Calculus in Cartesian Coordinates

For the scalar function \(F\) or vector field \(\vec{f} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}\),

- **gradient**: \(\nabla F = \frac{\partial F}{\partial x} \vec{i} + \frac{\partial F}{\partial y} \vec{j} + \frac{\partial F}{\partial z} \vec{k}\)
- **divergence**: \(\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\)
- **curl**: \(\nabla \times \vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) \vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) \vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \vec{k}\)
- **Laplacian**: \(\Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\)

---

Notes:
Key Idea 67  Vector Calculus in Cylindrical Coordinates
For the scalar function \( F \) or vector field \( \vec{f} = f_r \hat{e}_r + f_\theta \hat{e}_\theta + f_z \hat{e}_z \),

\[
\begin{align*}
\text{gradient:} & \quad \nabla F = \frac{\partial F}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{e}_\theta + \frac{\partial F}{\partial z} \hat{e}_z \\
\text{divergence:} & \quad \nabla \cdot \vec{f} = \frac{1}{r} \frac{\partial}{\partial r} (rf_r) + \frac{1}{r} \frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z} \\
\text{curl:} & \quad \nabla \times \vec{f} = \left( \frac{1}{r} \frac{\partial f_z}{\partial \theta} - \frac{\partial f_\theta}{\partial z} \right) \hat{e}_r + \left( \frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} (rf_\theta) - \frac{\partial f_r}{\partial \theta} \right) \hat{e}_z \\
\text{Laplacian:} & \quad \Delta F = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial^2 F}{\partial \phi^2}
\end{align*}
\]

The derivation of the above formulas for cylindrical and spherical coordinates is straightforward but extremely tedious. The basic idea is to take the Cartesian equivalent of the quantity in question and to substitute into that formula using the appropriate coordinate transformation. As an example, we will derive the formula for the gradient in spherical coordinates.

Example 3  Spherical Coordinates Gradient
Show that the gradient of a real-valued function \( F(\rho, \theta, \phi) \) in spherical coordinates is:

\[
\nabla F = \frac{\partial F}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial F}{\partial \theta} \hat{e}_\theta + \frac{1}{\rho \sin \phi} \frac{\partial F}{\partial \phi} \hat{e}_\phi
\]
SOLUTION The idea is that in the Cartesian gradient formula $\nabla F(x, y, z) = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k}$, we want to put the Cartesian basis vectors $\hat{i}, \hat{j}, \hat{k}$ in terms of the spherical coordinate basis vectors $\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi$ and functions of $\rho, \theta$ and $\phi$. Then put the partial derivatives $\frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \phi}$ in terms of $\frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \phi}$ and functions of $\rho, \theta$ and $\phi$.

Our first step is to get formulas for $\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi$ in terms of $\hat{i}, \hat{j}, \hat{k}$.

We can see from Figure 15.23(b) that the unit vector $\hat{e}_\rho$ in the $\rho$ direction at a general point $(\rho, \theta, \phi)$ is $\hat{e}_\rho = \frac{\hat{r}}{||\hat{r}||}$, where $\hat{r} = x \hat{i} + y \hat{j} + z \hat{k}$ is the position vector of the point in Cartesian coordinates. Thus,

$$\hat{e}_\rho = \frac{\hat{r}}{||\hat{r}||} = \frac{x \hat{i} + y \hat{j} + z \hat{k}}{\sqrt{x^2 + y^2 + z^2}}.$$ 

So using $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$, and $\rho = \sqrt{x^2 + y^2 + z^2}$, we get:

$$\hat{e}_\rho = \sin \phi \cos \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \phi \hat{k} \quad (15.23)$$

Now, since the angle $\theta$ is measured in the $xy$-plane, then the unit vector $\hat{e}_\theta$ in the $\theta$ direction must be parallel to the $xy$-plane. That is, $\hat{e}_\theta$ is of the form $a \hat{i} + b \hat{j} + 0 \hat{k}$. To figure out what $a$ and $b$ are, note that since $\hat{e}_\theta \perp \hat{e}_\rho$, then in particular $\hat{e}_\theta \perp \hat{e}_\rho$, when $\hat{e}_\rho$ is in the $xy$-plane. That occurs when the angle $\phi$ is $\pi/2$. Putting $\phi = \pi/2$ into the formula for $\hat{e}_\rho$ gives $\hat{e}_\rho = \cos \theta \hat{i} + \sin \theta \hat{j} + 0 \hat{k}$, and we see that a vector perpendicular to that is $-\sin \theta \hat{i} + \cos \theta \hat{j} + 0 \hat{k}$. Since this vector is also a unit vector and points in the (positive) $\theta$ direction, it must be $\hat{e}_\theta$:

$$\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} + 0 \hat{k} \quad (15.24)$$

Lastly, since $\hat{e}_\phi = \hat{e}_\theta \times \hat{e}_\rho$, we get:

$$\hat{e}_\phi = \cos \phi \cos \theta \hat{i} + \cos \phi \sin \theta \hat{j} - \sin \phi \hat{k} \quad (15.25)$$

Now that we have formulas for the three spherical unit vectors, our next step is to solve those for $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi$.

This comes down to solving a system of three equations in three unknowns. There are many ways of doing this, but we will do it by combining the formulas for $\hat{e}_\rho$ and $\hat{e}_\phi$ to eliminate $\hat{k}$, which will give us an equation involving just $\hat{i}$ and $\hat{j}$.

This, with the formula for $\hat{e}_\theta$, will then leave us with a system of two equations in two unknowns ($\hat{i}$ and $\hat{j}$), which we will use to solve first for $\hat{j}$ then for $\hat{i}$. Lastly, we will solve for $\hat{k}$.

First, note that

$$\sin \phi \hat{e}_\rho + \cos \phi \hat{e}_\phi = \cos \theta \hat{i} + \sin \theta \hat{j}$$

Notes:
so that
\[ \sin \theta (\sin \phi \vec{e}_\rho + \cos \phi \vec{e}_\phi) + \cos \theta \vec{e}_\theta = (\sin^2 \theta + \cos^2 \theta)\vec{j} = \vec{j}, \]
and so:
\[ \vec{j} = \sin \phi \sin \theta \vec{e}_\rho + \cos \theta \vec{e}_\theta + \cos \phi \sin \theta \vec{e}_\phi \quad (15.26) \]

Likewise, we see that
\[ \cos \theta (\sin \phi \vec{e}_\rho + \cos \phi \vec{e}_\phi) - \sin \theta \vec{e}_\theta = (\cos^2 \theta + \sin^2 \theta)\vec{i} = \vec{i}, \]
and so:
\[ \vec{i} = \sin \phi \cos \theta \vec{e}_\rho - \sin \theta \vec{e}_\theta + \cos \phi \cos \theta \vec{e}_\phi \quad (15.27) \]

Lastly, we see that:
\[ \vec{k} = \cos \phi \vec{e}_\rho - \sin \phi \vec{e}_\phi \quad (15.28) \]

Now that we have formulas for the three Cartesian unit vectors, our next step is to get formulas for \( \frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \phi} \) in terms of \( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \).

By the Chain Rule, we have
\[
\begin{align*}
\frac{\partial F}{\partial \rho} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial \rho}, \\
\frac{\partial F}{\partial \theta} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial \theta}, \\
\frac{\partial F}{\partial \phi} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \phi} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial \phi},
\end{align*}
\]
which yields:
\[
\begin{align*}
\frac{\partial F}{\partial \rho} &= \sin \phi \cos \theta \frac{\partial F}{\partial x} + \sin \phi \sin \theta \frac{\partial F}{\partial y} + \cos \phi \frac{\partial F}{\partial z} \quad (15.29) \\
\frac{\partial F}{\partial \theta} &= -\rho \sin \phi \sin \theta \frac{\partial F}{\partial x} + \rho \sin \phi \cos \theta \frac{\partial F}{\partial y} \quad (15.30) \\
\frac{\partial F}{\partial \phi} &= \rho \cos \phi \cos \theta \frac{\partial F}{\partial x} + \rho \cos \phi \sin \theta \frac{\partial F}{\partial y} - \rho \sin \phi \frac{\partial F}{\partial z} \quad (15.31)
\end{align*}
\]

Our next step is to invert the previous relation to solve for \( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \) in terms of \( \frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \phi} \).

Again, this involves solving a system of three equations in three unknowns.

Notes:
Using a similar process of elimination as before, we get:

\[
\frac{\partial F}{\partial x} = \frac{1}{\rho \sin \phi} \left( \rho \sin^2 \phi \cos \theta \frac{\partial F}{\partial \rho} - \sin \theta \frac{\partial F}{\partial \theta} + \sin \phi \cos \theta \frac{\partial F}{\partial \phi} \right)
\]

\[
\frac{\partial F}{\partial y} = \frac{1}{\rho \sin \phi} \left( \rho \sin^2 \phi \sin \theta \frac{\partial F}{\partial \rho} + \cos \theta \frac{\partial F}{\partial \theta} - \sin \phi \cos \theta \frac{\partial F}{\partial \phi} \right)
\]

\[
\frac{\partial F}{\partial z} = \frac{1}{\rho} \left( \rho \cos \phi \frac{\partial F}{\partial \rho} - \sin \phi \frac{\partial F}{\partial \phi} \right)
\]

(15.32) (15.33) (15.34)

Finally, we substitute the Equations (15.26), (15.27), (15.28), and (15.34) into the Cartesian gradient formula \( \nabla F(x, y, z) = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k} \).

Doing this last step is perhaps the most tedious, since it involves simplifying \( 3 \times 3 + 3 \times 3 + 2 \times 2 = 22 \) terms! Namely,

\[
\nabla F = \frac{1}{\rho \sin \phi} \left( \rho \sin^2 \phi \cos \theta \frac{\partial F}{\partial \rho} - \sin \theta \frac{\partial F}{\partial \theta} + \sin \phi \cos \theta \frac{\partial F}{\partial \phi} \right) (\sin \phi \cos \theta \hat{e}_\rho - \sin \theta \hat{e}_\theta)
\]

\[
+ \frac{1}{\rho \sin \phi} \left( \rho \sin^2 \phi \sin \theta \frac{\partial F}{\partial \rho} + \cos \theta \frac{\partial F}{\partial \theta} - \sin \phi \cos \theta \frac{\partial F}{\partial \phi} \right) (\sin \phi \sin \theta \hat{e}_\rho + \cos \theta \hat{e}_\theta)
\]

\[
+ \frac{1}{\rho} \left( \rho \cos \phi \frac{\partial F}{\partial \rho} - \sin \phi \frac{\partial F}{\partial \phi} \right) (\cos \phi \hat{e}_\rho - \sin \phi \hat{e}_\phi),
\]

which we see has 8 terms involving \( \hat{e}_\rho \), 6 terms involving \( \hat{e}_\theta \), and 8 terms involving \( \hat{e}_\phi \). But the algebra is straightforward and yields the desired result:

\[
\nabla F = \frac{\partial F}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho \sin \phi} \frac{\partial F}{\partial \theta} \hat{e}_\theta + \frac{1}{\rho} \frac{\partial F}{\partial \phi} \hat{e}_\phi
\]

Example 4 Practicing in Spherical Coordinates

In Example 1 we showed that \( \| \mathbf{r} \|^2 = 2 \mathbf{r} \) and \( \Delta \| \mathbf{r} \|^2 = 6 \), where \( \mathbf{r}(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \) in Cartesian coordinates. Verify that we get the same answers if we switch to spherical coordinates.

Solution Since \( \| \mathbf{r} \|^2 = x^2 + y^2 + z^2 = \rho^2 \) in spherical coordinates, let \( F(\rho, \theta, \phi) = \rho^2 \) (so that \( F(\rho, \theta, \phi) = \| \mathbf{r} \|^2 \)). The gradient of \( F \) in spherical...
coordinates is
\[
\nabla F = \frac{\partial F}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho \sin \phi} \frac{\partial F}{\partial \theta} \vec{e}_\theta + \frac{1}{\rho} \frac{\partial F}{\partial \phi} \vec{e}_\phi
\]
\[
= 2\rho \vec{e}_\rho + \frac{1}{\rho \sin \phi} (0) \vec{e}_\theta + \frac{1}{\rho} (0) \vec{e}_\phi
\]
\[
= 2\rho \vec{e}_\rho = 2\rho \frac{\vec{r}}{||\vec{r}||}, \text{ as we showed earlier, so}
\]
\[
= 2\rho \frac{\vec{r}}{\rho} = 2\vec{r}, \text{ as expected. And the Laplacian is}
\]
\[
\Delta F = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial F}{\partial \rho} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial F}{\partial \phi} \right)
\]
\[
= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial F}{\partial \rho} \right) + \frac{1}{\rho^2 \sin^2 \phi} (0) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi (0) \right)
\]
\[
= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (2\rho^3) + 0 + 0
\]
\[
= \frac{1}{\rho^2} (6\rho^3) = 6, \text{ as expected.}
\]
**Exercises 15.6**

**Problems**

In Exercises 1–6, find the Laplacian of the function \(f(x, y, z)\) in Cartesian coordinates.

1. \(f(x, y, z) = x + y + z\)
2. \(f(x, y, z) = x^2\)
3. \(f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}\)
4. \(f(x, y, z) = e^{x+y+z}\)
5. \(f(x, y, z) = x^3 + y^3 + z^3\)
6. \(f(x, y, z) = e^{-x^2-y^2-z^2}\)

7. Find the Laplacian of the function in Exercise 3 in spherical coordinates.
8. Find the Laplacian of the function in Exercise 6 in spherical coordinates.
9. Let \(f(x, y, z) = \frac{z}{x^2+y^2}\) in Cartesian coordinates. Find \(\nabla f\) in cylindrical coordinates.
10. For \(\vec{f}(r, \theta, z) = r \vec{e}_r + z \sin \theta \vec{e}_\theta + rz \vec{e}_z\) in cylindrical coordinates, find \(\text{div} \vec{f}\) and \(\text{curl} \vec{f}\).
11. For \(\vec{f}(\rho, \theta, \phi) = \vec{e}_\rho + \rho \cos \theta \vec{e}_\theta + \rho \vec{e}_\phi\) in spherical coordinates, find \(\text{div} \vec{f}\) and \(\text{curl} \vec{f}\).

In Exercises 12–23, prove the given formula \((r = \|\vec{r}\|)\) is the length of the position vector field \(\vec{r}(x, y, z) = \vec{x} + \vec{y} + \vec{z}\).

12. \(\nabla (1/r) = -\vec{r}/r^3\)
13. \(\Delta (1/r) = 0\)
14. \(\nabla \cdot (\vec{r}/r^3) = 0\)
15. \(\nabla (\ln r) = \vec{r}/r^2\)
16. \(\text{div} (\vec{F} + \vec{G}) = \text{div} \vec{F} + \text{div} \vec{G}\)
17. \(\text{curl} (\vec{F} + \vec{G}) = \text{curl} \vec{F} + \text{curl} \vec{G}\)
18. \(\text{div} (f \vec{F}) = f \text{div} \vec{F} + \vec{F} \cdot \nabla f\)
19. \(\text{div} (\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl} \vec{F} - \vec{F} \cdot \text{curl} \vec{G}\)
20. \(\text{div} (\nabla \nabla g) = 0\)
21. \(\text{curl} (f \vec{F}) = f \text{curl} \vec{F} + (\nabla f) \times \vec{F}\)
22. \(\text{curl} (\text{curl} \vec{F}) = \nabla (\text{div} \vec{F}) - \Delta \vec{F}\)
23. \(\Delta (fg) = f \Delta g + g \Delta f + 2(\nabla f \cdot \nabla g)\)
25. Derive the gradient formula in cylindrical coordinates: \(\nabla \vec{F} = \frac{\partial F_1}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial F_1}{\partial \theta} \vec{e}_\theta + \frac{\partial F_1}{\partial z} \vec{e}_z\)
26. Use \(\vec{f} = u \nabla v\) in the Divergence Theorem to prove:

   (a) Green’s first identity: \(\iiint_S (u \Delta v + (\nabla u) \cdot (\nabla v)) \, dV = \oint_S (u \nabla v) \cdot d\sigma\)

   (b) Green’s second identity: \(\iiint_S (u \Delta v - v \Delta u) \, dV = \oint_S (u \nabla v - v \nabla u) \cdot d\sigma\)

27. Suppose that \(\Delta u = 0\) (i.e., \(u\) is harmonic) over \(\mathbb{R}^3\). Define the normal derivative \(\frac{\partial u}{\partial n}\) of \(u\) over a closed surface \(\Sigma\) with outward unit normal vector \(\vec{n}\) by \(\frac{\partial u}{\partial n} = D_n u = \vec{n} \cdot \nabla u\). Show that \(\int_{\Sigma} \frac{\partial u}{\partial n} \, d\sigma = 0\). (Hint: Use Green’s second identity.)

991
A: SOLUTIONS TO SELECTED PROBLEMS

Chapter 11

Exercises 11.1

1. right hand
3. curve (a parabola); surface (a cylinder)
5. a hyperboloid of two sheets
7. \( \| \mathbf{AB} \| = \sqrt{6}; \| \mathbf{BC} \| = \sqrt{17}; \| \mathbf{AC} \| = \sqrt{11} \). Yes, it is a right triangle as \( \| \mathbf{AB} \|^2 + \| \mathbf{AC} \|^2 = \| \mathbf{BC} \|^2 \).
9. Center at \((4, -1, 0)\); radius = 3
11. Interior of a sphere with radius 1 centered at the origin.
13. The first octant of space along with its adjacent quarter planes; all points \((x, y, z)\) where each of \(x, y\) and \(z\) are positive or zero. (Analogous to the first quadrant in the plane.)

15.

17.

19. \( y^2 + z^2 = x^2 \)
21. \( z = (\sqrt{x^2 + y^2})^2 = x^2 + y^2 \)
23. (a) \( x = y^2 + \frac{z^2}{9} \)
25. (b) \( x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \)

Exercises 11.2

1. Answers will vary.
3. A vector with magnitude 1.
5. It stretches the vector by a factor of 2, and points it in the opposite direction.
7. \( \mathbf{PQ} = \langle 4, -4 \rangle = 4\mathbf{i} - 4\mathbf{j} \)
9. \( \mathbf{PQ} = \langle 2, 2, 0 \rangle = 2\mathbf{i} + 2\mathbf{j} \)
11. (a) \( \mathbf{u} + \mathbf{v} = \langle 3, 2, 1 \rangle; \mathbf{u} - \mathbf{v} = \langle -1, 0, -3 \rangle; \pi \mathbf{u} - \sqrt{2} \mathbf{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2} \rangle \).
(c) \( \mathbf{x} = \langle -1, 0, -3 \rangle \).
Exercises 11.3

1. Scalar
3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.

5. $-22$
7. 3
9. not defined
11. Answers will vary.
13. $\theta = 0.3218 \approx 18.43^\circ$
15. $\theta = \pi/4 = 45^\circ$
17. Answers will vary; two possible answers are $\langle -7, 4 \rangle$ and $\langle 14, -8 \rangle$.
19. Answers will vary; two possible answers are $\langle 1, 0, -1 \rangle$ and $\langle 4, 5, -9 \rangle$.
21. $\text{proj}_v \vec{u} = \langle -1/2, 3/2 \rangle$.

23. $\text{proj}_x \vec{u} = \langle -1/2, -1/2 \rangle$.
25. $\text{proj}_z \vec{u} = \langle 1, 2, 3 \rangle$.
27. $\vec{u} = \langle -1/2, 3/2 \rangle + \langle 3/2, 1/2 \rangle$.
29. $\vec{u} = \langle -1/2, -1/2 \rangle + \langle -5/2, 5/2 \rangle$.
31. $\vec{u} = \langle 1, 2, 3 \rangle + \langle 0, 3, -2 \rangle$.
33. 1.96lb
35. 141.42ft–lb
37. 500ft–lb
39. 500ft–lb

Exercises 11.4

1. vector
3. “Perpendicular” is one answer.
5. Torque
7. $\vec{u} \times \vec{v} = \langle 12, -15, 3 \rangle$
9. $\vec{u} \times \vec{v} = \langle -5, -31, 27 \rangle$
11. $\vec{u} \times \vec{v} = \langle 0, -2, 0 \rangle$
13. $\vec{t} \times \vec{j} = \vec{k}$
15. $\vec{j} \times \vec{k} = \vec{i}$
17. Answers will vary.
19. 21
21. 5

23. $\sqrt{230}$
25. 6
27. $3\sqrt{30}$
29. $5/2$
31. $8\sqrt{7/2}$

33. 15
35. $\pm \frac{1}{\sqrt{2}} \langle -2, 1, 4 \rangle$
37. any unit vector orthogonal to $\vec{u}$ works (such as $\frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$).
39. $43.75\sqrt{3} \approx 75.78$ft–lb
41. 11.58ft–lb
43. With $\vec{u} = \langle u_1, u_2, u_3 \rangle$, we have

$$\vec{u} \times \vec{u} = \langle u_2u_3 - u_3u_2, -u_1u_3 + u_3u_1, u_1u_2 - u_2u_1 \rangle = \langle 0, 0, 0 \rangle = \vec{0}.$$
Exercises 11.5
1. A point on the line and the direction of the line.
3. parallel, skew
5. vector: \( \ell(t) = (2, -4, 1) + t (9, 2, 5) \)
   parametric: \( x = 2 + 9t, \ y = -4 + 2t, \ z = 1 + 5t \)
   symmetric: \( (x - 2)/9 = (y + 4)/2 = (z - 1)/5 \)
7. Answers may vary; vector: \( \ell(t) = (2, 1, 5) + t (-5, -3, -1) \)
   parametric: \( x = 2 + 5t, \ y = 1 - 3t, \ z = 5 - t \)
   symmetric: \( (x - 2)/5 = -(y - 1)/3 = -(z - 5) \)
9. Answers can vary; here the direction is given by \( \vec{d}_1 \times \vec{d}_2 \): vector: \( \ell(t) = (0, 1, 2) + t (-10, 43, 9) \)
   parametric: \( x = -10t, \ y = 1 + 43t, \ z = 2 + 9t \)
   symmetric: \( x/10 = (y - 1)/43 = (z - 2)/9 \)
11. Answers can vary; here the direction is given by \( \vec{d}_1 \times \vec{d}_2 \): vector: \( \ell(t) = (7, 2, -1) + t (1, 1, 2) \)
   parametric: \( x = 7 + t, \ y = 2 - t, \ z = 2 + 2t \)
   symmetric: \( x - 2 = y = (z - 1)/2 \)
13. vector: \( \ell(t) = (1, 1) + t (2, 2) \)
   parametric: \( x = 1 + 2t, \ y = 1 + 3t \)
   symmetric: \( (x - 1)/2 = (y - 1)/3 \)
15. parallel
17. intersecting; \( \vec{e}_1(3) = \vec{e}_2(4) = (9, -5, 13) \)
19. skew
21. same
3. \( \sqrt{41}/3 \)
25. \( \sqrt{2}/3 \)
27. \( 3/\sqrt{2} \)
29. Since both \( P \) and \( Q \) are on the line, \( \vec{PQ} \) is parallel to \( \vec{d} \). Thus \( \vec{PQ} \times \vec{d} = \vec{0} \), giving a distance of 0.
31. (a) The distance formula cannot be used because since \( d_1 \) and \( d_2 \) are parallel, \( \vec{c} \) is \( \vec{0} \) and we cannot divide by \( \| \vec{0} \| \).
   (b) Since \( \vec{d}_1 \) and \( \vec{d}_2 \) are parallel, \( \vec{P_1P_2} \) lies in the plane formed by the two lines. Thus \( \vec{P_1P_2} \times \vec{d}_2 \) is orthogonal to this plane, and \( \vec{c} = (\vec{P_1P_2} \times \vec{d}_2) / \| \vec{d}_2 \| \) is parallel to the plane, but still orthogonal to both \( d_1 \) and \( d_2 \). We desire the length of the projection of \( \vec{P_1P_2} \) onto \( \vec{c} \), which is what the formula provides.
   (c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let \( P = P_1 \) and \( Q = P_2 \) as given by the equations of the lines, and apply the formula for distance between a point and a line.

Exercises 11.6
1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).
3. Answers will vary.
5. Answers will vary.
7. Standard form: \( 3(x - 2) - (y - 3) + 7(z - 4) = 0 \)
   general form: \( 3x - y + 7z = 31 \)
9. Answers may vary; Standard form: \( 8(x - 1) + 4(y - 2) - 4(z - 3) = 0 \)
   general form: \( 8x + 4y - 4z = 4 \)
11. Answers may vary; Standard form: \( -7(x - 2) + 2(y - 1) + (z - 2) = 0 \)
   general form: \( -7x + 2y + z = -10 \)
13. Answers may vary; Standard form: \( 2(x - 1) - (y - 1) = 0 \)
   general form: \( 2x - y = 1 \)
15. Answers may vary; Standard form: \( 2(x - 2) - (y + 6) - 4(z - 1) = 0 \)
   general form: \( 2x - y - 4z = 6 \)
17. Answers may vary; Standard form: \( (x - 5) + (y - 7) + (z - 3) = 0 \)
   general form: \( x + y + z = 15 \)
19. Answers may vary; Standard form: \( 3(x + 4) + 8(y - 7) - 10(z - 2) = 0 \)
   general form: \( 3x + 8y - 10z = 24 \)
21. Answers may vary; \( \ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases} \)
23. \( (-3, -7, -5) \)
25. No point of intersection; the plane and line are parallel.
27. \( \sqrt{5/7} \)
29. \( 1/\sqrt{3} \)
31. If \( P \) is any point in the plane, and \( Q \) is also in the plane, then \( \vec{PQ} \) lies parallel to the plane and is orthogonal to \( \vec{n} \), the normal vector. Thus \( \vec{n} \cdot \vec{PQ} = 0 \), giving the distance as 0.

Exercises 11.7
1. (a) \((4, \frac{9}{3}, -1)\); (b) \((\sqrt{17}, \frac{\sqrt{2}}{1}, 1.816)\)
3. (a) \((2\sqrt{7}, \frac{13}{2}, 0)\); (b) \((2\sqrt{7}, \frac{13}{2}, \frac{13}{2})\)
5. (a) \(r^2 + z^2 = 25\); (b) \(r = 5\)
7. (a) \(r^2 + 9z^2 = 36\); (b) \(r^2(1 + 8\cos^2 \phi) = 36\)
9. 11. 13.

Chapter 12
Exercises 12.1
1. parametric equations
3. displacement
Both derivatives return \( \vec{r}' = \langle 3 \sin t + 3, 2 \cos t - 2 \rangle \).

23. Answers may vary, though most direct solutions are
   \( \vec{r}(t) = \langle -3 \cos t + 3, 2 \sin t - 2 \rangle \),
   \( \vec{r}(t) = \langle 3 \cos t + 3, -2 \sin t - 2 \rangle \) and
   \( \vec{r}(t) = \langle 3 \sin t + 3, 2 \cos t - 2 \rangle \).

25. Answers may vary, though most direct solutions are
   \( \vec{r}(t) = \langle t, -1/2(t - 1) + 5 \rangle \),
   \( \vec{r}(t) = \langle t + 1, -1/2t + 5 \rangle \),
   \( \vec{r}(t) = \langle -2t + 1, t + 5 \rangle \) and
   \( \vec{r}(t) = \langle 2t + 1, -t + 5 \rangle \).

27. Answers may vary, though most direct solution is
   \( \vec{r}(t) = \langle 3 \cos(4\pi t), 3 \sin(4\pi t), 3t \rangle \).

29. (1, 1)
31. (1, 2, 7)

Exercises 12.2
1. component
3. It is difficult to identify the points on the graphs of \( \vec{r}(t) \) and \( \vec{r}'(t) \) that correspond to each other.
5. \( \langle e^t, 0 \rangle \)
7. \( \langle 2t, 1, 0 \rangle \)
9. \( \langle 0, \infty \rangle \)
11. \( \vec{r}'(t) = \langle -1/t^2, 5/(3t + 1)^2, \sec^2 t \rangle \)
13. \( \vec{r}'(t) = \langle 2t, 1 \rangle \cdot \langle \sin t, 2t + 5 \rangle + \langle t^2 + 1, t - 1 \rangle \cdot \langle \cos t, 2 \rangle =
   \langle (t^2 + 1) \cos t + 2t \sin t + 4t + 3 \rangle \)
15. \( \vec{r}'(t) = \langle 2t + 1, 2t - 1 \rangle \)
17. \( \vec{r}'(t) = \langle 2t, 3t^2 - 1 \rangle \)
19. \( \ell(t) = \langle 2, 0 \rangle + t \langle 3, 1 \rangle \)
21. \( \ell(t) = \langle -3, 0 \rangle + t \langle 0, -3, 1 \rangle \)
23. \( t = 2n \pi \), where \( n \) is an integer;
so \( t = \ldots -4 \pi, -2 \pi, 0, 2 \pi, 4 \pi, \ldots \)
25. \( \vec{r}(t) \) is not smooth at \( t = 3 \pi/4 + n \pi \), where \( n \) is an integer
27. Both derivatives return \( \langle 5t^4, 4t^3 - 3t^2, 3t^3 \rangle \).
29. Both derivatives return
   \( \langle 2t - e^t - 1, \cos t - 3t^2, (t^2 + 2t)e^t - (t - 1) \cos t - \sin t \rangle \).
31. \( \langle \tan^{-1} t, \tan t \rangle + \vec{C} \)
33. \( \langle 4, -4 \rangle \)
35. \( \vec{r}(t) = (\ln |t+1| + 1, -\ln |\cos t| + 2) \)
37. \( \vec{r}(t) = \langle -\cos t + 1, t - \sin t, e^t - t - 1 \rangle \)
39. \( 10\pi^2 \)
41. \( \sqrt{2}(1 - e^{-1}) \)

**Exercises 12.3**

1. Velocity is a vector, indicating an object's direction of travel and its rate of distance change (i.e., its speed). Speed is a scalar.
2. The average velocity is found by dividing the displacement by the time traveled — it is a vector. The average speed is found by dividing the distance traveled by the time traveled — it is a scalar.
3. One example is traveling at a constant speed \( s \) in a circle, ending at the starting position. Since the displacement is \( \vec{0} \), the average velocity is \( \vec{0} \), hence \( \|\vec{0}\| = 0 \). But traveling at constant speed \( s \) means the average speed is also \( s > 0 \).
4. \( \vec{v}(t) = (2, 5, 0), \vec{a}(t) = (0, 0, 0) \)
5. \( \vec{v}(t) = (-\sin t, \cos t), \vec{a}(t) = (-\cos t, -\sin t) \)
6. \( \vec{v}(t) = (1, \cos t), \vec{a}(t) = (0, -\sin t) \)

![Graph of \( \vec{v}(\pi/4) \) vs \( \pi/4 \) on the x-axis and \( \langle 1, 0 \rangle \) on the y-axis.]

7. \( \vec{v}(t) = (2t + 1, -2t + 2), \vec{a}(t) = (2, -2) \)

![Graph of \( \vec{v}(1) \) vs \( 6 \) on the x-axis and \( \vec{a}(1) \) on the y-axis.]

8. \( ||\vec{v}(t)|| = \sqrt{4t^2 + 1}. \)

Min at \( t = 0 \); Max at \( t = \pm 1 \).

9. \( ||\vec{v}(t)|| = \frac{4t}{\sqrt{4t^2 - 1}} \) \( + \frac{2t - 1}{\sqrt{4t^2 - 1}} \).

Min: \( t = 0 \); Max: \( t = \pi/4 \).

10. \( ||\vec{v}(t)|| = 13. \)

Speed is constant, so there is no difference between min/max

11. \( ||\vec{v}(t)|| = \sqrt{4t^2 + 1 + t^2/(1-t^2)} \).

Min: \( t = 0 \); Max: \( t \to \pm 1 \);

12. \( ||\vec{v}(t)|| = 2/\sqrt{4 + \cos^2 t - \sin^2 t} \).

13. \( ||\vec{v}(t)|| = 2/\sqrt{\cos^2 t - \sin^2 t} \).

14. \( \vec{a}(t) = (0, 0, 0) \) when \( t = 7.9; \vec{r}(7.9) = (1739.25, 0) \), meaning the box will travel about 1740ft horizontally before it lands.

**Exercises 12.4**

1. \( 1 \)
2. \( \vec{T}(t) \) and \( \vec{N}(t) \).
3. \( \vec{T}(t) = \langle \frac{4t}{\sqrt{20t^2 - 4t + 1}}, \sqrt{20t^2 - 4t + 1} \rangle \);

\( \vec{T}(1) = \langle 2, 1/\sqrt{2} \rangle \).

4. \( \vec{t}(t) = \langle \cos t, \sin t \rangle \).

Be careful; this cannot be simplified as just \( \langle -\cos t, \sin t \rangle \) as \( \sqrt{\cos^2 t \sin^2 t} \neq \cos t \sin t \).

5. \( \vec{t}(t) = \langle -\cos t, \sin t \rangle \).

6. \( \vec{\ell}(t) = \langle 2, 0 \rangle + t \langle 4/\sqrt{17}, 1/\sqrt{17} \rangle \); in parametric form,

\( \vec{\ell}(t) = \langle x = 2 + 4t/\sqrt{17}, y = t/\sqrt{17} \rangle \).

7. \( \vec{\ell}(t) = \langle \sqrt{2}/4, \sqrt{2}/4 \rangle + t \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle \); in parametric form,

\( \vec{\ell}(t) = \langle x = \sqrt{2}/4 - \sqrt{2}t, y = \sqrt{2}/4 + \sqrt{2}t \rangle \).

8. \( \vec{T}(t) = \langle -\cos t, \sin t \rangle \); \( \vec{N}(t) = \langle -\cos t, -\sin t \rangle \).

9. \( \vec{T}(t) = \langle \sin t, \sqrt{4 + \cos^2 t - \sin^2 t} \rangle \).

10. \( \vec{N}(t) = \langle -\frac{2 \cos t}{\sqrt{4 + \cos^2 t - \sin^2 t}}, \frac{\sin t}{\sqrt{4 + \cos^2 t - \sin^2 t}} \rangle \);
Exercises 12.5

1. time and/or distance
2. Answers may include lines, circles, helixes
3. \( \kappa \)
4. \( \quad s = 3t, \text{so } \vec{r}(s) = (2s/3, s/3, -2s/3) \)
5. \( s = \sqrt{3}t, \text{so } \vec{r}(s) = (3 \cos(s/\sqrt{3}), 3 \sin(s/\sqrt{3}), 2s/\sqrt{3}) \)
6. \( \kappa = \frac{|s|}{(s^2 + (\sin^2 s)^{3/2}), \text{or } \kappa(0) = 0, \kappa(1/2) = 192/17 \sqrt{17} \approx 2.74} \)
7. \( \kappa = \frac{|x \cos(\sin x)|}{(1+\sin^2 x)^{1/2}}, \text{or } \kappa(0) = 1, \kappa(\pi/2) = 0 \)
8. \( \kappa = \frac{|2 \cos t \cos 2t + 4 \sin t \sin 2t|}{(4 \cos^2 t + 4 \sin t)^{3/2}}, \text{or } \kappa(0) = 1/4, \kappa(\pi/4) = 8 \)
9. \( \kappa = \frac{|s^2 + |z|}{(s^2 + (3s^2 - 1)^2)^{3/2}}, \text{or } \kappa(0) = 2, \kappa(5) = 19/1394 \sqrt{1394} \approx 0.0004 \)
10. \( \kappa = 0; \text{or } \kappa(0) = 0, \kappa(1) = 0 \)

Exercises 13.1

1. Answers will vary.
2. topographical
3. surface
4. domain: \( \mathbb{R}^2 \)
5. range: \( z \geq 2 \)
6. domain: \( \mathbb{R}^2 \)
7. range: \( 0 < z \leq 1 \)
8. domain: \( \{(x, y) \mid x^2 + y^2 \leq 9 \}, \text{i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.} \)
9. range: \( 0 \leq z \leq 3 \)

10. Level curves are lines \( y = (3/2)x - c/2. \)
11. \( \kappa = \frac{1}{r}; \kappa(0) = 3/13, \kappa(\pi/2) = 3/13 \)
12. maximized at \( x = \pm \frac{\sqrt{17}}{5} \)
13. maximized at \( t = 1/4 \)
14. radius of curvature is \( 5 \sqrt{5}/4. \)
15. radius of curvature is \( 9. \)
16. \( x^2 + (y - 1/2)^2 = 1/4, \text{or } \vec{c}(t) = \left( \frac{1}{2} \cos t, \frac{1}{2} \sin t + 1/2 \right) \)
17. \( x^2 + (y + 8)^2 = 81, \text{or } \vec{c}(t) = (9 \cos t, 9 \sin t - 8) \)
18. Let \( \vec{r}(t) = (\chi(t), y(t), 0) \) and apply the second formula of part 3.

Chapter 13

Exercises 13.1

1. Answers will vary.
2. topographical
3. surface
4. domain: \( \mathbb{R}^2 \)
5. range: \( z \geq 2 \)
6. domain: \( \mathbb{R}^2 \)
7. range: \( 0 < z \leq 1 \)
8. domain: \( \{(x, y) \mid x^2 + y^2 \leq 9 \}, \text{i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.} \)
9. range: \( 0 \leq z \leq 3 \)
10. Level curves are lines \( y = (3/2)x - c/2. \)

Exercises 13.5

1. Be sure to show work
2. \( \vec{N}(\pi/4) = \langle -5/\sqrt{34}, -3/\sqrt{34} \rangle \)
3. Be sure to show work
4. \( \vec{N}(0) = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \)
5. \( \vec{r}(t) = \frac{1}{\sqrt{3}} (2, \cos t, -\sin t); \vec{N}(t) = (0, -\sin t, -\cos t) \)
6. \( \vec{r}(t) = \frac{1}{\sqrt{3}} (-a \sin t, a \cos t, b); \vec{N}(t) = (-\cos t, -\sin t, 0) \)
7. \( a_T = \frac{2a}{\sqrt{1+a^2}} \text{ and } a_N = \sqrt{4 \frac{a}{1+a^2}}. \)
8. At \( t = 0, a_T = 0 \text{ and } a_N = 2; \)
9. At \( t = \pi/2, a_T = 0 \text{ and } a_N = 2. \)
10. The object moves at constant speed, so all acceleration comes from changing direction, hence \( a_T = 0. \) \( a(t) \) is always parallel to \( \vec{N}(t), \) but twice as long, hence \( a_N = 2. \)
11. \( a_T = 0 \text{ and } a_N = a \)
12. At \( t = 0, a_T = 0 \text{ and } a_N = a; \)
13. At \( t = \pi/2, a_T = 0 \text{ and } a_N = a. \)
14. The object moves at constant speed, meaning that \( a_T \) is always 0. The object “rises” along the z-axis at a constant rate, so all acceleration comes in the form of changing direction circling the z-axis. The greater the radius of this circle the greater the acceleration, hence \( a_N = a. \)
19. Level curves are circles, centered at \((1/c, -1/c)\) with radius \(\sqrt{2/c^2 - 1}\). When \(c = 0\), the level curve is the line \(y = x\).

21. Level curves are ellipses of the form \(\frac{x^2}{c} + \frac{y^2}{c^2/2} = 1\), i.e., \(a = c\) and \(b = c/2\).

23. Domain: \(x + 2y - 4z \neq 0\); the set of points in \(\mathbb{R}^3\) NOT in the domain form a plane through the origin. 
   Range: \(\mathbb{R}\)

25. Domain: \(z \geq x^2 - y^2\); the set of points in \(\mathbb{R}^3\) above (and including) the hyperbolic paraboloid \(z = x^2 - y^2\). 
   Range: \([0, \infty)\)

27. The level surfaces are spheres, centered at the origin, with radius \(\sqrt{c}\).

29. The level surfaces are paraboloids of the form \(z = \frac{x^2}{c} + \frac{y^2}{c^2/2}\); the larger \(c\), the "wider" the paraboloid.

31. The level curves for each surface are similar; for \(z = \sqrt{x^2 + 4y^2}\) the level curves are ellipses of the form \(\frac{x^2}{c^2} + \frac{y^2}{c^2/2} = 1\), i.e., \(a = c\) and \(b = c/2\); whereas for \(z = x^2 + 4y^2\) the level curves are ellipses of the form \(\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1\), i.e., \(a = \sqrt{c}\) and \(b = \sqrt{c}/2\). The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as \(c\) grows, meaning the function grows faster and faster as \(c\) increases. 
   The function \(z = \sqrt{x^2 + 4y^2}\) can be rewritten as \(z^2 = x^2 + 4y^2\), an elliptic cone; the function \(z = x^2 + 4y^2\) is a paraboloid, each matching the description above.

Exercises 13.2

1. Answers will vary.

3. Answers will vary. 
   One possible answer: \(\{(x, y)|x^2 + y^2 \leq 1\}\)

5. Answers will vary. 
   One possible answer: \(\{(x, y)|x^2 + y^2 < 1\}\)
Exercises 13.3

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3. $f_x$

5. $f_x = 2xy - 1, f_y = x^2 + 2$

12. $f_x = 2x + 6y, f_y = x^2 + 4$

18. $f_x = 2xe^{2xy}, f_y = 2ye^{2xy}$

25. $f_x = 1/6x, f_y = -x^2/3$

27. $f(x, y) = x \sin y + x + C$, where $C$ is any constant.

29. $f(x, y) = 3x^2 - 4y^2 + 2y + C$, where $C$ is any constant.

31. $f_x = 2xe^{2xy}, f_y = 2xe^{2xy}$

33. $f_x = 6x, f_y = 0$

Exercises 13.4

1. $T$

3. $T$

5. $dz = (\sin y + 2x)dx + (x \cos y)dy$

7. $dz = 5dx - 7dy$

9. $dz = \frac{4}{\sqrt{x^2 + y^2}} dx + \frac{6x}{\sqrt{x^2 + y^2}} dy$, with $dx = -0.05$ and $dy = 0.1$. At $(3, 7)$, $dz = 3/4(-0.05) + 1/8(1) = -0.025$, so $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$.

11. $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$, with $dx = 0.04$ and $dy = 0.06$. At $(2, 3)$, $dz = 3(0.04) + (\theta)(0.06) = -0.36$, so $f(2.04, 3.06) \approx -0.36 - 6 = -6.36$.

13. The total differential of volume is $dV = 4\pi dr + \pi dh$. The coefficient of $dr$ is greater than the coefficient of $dh$, so the volume is more sensitive to changes in the radius.

15. Using trigonometry, $\ell = x\tan \theta$, so $d\ell = \tan \theta dx + x\sec^2 \theta d\theta$. With $\theta = 85^\circ$ and $x = 30$, we have $d\ell = 11.43dx + 3949.38d\theta$. The measured length of the wall is much more sensitive to errors in $\theta$ than in $x$. While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like “comparing apples to oranges”), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.

17. $dw = 2xy^2 dx + x^2 z^2 dy + 3x^2 y^2 dz$

Exercises 13.5

1. Because the parametric equations describe a level curve, $z$ is constant for all $t$. Therefore $\frac{dz}{dt} = 0$.

3. $\frac{dx}{dt}$ and $\frac{dy}{dt}$

5. $F$

7. $\frac{dz}{dt} = 3(2t) + 4(2) = 6t + 8$.

9. $\frac{dz}{dt} = 5(-2 \sin t) + 2(\cos t) = -10 \sin t + 2 \cos t$

11. $\frac{dz}{dt} = 2x(\cos t) + 4y(3 \cos t)$.

13. $t = -4/3$; this corresponds to a minimum.

15. $t = \tan^{-1}(1/5) + n\pi$, where $n$ is an integer.
17. We find that 
\[ \frac{dz}{dt} = 38 \cos t \sin t. \]
Thus \( \frac{dz}{dt} = 0 \) when \( t = \pi n \) or \( \pi n + \pi/2 \), where \( n \) is any integer.

19. \( \frac{dz}{dt} = 2xy(1) + x^2(2) = 2xy + 2x^2; \]
\( \frac{dy}{dt} = 2x(y) - x^2(4) = -2xy + 4x^2 \)
(a) With \( s = 1, t = 0, x = 1 \) and \( y = 2 \). Thus \( \frac{dz}{dt} = 6 \) and \( \frac{dy}{dt} = 0 \)

21. \( \frac{dz}{dt} = 2x(\cos t) + 2y(\sin t) = 2x \cos t + 2y \sin t; \]
\( \frac{dy}{dt} = 2x(-s \sin t) + 2y(s \cos t) = -2xs \sin t + 2ys \cos t \)
(b) With \( s = 2, t = \pi/4, x = \sqrt{2} \) and \( y = \sqrt{2} \). Thus \( \frac{dz}{dt} = 4 \) and \( \frac{dy}{dt} = 0 \)

23. \( f_x = 2x \tan y, f_y = x^2 \sec^2 y; \]
\( \frac{dy}{dx} = \frac{2 \tan y}{x \sec^2 y} \)

25. \( f_x = \frac{(x + y^2)(2x) - (x^2 + y)(1)}{(x + y^2)^2} \)
\( f_y = \frac{(x + y^2)(1) - (x^2 + y)(2y)}{(x + y^2)^2} \)
\( \frac{dy}{dx} = \frac{-2x(x + y^2) - (x^2 + y)}{x + y^2 - 2y} \)

Exercises 13.6
1. A partial derivative is essentially a special case of a directional derivative; it is the directional derivative in the direction of \( x \) or \( y \), i.e., \( \langle 1, 0 \rangle \) or \( \langle 0, 1 \rangle \).
3. \( \vec{u} = (0, 1) \)
5. maximal, or greatest
7. \( \nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle \)
9. \( \nabla f = \left(\frac{-2x}{(x^2 + y^2)^2}, \frac{-2y}{(x^2 + y^2)^2}\right) \)
11. \( \nabla f = \langle 2x - y - 7, 4y - x \rangle \)
13. \( \nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle; \quad \nabla f(2, 1) = \langle -2, 2 \rangle \) Be sure to change all directions to unit vectors.
(a) \( 2/5 \vec{u} = (3/5, 4/5) \)
(b) \( -2/\sqrt{5} \vec{u} = (-1/\sqrt{5}, -2/\sqrt{5}) \)

15. \( \nabla f = \left(\frac{-2x}{(x^2 + y^2 + 1)^2}, \frac{-2y}{(x^2 + y^2 + 1)^2}\right); \quad \nabla f(1, 1) = \langle -2/9, -2/9 \rangle \) Be sure to change all directions to unit vectors.
(a) \( 0 \vec{u} = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle \)
(b) \( 2\sqrt{2}/9 \vec{u} = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle \)

17. \( \nabla f = \langle 2x - y - 7, 4y - x \rangle; \quad \nabla f(4, 1) = \langle 0, 0 \rangle \)
(a) \( 0 \)
(b) \( 0 \)

19. \( \nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle \)
(a) \( \nabla f(2, 1) = \langle -2, 2 \rangle \)
(b) \( ||\nabla f(2, 1)|| = \sqrt{2} \)
(c) \( \langle 2, -2 \rangle \)
(d) \( \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \)

21. \( \nabla f = \left\langle \frac{-2x}{(x^2 + y^2 + 1)^2}, \frac{-2y}{(x^2 + y^2 + 1)^2}\right\rangle \)
(a) \( \nabla f(1, 1) = \langle -2/9, -2/9 \rangle \)
(b) \( ||\nabla f(1, 1)|| = \sqrt{2}/9 \)
(c) \( \langle 2/9, 2/9 \rangle \)
(d) \( \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle \)

23. \( \nabla f = \langle 2x - y - 7, 4y - x \rangle \)
(a) \( \nabla f(4, 1) = \langle 0, 0 \rangle \)
(b) \( 0 \)
(c) \( 0 \)
(d) All directions give a directional derivative of \( 0 \).

25. \( \nabla F(x, y, z) = \langle 6xz^3 + 4y, 4x, 9x^2z^2 - 6z \rangle \)
\( 113/\sqrt{3} \)

27. \( \nabla F(x, y, z) = \langle 2xy^2, 2y(x^2 - z^2), -2y^2z \rangle \)
(b) \( 0 \)

Exercises 13.7
1. Answers will vary. The displacement of the vector is one unit in the \( x \)-direction and 3 units in the \( z \)-direction, with no change in \( y \). Thus along a line parallel to \( \vec{v} \), the change in \( z \) is 3 times the change in \( x \) i.e., a "slope" of 3. Specifically, the line in the \( x-z \) plane parallel to \( z \) has a slope of \( 3 \).
3. \( \vec{T} \)

5. \( \ell_x(t) = \left\{ \begin{array}{l} x = 2 + t \\
y = 3 \\
z = -48 - 12t \end{array} \right\} \)
(b) \( \ell_y(t) = \left\{ \begin{array}{l} x = 2 \\
y = 3 + t \\
z = -48 - 40t \end{array} \right\} \)
(c) \( \ell_z(t) = \left\{ \begin{array}{l} x = 2 + t/\sqrt{10} \\
y = 3 + 3t/\sqrt{10} \\
z = -48 - 66/\sqrt{250} t \end{array} \right\} \)
7.\[\ell_x(t) = \begin{cases} x = 4 + t \\ y = 2 \\ z = 2 + 3t \end{cases}\]
\[\ell_y(t) = \begin{cases} x = 4 \\ y = 2 + t \\ z = 2 - 5t \end{cases}\]
\[\ell_z(t) = \begin{cases} x = 4 + t/\sqrt{2} \\ y = 2 + t/\sqrt{2} \\ z = 2 - \sqrt{2}t \end{cases}\]

9. \[\ell_x(t) = \begin{cases} x = 2 - 12t \\ y = 3 - 40t \\ z = -48 - t \end{cases}\]

11. \[\ell_x(t) = \begin{cases} x = 4 + 3t \\ y = 2 - 5t \\ z = 2 - t \end{cases}\]

13. \[(1.425, 1.085, -48.078), (2.575, 4.915, -47.952)\]

15. \[(5.014, 0.31, 1.662) \text{ and } (2.986, 3.690, 2.338)\]

17. \[-12(x - 2) - 40(y - 3) - (z + 48) = 0\]

19. \[3(x - 4) - 5(y - 2) - (z - 2) = 0\] (Note that this tangent plane is the same as the original function, a plane.)

21. \[\nabla F = \langle x/4, y/2, z/8 \rangle; \text{ at } P, \nabla F = \langle 1/4, \sqrt{2}/2, \sqrt{6}/8 \rangle\]

(a) \[\ell_x(t) = \begin{cases} x = 1 + t/4 \\ y = \sqrt{2} + \sqrt{2}t/2 \\ z = \sqrt{6} + \sqrt{6}t/8 \end{cases}\]

(b) \[\frac{1}{4}(x - 1) + \frac{\sqrt{2}}{2}(y - \sqrt{2}) + \frac{\sqrt{6}}{2}(z - \sqrt{6}) = 0.\]

23. \[\nabla F = \langle y^2 - x^2, 2xy, -2x^2 \rangle; \text{ at } P, \nabla F = \langle 0, 4, 4 \rangle\]

(a) \[\ell_y(t) = \begin{cases} x = 2 \\ y = 1 + 4t \\ z = -1 + 4t \end{cases}\]

(b) \[4(y - 1) + 4(z + 1) = 0.\]

Exercises 13.8

1. F; it is the “other way around.”

3. T

5. One critical point at \((-4, 2); f_x = 1 \text{ and } D = 4, \text{ so this point corresponds to a relative minimum.}\)

7. One critical point at \((6, -3); D = -4, \text{ so this point corresponds to a saddle point.}\)

9. Two critical points: at \((0, -1); f_x = 2 \text{ and } D = -12, \text{ so this point corresponds to a saddle point; at } (0, 1), f_x = 2 \text{ and } D = 12, \text{ so this corresponds to a relative minimum.}\)

11. Critical points when \(x \text{ or } y \) are 0. \(D = -12x^2y^2, \) so the test is inconclusive. (Some elementary thought shows that these are absolute minima.)

13. One critical point: \(f_z = 0 \text{ when } x = 3; f_y = 0 \text{ when } y = 0, \text{ so one critical point at } (3, 0), \text{ which is a relative maximum, where } f_{xx} = \frac{y^4 - 16}{(16 - (x-3)^2 - y^2)^{3/2}} \text{ and } D = \frac{16}{(16 - (x-3)^2 - y^2)^{3/2}}.\) Both \(f_x \text{ and } f_y \) are undefined along the circle \((x-3)^2 + y^2 = 16\); at any point along this curve, \(f(x, y) = 0, \text{ the absolute minimum of the function.}\)

15. The triangle is bound by the lines \(y = -1, y = 2x + 1 \text{ and } y = -2x + 1.\) Along \(y = -1, \) there is a critical point at \((0, -1).\)

Along \(y = 2x + 1, \) there is a critical point at \((-3/5, -1/5).\)

Along \(y = -2x + 1, \) there is a critical point at \((3/5, -1/5).\)

The function \(f \) has one critical point, irrespective of the constraint, at \((0, -1/2).\)

Checking the value of \(f \) at these four points, along with the three vertices of the triangle, we find the absolute maximum is at \((0, 1, 3) \) and the absolute minimum is at \((0, -1/2, 3/4).\)

17. The region has no “corners” or “vertices,” just a smooth edge.

To find critical points along the circle \(x^2 + y^2 = 4, \) we solve for \(y': y' = 4 - x^2. \) We can go further and state \(y = \pm \sqrt{4 - x^2}. \)

We can rewrite \(f \) as \(f(x) = x^2 + 2x + (4 - x^2) + \sqrt{4 - x^2} = 2x + 4 + \sqrt{4 - x^2}. \) (We will return and use \(-\sqrt{4 - x^2} \) later.) Solving \(f'(x) = 0, \) we get \(x = \sqrt{2} \Rightarrow y = \sqrt{2}. \)

The function \(f \) itself has a critical point at \((-1, -1).\)

Checking the value of \(f \) at \((-1, 1), (-\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}), (2, 0) \) and \((-2, 0), \) we find the absolute maximum is at \((\sqrt{2}, \sqrt{2}, 4 + 4\sqrt{2}) \) and the absolute minimum is at \((-1, -1, -2).\)

Exercises 13.9

1. \(\pm 2\sqrt{5} \text{ at } (\pm 4/\sqrt{5}, \pm 2/\sqrt{5})\)

3. \((\pm 20/\sqrt{13}, \pm 30/\sqrt{13})\)

5. \(8abc/3\sqrt{3}\)

7. Length 130/3, height and width 65/3.

9. \((0, \pm 1, 0)\)

11. \((2, 1, 2)\)

13. Max: \(5 \text{ at } (2, 2), \text{ min: } -9/2 \text{ at } (3/\sqrt{2}, -3/\sqrt{2}).\)

15. \((2/\sqrt{3})^3 = 8/3\sqrt{3}\)

Chapter 14

Exercises 14.1

1. \(C(y), \text{ meaning that instead of being just a constant, like the number } 5, \text{ it is a function of } y, \text{ which acts like a constant when taking derivatives with respect to } x.\)

3. curve to curve, then from point to point

5. \[a \text{ 18}x^2 + 42x - 117\]

(b) \(-108\)

7. \[a x^2/2 - x^2 + 2x - 3/2\]

(b) \(23/15\)
9. (a) \( \sin^2 y \)
(b) \( \pi/2 \)

11. \( \int_{-1}^{1} \int_{-1}^{1} dy \, dx \) and \( \int_{-1}^{1} \int_{-1}^{1} dx \, dy \).
area of \( R \) = 9 units²

13. \( \int_{0}^{1} \int_{0}^{1-x} dy \, dx \). The order \( dx \, dy \) needs two iterated integrals as \( x \) is bounded above by two different functions. This gives:
\[ \int_{1}^{3} \int_{2}^{y+1} dx \, dy + \int_{3}^{7} \int_{2}^{1-y} dx \, dy. \]
area of \( R \) = 4 units²

15. \( \int_{0}^{1} \int_{0}^{x^2} dy \, dx \) and \( \int_{0}^{1} \int_{0}^{y^2} dx \, dy \)
area of \( R \) = 7/15 units²

17. \( y = 4 - x^2 \)
area of \( R \) = \( \int_{0}^{1} \int_{\sqrt{4-y}}^{1} dx \, dy \)

19. \( x^2/16 + y^2/4 = 1 \)
area of \( R \) = \( \int_{0}^{1} \int_{-\sqrt{3-x^2}/4}^{\sqrt{3-x^2}/4} dy \, dx \)

21. \( y = x^2 \)
area of \( R \) = \( \int_{-1}^{1} \int_{x^2}^{x+1} dy \, dx \)

Exercises 14.2

1. volume
A.12

17. \( y \)
   \[ \begin{array}{c}
   \text{(a)}
   \end{array} \]
   \[ \begin{array}{c}
   \text{R}
   \end{array} \]
   \[ \begin{array}{c}
   \text{x}
   \end{array} \]
   \[ \begin{array}{c}
   \text{y}
   \end{array} \]

\( (b) \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} (x^2 y - x) \, dy \, dx = \int_{0}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 y - x) \, dy \, dx \)

\( (c) 0 \)

19. Integrating \( e^x \) with respect to \( x \) is not possible in terms of elementary functions. \( \int_{0}^{2} e^x \, dx = e^x - 1 \).

21. Integrating \( \int_{1}^{\infty} \frac{2y}{x^2 + y^2} \, dx \) gives \( \tan^{-1}(1/y) - \pi/4 \); integrating \( \tan^{-1}(1/y) \) is hard.

23. average value of \( f = 6/2 = 3 \)

25. average value of \( f = \frac{112/3}{4} = 28/3 \)

Exercises 14.3

1. \( f(r \cos \theta, r \sin \theta), r \, dr \, d\theta \)

3. \( \int_{0}^{2\pi} \int_{0}^{\pi} (3r \cos \theta - r \sin \theta + 4) \, r \, dr \, d\theta = 4\pi \)

5. \( \int_{0}^{\pi} \int_{0}^{3 \cos \theta} (8 - r \sin \theta) \, r \, dr \, d\theta = 16\pi \)

7. \( \int_{0}^{2\pi} \int_{1}^{(\ln(r^2))} r \, dr \, d\theta = 2\pi (\ln 16 - 3/2) \)

9. \( \int_{-\pi/2}^{\pi/2} \int_{0}^{6} (r^2 \cos^2 \theta - r^2 \sin^2 \theta) \, r \, dr \, d\theta \)
   \( = \int_{-\pi/2}^{\pi/2} \int_{0}^{6} (r^2 \cos(2\theta)) \, r \, dr \, d\theta = 0 \)

11. \( \int_{-\pi/2}^{\pi/2} \int_{0}^{5} (r^2) \, dr \, d\theta = 125\pi/3 \)

13. \( \int_{0}^{\pi/4} \int_{0}^{2\pi} (r \cos \theta + r \sin \theta) \, r \, dr \, d\theta = 16\sqrt{2}/3 \)

Exercises 14.4

1. Because they are scalar multiples of each other.

3. “little masses”

5. \( M_y \) measures the moment about the \( x \)-axis, meaning we need to measure distance from the \( x \)-axis. Such measurements are measures in the \( y \)-direction.

7. \( \bar{x} = 5.25 \)

9. \((\bar{x}, \bar{y}) = (0.3)\)

11. \( M = 150g; \)

13. \( M = 2lb \)

15. \( M = 16\pi \approx 50.27kg \)

17. \( M = 54\pi \approx 169.65lb \)

19. \( M = 150g; M_y = 600; M_x = -75; (\bar{x}, \bar{y}) = (4, -0.5) \)

21. \( M = 2lb; M_y = 0; M_x = 2/3; (\bar{x}, \bar{y}) = (0, 1/3) \)

23. \( M = 16\pi \approx 50.27kg; M_y = 4\pi; M_x = 4\pi; (\bar{x}, \bar{y}) = (1/4, 1/4) \)

25. \( M = 54\pi \approx 169.65lb; M_y = 0; M_x = 504; (\bar{x}, \bar{y}) = (0, 2.97) \)

27. \( I_x = 64/3; I_y = 64/3; I_o = 128/3 \)

29. \( I_x = 16/3; I_y = 64/3; I_o = 80/3 \)

Exercises 14.5

1. arc length

3. surface areas

5. Intuitively, adding \( h \) to \( f \) only shifts \( f \) up (i.e., parallel to the \( z \)-axis) and does not change its shape. Therefore it will not change the surface area over \( R \).

Analytically, \( f_s = g_x \) and \( f_t = g_y \); therefore, the surface area of each is computed with identical double integrals.

7. \( S = \int_{0}^{2\pi} \int_{0}^{2} \sqrt{1 + \cos^2 \theta \cos^2 \psi + \sin^2 \theta \sin^2 \psi \, dy \, dy} \)

9. \( S = \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy \)

11. \( S = \int_{0}^{3} \int_{-1}^{1} \sqrt{1 + 9 + 49} \, dx \, dy = 6\sqrt{59} \approx 46.09 \)
13. This is easier in polar:

\[ S = \int_0^{2\pi} \int_0^R \sqrt{1 + 4r^2 \cos^2 t + 4r^2 \sin^2 t} \, dr \, dt \]
\[ = \int_0^{2\pi} \int_0^R \sqrt{1 + 4r^2} \, dr \, dt \]
\[ = \frac{\pi}{6} (65\sqrt{65} - 1) \approx 273.87 \]

15.

\[ S = \int_0^2 \int_0^{2\pi} \sqrt{1 + 4x^2} \, dy \, dx \]
\[ = \int_0^2 (2\sqrt{2} + 4x^2) \, dx \]
\[ = \frac{26}{3} \sqrt{2} \approx 12.26 \]

17. This is easier in polar:

\[ S = \int_0^{2\pi} \int_0^R \sqrt{1 + 4r^2 \cos^2 t + 4r^2 \sin^2 t} \, dr \, dt \]
\[ = \int_0^{2\pi} \int_0^R \sqrt{1 + 4r^2} \, dr \, dt \]
\[ = 25\pi \sqrt{5} \approx 175.62 \]

19. Integrating in polar is easiest considering \( R \):

\[ S = \int_0^{2\pi} \int_0^{\sqrt{1 + c^2 + d^2}} r \sqrt{1 + c^2 + d^2} \, dr \, d\theta \]
\[ = \int_0^{2\pi} \frac{1}{2} \left( \sqrt{1 + c^2 + d^2} \right) \, d\theta \]
\[ = \pi \sqrt{1 + c^2 + d^2}. \]

The value of \( h \) does not matter as it only shifts the plane vertically (i.e., parallel to the z-axis). Different values of \( h \) do not create different ellipses in the plane.

**Exercises 14.6**

1. surface to surface, curve to curve and point to point

3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.

5. \( V = \int_{-1}^1 \int_{-1}^{1-x^2} (8 - x^2 - y^2 - (2x + y)) \, dx \, dy = 88/3 \)

7. \( V = \int_0^\pi \int_0^\pi (\cos x \sin y + 2 - \sin x \cos y) \, dy \, dx = \pi^2 - \pi \approx 6.728 \)

9. \( dz \, dy \, dx: \int_0^{2\pi} \int_0^R \int_0^1 r \sqrt{1 + 4r^2} \, dr \, d\theta \, dz \)
\[ = \frac{26}{3} \pi \approx 12.26 \]

11. \( dz \, dy \, dx: \int_0^{2\pi} \int_0^R \int_0^{R/2} \sqrt{1 + 4r^2} \, dr \, d\theta \, dz \)
\[ = \frac{25}{3} \pi \sqrt{5} \approx 175.62 \]

13. \( dz \, dy \, dx: \int_0^{2\pi} \int_0^R \int_0^{1-\frac{1}{2}} r \sqrt{1 + 4r^2} \, dr \, d\theta \, dz \)
\[ = \frac{26}{3} \pi \approx 12.26 \]
15. \(dz\ dy\ dx: \int_0^1 \int_0^{1-x^2} \int_0^{1-y^2} dz\ dy\ dx\)
\(dz\ dx\ dy: \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dz\ dx\ dy\)
\(dy\ dz\ dx: \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dy\ dz\ dx\)
\(dy\ dx\ dz: \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dy\ dx\ dz\)
\(dx\ dy\ dz: \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} dx\ dy\ dz\)

Answers will vary. Neither order is particularly “hard.” The order \(dz\ dy\ dx\) requires integrating a square root, so powers can be messy; the order \(dy\ dz\ dx\) requires two triple integrals, but each uses only polynomials.

17. 8

19. \(\pi\)

21. \(M = 10, M_{x2} = 15/2, M_{y2} = 5/2, M_{y} = 5; \quad (x, y, z) = (3/4, 1/4, 1/2)\)

23. \(M = 16/5, M_{x2} = 16/3, M_{y2} = 104/45, M_{z2} = 32/9; \quad (x, y, z) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)\)

Exercises 14.7

1. \(8\pi\)

3. \(4\pi(8 - 3^{3/2})/3\)

5.

7. \(1 - \sin 2/2\)

9. \(2\pi ab\)

11.

Chapter 15

Exercises 15.1

1. \(1/2\)

3. \(23\)

5. \(24\pi\)

7. \(-2\pi\)

9. \(2\pi\)

11. 0

13.

15.

17.

Exercises 15.2

1. 0

3. No.

5. No.

7.

9. (b) No. \textit{Hint:} Think of how \(F\) is defined.

Exercises 15.3

1. \(16/15\)

3. \(-5\pi\)

5. Yes. \(F(x, y) = xy^2 + x^3\)

7. Yes. \(F(x, y) = 4x^2y + 2y^2 + 3x\)

9.

11.

Exercises 15.4

1. \(216\pi\)

3. \(12\pi/5\)

5.

7. \(15/4\)

9.

11.

Exercises 15.5

1. \(2\sqrt{2}\pi^2\)

3. \(2/5\)

5. \(2\pi(\pi - 1)\)

7. \(67/15\)

9. \(6\)

11. Yes

13. No

15.

17.

19. \textit{Hint:} Think of how a vector field \(f(x, y) = P(x, y)i + Q(x, y)j\)

in \(\mathbb{R}^2\) can be extended in a natural way to be a vector field

in \(\mathbb{R}^3\).

Exercises 15.6

1. 0

3. \(12\sqrt{x^2 + y^2 + z^2}\)

5. \(6(x + y + z)\)

7. \(12\rho\)

9. \(-2r^2\theta\sin\theta + r^{-2}\theta\sin\theta\)

11. \(\text{div}\vec{f} = 2\rho^{-1} - \sin\theta\cos\phi + \cot\phi + \cos\theta\cot\phi\); \(\text{curl}\vec{f} = \cos\theta\sin\phi\theta\sin\phi - 2\sin\theta\theta\sin\phi\)

13.

15.

17.

19.

21.

23.

25. \textit{Hint:} Start by showing that \(\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}, \hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}, \hat{e}_\phi = \hat{k}.

27.
Index

\(\nabla\), 981
\(\frac{\partial(x, y, z)}{\partial(u, v, w)}\), 924
\(\int_C\), 930, 933
\(\nabla\), 820
\(\nabla\), 961, 970, 979
\(\nabla^2\), 981
\(\frac{\partial^2}{\partial x^2}\), 956, 959
\(\frac{\partial}{\partial x}\), 941
\(e_r, e_\theta, e_\phi, \vec{\imath}, \vec{\jmath}, \vec{k}\), 985
d\(\vec{r}\), 934

acceleration, 731
\(a_N\), 749, 760
angle of elevation, 736
annulus, 950
arc length, 729, 753
arc length parameter, 753, 755
\(a_t\), 749, 760
average rate of change, 715
average value of a function, 870

Beta function, 927
boundary point, 773
bounded set, 773
capping surface, 976
center of mass, 885, 886, 888, 889, 916
Chain Rule
  multivariable, 804, 808
change of variable, 921, 923
circle of curvature, 759
circulation, 975
closed, 773
closed curve, 941
closed disk, 773
closed surface, 960
conical helix, 967
conservative field, 945
constrained critical point, 844
constrained optimization, 839
continuous function, 780
  properties, 781
  vector–valued, 719, 720
contour lines, 767
coordinates
  curvilinear, 705
cylindrical, 705, 986
ellipsoidal, 964
  polar, 705
  spherical, 705, 986
critical point, 833, 834, 836
cross product
  and derivatives, 724
  applications, 681
    area of parallelogram, 681
torque, 684
    volume of parallelepiped, 683
definition, 676
    properties, 679, 680
curl, 970, 980, 986
curvature, 756
  and motion, 760
  equations for, 757
  of circle, 758, 759
  radius of, 759
cycloid, 713
cylinder, 637
derivative
  Chain Rule, 804, 808
  directional, 813, 816, 817, 820, 821
  implicit, 811
  mixed partial, 789
  multivariable differentiability, 797, 802
  partial, 785, 792
  vector–valued functions, 721, 724
differentiable, 797, 802
differential, 934
differential form, 934
directed curve, 939
directional derivative, 813, 816, 817, 820, 821
directrix, 637
displacement, 714, 729
distance
  between lines, 694
  between point and line, 694
  between point and plane, 702
  between points in space, 634
  traveled, 739
divergence, 961, 980, 986
Divergence Theorem, 961
dot product
  and derivatives, 724
  definition, 663
  properties, 664
double integral, 862, 863
  in polar, 874
  properties, 867
deviation of, 724
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
deviation, 986
absolute, 833
relative, 833, 834
Extreme Value Theorem, 840

flux, 961
Fubini’s Theorem, 863
function
of three variables, 769
of two variables, 765
vector–valued, 711

gradient, 815–817, 820, 821, 986
and level curves, 817
and level surfaces, 821

Green’s identities, 991
Green’s Theorem, 947
harmonic, 991

Head To Tail Rule, 653
helicoid, 708
helix, 967

implicit differentiation, 811
initial point, 649
integral
surface, 954, 956
integration
area, 854, 855
distance traveled, 739
double, 862
iterated, 854
multiple, 854
notation, 854
of multivariable functions, 851
of vector–valued functions, 727
triple, 902, 913, 915

interior point, 773
irrotational, 975
iterated integration, 854, 862, 863, 902, 913, 915
changing order, 857
properties, 867, 908

Jacobian, 923
Lagrange multiplier, 844
lamina, 881
Laplacian, 981, 986
level curves, 767, 817
level surface, 770, 821

limit
of multivariable function, 775, 776, 783
of vector–valued functions, 718, 719
properties, 776

line integral, 930, 933
lines, 687
distances between, 694
equations for, 689
intersecting, 691
parallel, 691
skew, 691

mass, 881, 882, 916
center of, 885
maximum
absolute, 833
relative/local, 833, 837
minimum
absolute, 833
relative/local, 833, 837
Möbius strip, 969
moment, 887, 889, 916
multiple integration, see iterated integration
multiply connected, 951

multivariable function, 765, 769
continuity, 780, 781, 783, 798, 802
differentiability, 797, 798, 802
domain, 765, 769
level curves, 767
level surface, 770
limit, 775, 776, 783
range, 765, 769

n-positive direction, 969
norm, 649
normal derivative, 991
normal line, 827
normal vector, 697
normal vector field, 968
open, 773
open ball, 783
open disk, 773

optimization
constrained, 839
orientable, 968
orthogonal, 667, 827
decomposition, 670
orthogonal decomposition of vectors, 670
orthogonal projection, 669
osculating circle, 759
outward normal, 958

parallel vectors, 657
Parallelogram Law, 653
partial derivative, 785, 792
high order, 793
meaning, 787
mixed, 789
second derivative, 789
total differential, 796, 801
path independence, 942, 952, 976
perpendicular, see orthogonal
piecewise smooth curve, 936
planes
coordinate plane, 636
distance between point and plane, 702
equations of, 698
introduction, 636
normal vector, 697
tangent, 829

position vector, 933
potential, 944
projectile motion, 736, 751
quadric surface
definition, 640
elipsoid, 643
elliptic cone, 642
elliptic paraboloid, 642
gallery, 642–644
hyperbolic paraboloid, 644
hyperboloid of one sheet, 643
hyperboloid of two sheets, 644
sphere, 643
trace, 640
\( \mathbb{R} \), 649
radius of curvature, 759
Riemann integral, 929
right hand rule
doing of Cartesian coordinates, 634
saddle point, 836, 837
Second Derivative Test, 837
sensitivity analysis, 800
signed volume, 862, 863
simple closed curve, 941
simply connected, 952, 976
smooth, 724
solenoidal, 962
speed, 731
sphere, 635
Stokes' Theorem, 968, 970
surface
orientable, 968
two-sided, 969
surface area, 894
surface integral, 954, 956
surface of revolution, 639, 640
tangent line, 723
directional, 824
tangent plane, 829
terminal point, 649
torus, 957
total differential, 796, 801
sensitivity analysis, 800
trace, 640
triple integral, 902, 913, 915
cylindrical coordinates, 925
properties, 908
spherical coordinates, 926
unbounded set, 773
unit normal vector
\( n \), 749
definition, 744
in \( \mathbb{R}^2 \), 748
unit vector, 655
properties, 657
standard unit vector, 659
unit normal vector, 746
unit tangent vector, 744
vector
normal, 958
positive unit normal, 969
vector field, 933
normal, 968
smooth, 947
vector–valued function
algebra of, 713
arc length, 729
average rate of change, 715
continuity, 719, 720
definition, 711
derivatives, 721, 724
describing motion, 731
displacement, 714
distance traveled, 739
graphing, 711
integration, 727
limits, 718, 719
of constant length, 726, 736, 745
projectile motion, 736
smooth, 724
tangent line, 723
vectors, 649
algebra of, 652
algebraic properties, 655
component form, 650
cross product, 676, 679, 680
definition, 649
dot product, 663, 664
Head To Tail Rule, 653
magnitude, 649
norm, 649
normal vector, 697
orthogonal, 667
orthogonal decomposition, 670
orthogonal projection, 669
parallel, 657
Parallelogram Law, 653
resultant, 653
standard unit vector, 659
unit vector, 655, 657
zero vector, 653
velocity, 731
volume, 862, 863, 900
work, 673, 929, 966
zenith angle, 705
Differentiation Rules

1. \( \frac{d}{dx}(cx) = c \)
2. \( \frac{d}{dx}(u \pm v) = u' \pm v' \)
3. \( \frac{d}{dx}(uv) = u'v + uv' \)
4. \( \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v'u - u'v}{v^2} \)
5. \( \frac{d}{dx}(u(v)) = u'(v)v' \)
6. \( \frac{d}{dx}(c) = 0 \)
7. \( \frac{d}{dx}(x) = 1 \)
8. \( \frac{d}{dx}(e^x) = e^x \)
9. \( \frac{d}{dx}(\sin x) = \cos x \)
10. \( \frac{d}{dx}(\cos x) = -\sin x \)
11. \( \frac{d}{dx}(\tan x) = \sec^2 x \)
12. \( \frac{d}{dx}(\sec x) = \sec x \tan x \)
13. \( \frac{d}{dx}(\csc x) = -\csc x \cot x \)
14. \( \frac{d}{dx}(\cot x) = -\csc^2 x \)
15. \( \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}} \)
16. \( \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2 - 1}} \)
17. \( \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \)
18. \( \frac{d}{dx}(e^x) = e^x \)
19. \( \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \)
20. \( \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}} \)
21. \( \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{x \sqrt{x^2 - 1}} \)
22. \( \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{x \sqrt{x^2 + 1}} \)
23. \( \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2} \)
24. \( \frac{d}{dx}(\tanh x) = \text{sech}^2 x \)
25. \( \frac{d}{dx}(\coth x) = -\text{csch}^2 x \)
26. \( \frac{d}{dx}(\ln x) = \frac{1}{x} \)
27. \( \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \)
28. \( \frac{d}{dx}(\sec x) = \sec x \tan x \)
29. \( \frac{d}{dx}(\csc x) = -\csc x \cot x \)
30. \( \frac{d}{dx}(\tan x) = \sec^2 x \)
31. \( \frac{d}{dx}(\cot x) = -\csc^2 x \)
32. \( \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}} \)
33. \( \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2 - 1}} \)
34. \( \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}} \)
35. \( \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2 - 1}} \)
36. \( \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \)
37. \( \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2} \)
38. \( \frac{d}{dx}(\cosh x) = \sinh x \)
39. \( \frac{d}{dx}(\sinh x) = \cosh x \)
40. \( \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{x \sqrt{x^2 - 1}} \)
41. \( \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{x \sqrt{x^2 + 1}} \)
42. \( \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2} \)
43. \( \frac{d}{dx}(\coth^{-1} x) = -\frac{1}{1 - x^2} \)

Integration Rules

1. \( \int c \cdot f(x) \, dx = c \int f(x) \, dx \)
2. \( \int f(x) \, dx = \int f(x) \, dx \)
3. \( \int 0 \, dx = C \)
4. \( \int 1 \, dx = x + C \)
5. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \)
6. \( \int e^x \, dx = e^x + C \)
7. \( \int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C \)
8. \( \int \frac{1}{x} \, dx = \ln |x| + C \)
9. \( \int \cos x \, dx = \sin x + C \)
10. \( \int \sin x \, dx = -\cos x + C \)
11. \( \int \tan x \, dx = -\ln |\cos x| + C \)
12. \( \int \sec x \, dx = \ln |\sec x + \tan x| + C \)
13. \( \int \csc x \, dx = -\ln |\csc x + \cot x| + C \)
14. \( \int \cot x \, dx = \ln |\sin x| + C \)
15. \( \int \sec^2 x \, dx = \tan x + C \)
16. \( \int \csc^2 x \, dx = -\cot x + C \)
17. \( \int \sec x \tan x \, dx = \sec x + C \)
18. \( \int \csc x \cot x \, dx = -\csc x + C \)
19. \( \int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \sin (2x) + C \)
20. \( \int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin (2x) + C \)
21. \( \int \frac{1}{x^2 + a^2} \, dx = \tan^{-1} \left( \frac{x}{a} \right) + C \)
22. \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + C \)
23. \( \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \ln \left| \frac{x}{a} + \sqrt{x^2 + a^2} \right| + C \)
24. \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x}{a} + \frac{1}{a} + \sqrt{x^2 + a^2} \right| + C \)
25. \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x}{a} - \frac{1}{a} - 1 + \sqrt{x^2 + a^2} \right| + C \)
26. \( \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \ln \left| \frac{x}{a} + \sqrt{x^2 + a^2} \right| + C \)
27. \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x}{a} - \frac{1}{a} + \sqrt{x^2 + a^2} \right| + C \)
28. \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x}{a} + \frac{1}{a} - 1 - \sqrt{x^2 + a^2} \right| + C \)
### The Unit Circle

![Unit Circle Diagram](image)

### Definitions of the Trigonometric Functions

#### Unit Circle Definition
- \( \sin \theta = \frac{y}{r} \)
- \( \cos \theta = \frac{x}{r} \)
- \( \tan \theta = \frac{y}{x} \)
- \( \cot \theta = \frac{x}{y} \)
- \( \sec \theta = \frac{r}{x} \)
- \( \csc \theta = \frac{r}{y} \)

#### Right Triangle Definition
- \( \sin \theta = \frac{O}{H} \)
- \( \cos \theta = \frac{A}{H} \)
- \( \tan \theta = \frac{O}{A} \)
- \( \cot \theta = \frac{A}{O} \)

### Common Trigonometric Identities

#### Pythagorean Identities
- \( \sin^2 x + \cos^2 x = 1 \)
- \( \tan^2 x + 1 = \sec^2 x \)
- \( 1 + \cot^2 x = \csc^2 x \)

#### Cofunction Identities
- \( \sin \left( \frac{\pi}{2} - x \right) = \cos x \)
- \( \cos \left( \frac{\pi}{2} - x \right) = \sin x \)
- \( \tan \left( \frac{\pi}{2} - x \right) = \cot x \)
- \( \csc \left( \frac{\pi}{2} - x \right) = \sec x \)
- \( \sec \left( \frac{\pi}{2} - x \right) = \csc x \)
- \( \cot \left( \frac{\pi}{2} - x \right) = \tan x \)

#### Double Angle Formulas
- \( \sin 2x = 2 \sin x \cos x \)
- \( \cos 2x = \cos^2 x - \sin^2 x \)
- \( = 2 \cos^2 x - 1 \)
- \( = 1 - 2 \sin^2 x \)
- \( \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \)

#### Sum to Product Formulas
- \( \sin x + \sin y = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \)
- \( \sin x - \sin y = 2 \sin \left( \frac{x - y}{2} \right) \cos \left( \frac{x + y}{2} \right) \)
- \( \cos x + \cos y = 2 \cos \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \)
- \( \cos x - \cos y = 2 \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right) \)

#### Power–Reducing Formulas
- \( \sin^2 x = \frac{1 - \cos 2x}{2} \)
- \( \cos^2 x = \frac{1 + \cos 2x}{2} \)
- \( \tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x} \)

#### Even/Odd Identities
- \( \sin(-x) = - \sin x \)
- \( \cos(-x) = \cos x \)
- \( \tan(-x) = - \tan x \)
- \( \sec(-x) = \sec x \)
- \( \csc(-x) = - \csc x \)
- \( \cot(-x) = - \cot x \)

#### Product to Sum Formulas
- \( \sin x \sin y = \frac{1}{2} \left( \cos(x - y) - \cos(x + y) \right) \)
- \( \cos x \cos y = \frac{1}{2} \left( \cos(x - y) + \cos(x + y) \right) \)
- \( \sin x \cos y = \frac{1}{2} \left( \sin(x + y) + \sin(x - y) \right) \)

#### Angle Sum/Difference Formulas
- \( \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \)
- \( \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \)
- \( \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \)
Areas and Volumes

**Triangles**

\[ h = a \sin \theta \]
\[ \text{Area} = \frac{1}{2}bh \]

Law of Cosines:
\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]

**Parallelograms** Area = \(bh\)

**Trapezoids** Area = \(\frac{1}{2}(a + b)h\)

**Circles**

\[ \text{Area} = \pi r^2 \]
\[ \text{Circumference} = 2\pi r \]

**Sectors of Circles**

\(\theta\) in radians
\[ \text{Area} = \frac{1}{2}\theta r^2 \]
\[ s = r\theta \]

**Right Circular Cone**

Volume = \(\frac{1}{3}\pi r^2 h\)

Surface Area = \(\pi r \sqrt{r^2 + h^2 + \pi r^2}\)

**Right Circular Cylinder**

Volume = \(\pi r^2 h\)

Surface Area = \(2\pi rh + 2\pi r^2\)

**Sphere**

Volume = \(\frac{4}{3}\pi r^3\)

Surface Area = \(4\pi r^2\)

**General Cone**

Area of Base = \(A\)

Volume = \(\frac{1}{3}Ah\)

**General Right Cylinder**

Area of Base = \(A\)

Volume = \(Ah\)
Algebra

Factors and Zeros of Polynomials

Let \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) be a polynomial. If \( p(a) = 0 \), then \( a \) is a zero of the polynomial and a solution of the equation \( p(x) = 0 \). Furthermore, \( (x - a) \) is a factor of the polynomial.

Fundamental Theorem of Algebra

An \( n \)th degree polynomial has \( n \) (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If \( p(x) = ax^2 + bx + c \), then the zeros of \( p \) are \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

Special Factoring

\[ \begin{align*}
  x^2 - a^2 &= (x - a)(x + a) \\
  x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\
  x^4 - a^4 &= (x^2 - a^2)(x^2 + a^2)
\end{align*} \]

Binomial Theorem

\[ \begin{align*}
  (x + y)^2 &= x^2 + 2xy + y^2 \\
  (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
  (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
  (x + y)^n &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i
\end{align*} \]

Rational Zero Theorem

If \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) has integer coefficients, then every rational zero of \( p \) is of the form \( x = \frac{r}{s} \), where \( r \) is a factor of \( a_0 \) and \( s \) is a factor of \( a_n \).

Factoring by Grouping

\[ \begin{align*}
  acx^3 + adx^2 + bcx + bd &= ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)
\end{align*} \]

Arithmetic Operations

\[ \begin{align*}
  ab + ac &= a(b + c) \\
  \frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\
  \frac{a}{b} \cdot \frac{a}{c} &= \frac{ac}{bc} \\
  a - b &= \frac{b - a}{d - c}
\end{align*} \]

Exponents and Radicals

\[ \begin{align*}
  a^0 &= 1, \quad a \neq 0 \\
  (ab)^x &= a^x b^x \\
  a^x a^y &= a^{x+y} \\
  \sqrt{a} &= a^{1/2} \\
  \frac{a^x}{a^y} &= a^{x-y} \\
  \sqrt[n]{a} &= a^{1/n}
\end{align*} \]
Additional Formulas

**Summation Formulas**

\[
\begin{align*}
\sum_{i=1}^{n} c &= cn \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left( \frac{n(n+1)}{2} \right)^2
\end{align*}
\]

**Trapezoidal Rule**

\[
\int_{a}^{b} f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_n) + f(x_{n+1}) \right]
\]

with Error \( \leq \frac{(b-a)^3}{12n^2} \max |f''(x)| \)

**Simpson's Rule**

\[
\int_{a}^{b} f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_1) + 4f(x_3) + 2f(x_5) + \cdots + 4f(x_{n-1}) + f(x_{n+1}) \right]
\]

with Error \( \leq \frac{(b-a)^5}{180n^4} \max |f^{(4)}(x)| \)

**Arc Length**

\[
L = \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx
\]

**Work Done by a Variable Force**

\[W = \int_{a}^{b} F(x) \, dx\]

**Force Exerted by a Fluid**

\[F = \int_{a}^{b} w \, d(y) \ell(y) \, dy\]

**Taylor Series Expansion for \( f(x) \)**

\[p_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n\]

**Standard Form of Conic Sections**

<table>
<thead>
<tr>
<th>Parabola</th>
<th>Ellipse</th>
<th>Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical axis</td>
<td></td>
<td>Foci and vertices on x-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{x^2}{4p} )</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 )</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x = \frac{y^2}{4p} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Summary of Tests for Series

<table>
<thead>
<tr>
<th>Test</th>
<th>Series</th>
<th>Condition(s) of Convergence</th>
<th>Condition(s) of Divergence</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$n^{th}$-Term Test for Divergence</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td></td>
<td>$\lim_{n \to \infty} a_n \neq 0$</td>
<td>cannot show convergence.</td>
</tr>
<tr>
<td>Geometric Series</td>
<td>$\sum_{n=0}^{\infty} ar^n$</td>
<td>$</td>
<td>r</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td>Telescoping Series</td>
<td>$\sum_{n=1}^{\infty} b_n - b_{n+m}$</td>
<td>$\lim_{n \to \infty} b_n = L$</td>
<td></td>
<td>$\text{Sum} = \left( \sum_{n=1}^{m} b_n \right) - L$</td>
</tr>
<tr>
<td>$p$-Series</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{(an + b)^p}$</td>
<td>$p &gt; 1$</td>
<td>$p \leq 1$</td>
<td>The base of the logarithm doesn't affect convergence.</td>
</tr>
<tr>
<td>$p$-Series For Logarithms</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{(an + b)(\log n)^p}$</td>
<td>$p &gt; 1$</td>
<td>$p \leq 1$</td>
<td></td>
</tr>
<tr>
<td>Integral Test</td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\int_1^{\infty} a(n) , dn$ converges</td>
<td>$\int_1^{\infty} a(n) , dn$ diverges</td>
<td>$a_n = a(n)$ must be continuous and decreasing</td>
</tr>
<tr>
<td>Direct Comparison</td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$</td>
<td>$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$</td>
<td></td>
</tr>
<tr>
<td>Limit Comparison</td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n \to \infty} a_n/b_n \geq 0$</td>
<td>$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n \to \infty} a_n/b_n &gt; 0$ or $\lim_{n \to \infty} a_n/b_n = 1$</td>
<td></td>
</tr>
<tr>
<td>Ratio Test</td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} &lt; 1$</td>
<td>$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} &gt; 1$ or $\lim_{n \to \infty} a_n = \infty$</td>
<td>${a_n}$ must be positive</td>
</tr>
<tr>
<td>Root Test</td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\lim_{n \to \infty} \left( a_n \right)^{1/n} &lt; 1$</td>
<td>$\lim_{n \to \infty} \left( a_n \right)^{1/n} &gt; 1$ or $\lim_{n \to \infty} a_n = \infty$</td>
<td>${a_n}$ must be positive</td>
</tr>
</tbody>
</table>